ON A TENSOR FIELD $\phi_i^{\ h}$ SATISFYING $\phi^p = \pm I$

SHIGERU ISHIHARA

Dedicated to Professor Kentaro Yano on his fiftieth birthday.

(Received June 30, 1961)

S.Tachibana [3][1] has recently studied linear connections with respect to which a tensor field $\phi_i^{\ h}$ satisfying $\phi^p = \pm I$, is parallel, and got some necessary and sufficient conditions for a linear connection to make such a structure parallel.

In the present paper, we shall study the integrability condition of such a structure. In case $p = 2$, $\phi_i^{\ h}$ is an almost complex structure, or an almost product structure. In case $p = 3$, $\phi_i^{\ h}$ gives a structure closely related to the almost contact structure or the so-called (F, ξ, η)-structure introduced by S.Sasaki in [2].

The tensor calculus developed in the present paper is quite similar to that given by M.Obata [1].

After giving some preliminaries in §1, we shall study in §2 the linear connection with respect to which a tensor field $\phi_i^{\ h}$, such that $\phi^p = \pm I$, is parallel. §3 is devoted to the study of relations between linear connections making $\phi_i^{\ h}$ parallel and a tensor $L_{ji}^{\ \ h}$ constructed only from $\phi_i^{\ h}$. In §4, we shall discuss the properties of a tensor field $\phi_i^{\ h}$ such that $\phi^3 = I$ as the simplest example for our structures and obtain an integrability condition of such a structure. In the last section the integrability condition for the general case will be given without proof.

1. Preliminaries. In an n-dimensional manifold,[2] a tensor field $\phi_i^{\ h}$ of type (1, 1) and a tensor field $T_{ji}^{\ \ h}$ of type (1, 2) are sometimes denoted respectively by

$$\phi = (\phi_i^{\ h}) \qquad \text{and} \qquad T = (T_{ji}^{\ \ h})$$

by making use of matrix notations with respect to the indices h and i [3]. Let $\psi = (\psi_i^{\ h})$ by an other tensor field of type (1,1). Then we shall use the following notation :

1) See the Bibliography at the end of the paper.

2) We restrict ourselves to differentiable manifolds of class C^∞ and we suppose for all quantities to be of class C^∞.

3) $a, b, c, h, i, j = 1, 2, \dots\dots\dots, n.$

$$\phi\cdot\psi = (\phi_i{}^a\psi_a{}^h),$$
$$T\cdot\psi = (T_{ji}{}^a\psi_a{}^h),\ \phi\cdot T = (\phi_i{}^b T_{jb}{}^h),$$
$$\phi\cdot T\cdot\psi = (\phi_i{}^b T_{jb}{}^a\psi_a{}^h).$$

The identity matrix I denotes obviously the numerical tensor field δ_i^h such that $\delta_i^h = 1$, if $h = i$, and $\delta_i^h = 0$, if $h \neq i$.

We suppose that on a differentiable manifold there is given a non-trivial tensor field $\phi = (\phi_i{}^h)$ of type $(1, 1)$ satisfying $\phi \neq \pm I$ and

$$\phi^p = \varepsilon I, \tag{1.1}$$

for some integer $p(\geqq 2)$, where ε is a constant $+1$ or -1 and ϕ^p denotes the p-th power of the matrix ϕ. Such a tensor field is briefly called a *(p,ε)-structure*. Because this ϕ is non-singular, it has the inverse tensor ϕ^{-1}, which we denote by $\psi = (\psi_i{}^h)$. Denoting ϕ^r and ψ^r respectively by

$$\overset{r}{\phi} = (\overset{r}{\phi}{}_i{}^h) \qquad \text{and} \qquad \overset{r}{\psi} = (\overset{r}{\psi}{}_i{}^h),$$

we have easily from the definition

$$\overset{r}{\phi}{}^{-1} = \overset{r}{\psi}, \qquad \overset{p}{\phi} = \overset{p}{\psi} = \varepsilon I, \tag{1.2}$$

where r is an arbitrary integer.

