ON THE EXISTENCE OF GREEN FUNCTION

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1. Let R be a Riemann surface whose boundary Γ consists of a finite number of analytic Jordan curves.

We call a function $g(p,q)$ satisfying following three conditions Green function of R with its pole at q:

1. $g(p,q)$ is harmonic on R except at a point q.
2. $g(p,q)=0$ on Γ.
3. In a neighbourhood of q which is mapped onto a local parameter disc $|z|<1+\varepsilon\ (\varepsilon>0)$ with the origin $z=0$ corresponding to the point q, it has the form

$$g(p,q)=\log\frac{1}{|z|}+h(z),$$

where $h(z)$ is harmonic in $|z|<1$.

The existence of such a function $g(p,q)$ is well known.

However we shall give here a simple proof.

Now consider a sequence of circumferences $|z|=r_n (n=0,1,\ldots\ldots; r_0=1, r_n\downarrow 0)$ in the parameter disc $|z|\leqq 1$ and denote by Γ_n the image of $|z|=r_n$ on R by mapping $p\leftrightarrow z$.

Let $u_n(p)$ be the single-valued harmonic function in the subdomain R_n of R bounded by Γ and Γ_n such that $u_n(p)$ equals to zero on Γ and to $\log\frac{1}{r_n}$ on Γ_n.

We shall prove the following

THEOREM 1 (Parreau*)). *The sequence of functions* $\{u_n(p)\}$ $(n=1,2,\ldots\ldots)$ *is monotonically increasing with n and converges to Green function* $g(p,q)$ *of R with q as its pole.*

PROOF. Since

$$\log\frac{1}{|z|}=\begin{cases}0 & \text{on } \Gamma_0\\ \log\dfrac{1}{r_{n+1}} & \text{on } \Gamma_{n+1},\end{cases}$$

we have, by the maximum principle,

$$u_{n+1}(p)\geqq\log\frac{1}{|z|} \tag{1}$$

in the annulus R'_{n+1} bounded by Γ_0 and Γ_{n+1}.

In particular, we obtain

*)M. PARREAU, Sur les moyennes des fonctions harmoniques et analytiques et la calssification des surfaces de Riemann, Ann. l'Inst. Fourier 3(1952).

$$u_{n+1}(p) \geqq \log \frac{1}{r_n}$$

on Γ_n. Hence, by applying the maximum principle, we have

$$u_{n+1}(p) \geqq u_n(p) \tag{2}$$

in the doamin R_n.

From (1), it follows that

$$\int_{\Gamma_n} \frac{\partial u_n}{\partial \nu} ds \leqq \int_{\Gamma_n} \frac{\partial}{\partial \nu} \log \frac{1}{|z|} ds = 2\pi,$$

where ν is the outer normal and the integal is taken in the positive sense with respect to R_n. Let $\omega_0(p)$ be the harmonic measure of Γ_0 with respect to R_0. Then, by Green's formula, we have

$$\int_{\Gamma_0} u_n \frac{\partial \omega_0}{\partial \nu} ds = \int_{\Gamma_0} \omega_0 \frac{\partial u_n}{\partial \nu} ds = \int_{\Gamma_n} \frac{\partial u_n}{\partial \nu} ds \leqq 2\pi.$$

Since $\frac{\partial \omega_0}{\partial \nu} > 0$ on Γ_0, there exists at least a point q_n on Γ_0 such that

$$u_n(q_n) \leqq \frac{2\pi}{\int_{\Gamma_0} \frac{\partial \omega_0}{\partial \nu} ds} \tag{3}$$

On the other hand, by Harnack's principle, we see that for any compact subdomain Δ of R_n and for any point q_0 of Δ, there exists a constant K depending only upon (q_0, Δ) such that

$$K\, u_n(q_0) \leqq u_n(p) \leqq \frac{1}{K} u_n(q_0) \tag{4}$$

for every point $p \in \Delta$.

From (3) and (4), we have

$$u_n(p) \leqq \frac{1}{K^2} \frac{2\pi}{\int_{\Gamma_0} \frac{\partial \omega_0}{\partial \nu} ds}$$

in Δ.

This shows that $\{u_n(p)\}$ is uniformly bounded in any compact subdomain of the domain R' which is obtained by deleting the point q from R.

Combining this with (2), we can conclude the following:

The sequence of $\{u_n(p)\}$ converges uniformly, in the wider sense, to a finite harmonic function $u(p)$ on the domain R'.

Now, the function

$$u'_n(p) = \lambda + \left(1 - \frac{\lambda}{\log \frac{1}{r_n}}\right) \log \frac{1}{|z|}, \qquad \lambda = \max_{\Gamma_0} u(p) \tag{5}$$

is harmonic in R'_n and its boundary values equal to λ on Γ_0 and to $\log \frac{1}{r_n}$ on Γ_n.

Then, since $u_n(p) \leqq u(p)$ in R'_n, it follows by using the maximum principle that

(6) $$u'_n(p) \geqq u_n(p)$$

in R'_n. From (1), (5) and (6), we have

$$\lambda + \left(1 - \frac{\lambda}{\log \frac{1}{r_n}}\right) \log \frac{1}{|z|} > u_n(p) > \log \frac{1}{|z|}$$

in R'_n. Hence, letting n tend to ∞, we obtain in $0 < |z| \leqq 1$

$$\lambda \geqq u(p) - \log \frac{1}{|z|} \geqq 0.$$

Thus the function $u(p) - \log \frac{1}{|z|}$ is bounded and harmonic in $0 < |z| \leqq 1$. Therefore, we can see that the function $u(p) - \log \frac{1}{|z|}$ is harmonic even at q.

It is easily seen that $u(p) = 0$ on Γ.

Hence $u(p)$ is Green function.

2. Next we consider the harmonic function $\omega_n(p)$ in R_n such that

$$\omega_n(p) = 0 \text{ on } \Gamma \quad \text{and} \quad \omega_n(p) = 1 \text{ on } \Gamma_n,$$

and let D_n be the Dirichlet integral of $\omega_n(p)$ taken over R_n.

Then

$$D_n = \int_{\Gamma_n} d\bar{\omega}_n = \int_{\Gamma_n} \frac{\partial \omega_n}{\partial \nu} ds,$$

where $\bar{\omega}_n$ is a conjugate harmonic function of ω_n.

THEOREM 2. *The sequence of functions* $\left\{\frac{2\pi\omega_n(p)}{D_n}\right\} (n = 1, 2, \ldots.)$ *converges to Green function of R with its pole at q.*

PROOF. Using Green's formula, we have

$$\int_{\Gamma_0+\Gamma_n} \omega_n \frac{\partial}{\partial \nu} \log \frac{1}{|z|} ds = \int_{\Gamma_0+\Gamma_n} \log \frac{1}{|z|} \frac{\partial \omega_n}{\partial \nu} ds.$$

Since $\frac{\partial}{\partial \nu} \log \frac{1}{|z|} < 0$ on Γ_0, there exists at least a point p_n on Γ_0, at which there holds

$$[1 - \omega_n(p_n)]\, 2\pi = \log \frac{1}{r_n} \cdot D_n.$$

Thus we get

$$\frac{2\pi\omega_n(p)}{D_n} = \frac{\log \frac{1}{r_n} \cdot \omega_n(p)}{1 - \omega_n(p_n)}.$$

Putting $u_n(p) = \log \frac{1}{r_n} \cdot \omega_n(p)$, we can easily see by Theorem 1 that $u_n(p)$ tends to Green function of R with its pole at q.

Further, since $\{\omega_n(p)\}$ $(n = 1, 2, \ldots.)$ converges to the constant zero on R' in the wider sense, $\omega_n(p_n)$ tends to zero as n tends to infinity.

Thus we have

$$\lim_{n\to\infty} \frac{2\pi\omega_n(p)}{D_n} = \lim_{n\to\infty} u_n(p) = g(p, q).$$

3. Now we shall consider a non-constant, positive harmonic function $u(p)$ on R' which equals to zero on Γ.

It is easy to see that $u(p)$ is not bounded in a neighborhood V of q. For, if not so, $u(p)$ is also harmonic and bounded in V and, necessarily, $u(p)$ is harmonic throughout R and $u(p)$ must be identically zero in R, which contradicts our assumption.

Hence there exists a sequence of points $\{p_n\}$ on R' such that $\lim_{n\to\infty} p_n = q$ and $\lim_{n\to\infty} u(p_n) = +\infty$.

Further, we can see that there exists no sequence of points $\{p_n'\}$ on R' such that $\lim_{n\to\infty} p_n' = q$ and $\lim_{n\to\infty} u(p_n') = M < +\infty$. For, if there exists such a sequence $\{p_n'\}$, the single-valued regular function

$$e^{\frac{2\pi}{m}(u(p)+iv(p))},$$

where v is a conjugate function of u and m is the period of v about q, has an essential singularity at q and hence, by Weierstrass' theorem, there exists a sequence of points $\{p_n''\}$ such that $\lim_{n\to\infty} p_n'' = q$ and $\lim_{n\to\infty} u(p_n'') = -\infty$, which contradicts our assumption.

Hence, by the usual manner, we obtain the fact that $u(p)$ has a logarithmic pole at the point q.

Thus we have the following proposition:

Let $u(p)$ be a non-constant positive harmonic function on R' which equals to zero on Γ. Then there holds

$$u(p) = k \cdot g(p, q),$$

where k is a positive constant and $g(p, q)$ is Green function of R with its pole at q.

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