We shall now define a correspondence Φ_1 which associates a tensor field $\Phi_1 T$ of type (1.2) to any tensor field T of the same type by the following formula:

$$\Phi_1 T = \frac{1}{p}\sum_{r=0}^{p-1} \overset{r}{\phi}\cdot T\cdot \overset{r}{\psi}. \tag{1.3}$$

The components of $\Phi_1 T$ are sometimes denoted by

$$\Phi_1 T = (\Phi_1 T_{ji}{}^h).$$

Then, taking account of (1.2), we see easily

$$\Phi_1\Phi_1 = \Phi_1.$$

Next, defining another correspondence Φ_2 by

$$\Phi_2 T = T - \Phi_1 T, \tag{1.4}$$

we obtain directly from (1.4)

$$\Phi_2\Phi_2 = \Phi_2,\ \Phi_1\Phi_2 = \Phi_2\Phi_1 = 0, \tag{1.5}$$

where 0 means the zero correspondence assigning the zero tensor field to any tensor of type (1.1). Taking account of (1.4) and (1.5), we have

LEMMA 1. *A tensor field T of type $(1, 2)$ satisfies the equation $\Phi_2 T = 0$*

if and only if there exists another tensor field S of the same type such that $T=\Phi_1 S$.

LEMMA 2. *Let A be a given tensor field of type* (1, 2) *and*

$$\Phi_2 T = A \tag{1.6}$$

be a linear equation with unknown tensor field T of the same type. Then (1.6) *has at least one solution if and only if* $\Phi_1 A=0$ (*or equivalently* $\Phi_2 A=A$). *If this is the case, the general solution of* (1.6) *is given by*

$$T = A + U,$$

where U is an arbitrary tensor field of type (1,2) *satisfying* $\Phi_2 U = 0$.

We now give two identities for the later use. For any tensor field T of type (1, 2) we have identities:

$$\Phi_2 T = \frac{1}{p}\sum_{s=1}^{p-1}\sum_{r=0}^{s-1}\overset{r}{\phi}\cdot(T-\phi\cdot T\cdot\psi)\cdot\overset{r}{\psi}, \tag{1.7}$$

$$T-\phi\cdot T\cdot\psi = \Phi_2 T - \phi\cdot(\Phi_2 T)\cdot\psi. \tag{1.8}$$

Let h_{ji} be a positive definite Riemannian metric. Then, as was proved in [3], it is easily verified that a tensor field g_{ji} defined by

$$g_{ji} = \frac{1}{p}\sum_{r=0}^{p-1}\overset{r}{\phi}{}_j{}^c h_{cb}\overset{r}{\phi}{}_i{}^b$$

is a positive definite Riemann metric satisfying

$$\phi_j{}^c g_{cb}\phi_i{}^b = g_{ji}. \tag{1.9}$$

2. ϕ-connections. Let Γ be a linear connection with respect to which the covariant derivative $\nabla_j v^h$ of a contravariant vector field v^h is given by

$$\nabla_j v^h = \partial_j v^h + \Gamma_{ja}^h v^a,$$

where Γ_{ji}^h are coefficients of the connection Γ. A linear connection Γ is called a *ϕ-connection* if it makes a (p,ε)-structure ϕ parallel, i. e. if $\nabla_j\phi^h{}_i=0$. We define a correspondence Φ associating a linear connection $\Phi\Gamma$ to any linear connection Γ by the formula

$$\Phi\Gamma_{ji}^h = \Gamma_{ji}^h + \frac{1}{p}\sum_{r=1}^{p-1}(\nabla_j\overset{r}{\phi}{}_i{}^a)\overset{r}{\psi}{}_a{}^h, \tag{2.1}$$

where $\Phi\Gamma_{ji}^h$ denote the coefficients of the new connection $\Phi\Gamma$. Let $T=(T_{ji}{}^h)$ be a tensor field of type (1,2). Then $\Gamma+T$ denotes a linear connection with coefficients $\Gamma_{ji}^h+T_{ji}{}^h$. Now, by making use of the definition (2. 1) of Φ and the definition (1. 3) of Φ_1, we have directly

LEMMA 3. *We have*

$$\Phi(\Gamma + T) = \Phi\Gamma + \Phi_1 T$$

for any linear connection Γ *and any tensor field* T *of type* $(1, 2)$.

S. Tachibana [3], basing on the notion of the infinitesimal connection defined in the principal tangent bundle, has proved

THEOREM 1. *A linear connection* Γ *is a* ϕ*-connection if and only if there exists another linear connection* $\hat{\Gamma}$ *such that*

$$\Gamma = \Phi\hat{\Gamma}.$$

Theorem 1 shows that *there exists always a* ϕ*-connection in any manifold admitting a* (p, ε)*-structure.* This theorem together with (2.1) implies that the correspondence Φ satisfies $\Phi\Phi = \Phi$ and that, for any ϕ-connection $\Gamma, \Phi\Gamma = \Gamma$ holds good.

Because of Lemmas 1 and 3, Theorem 1 implies

THEOREM 2. *Let* $\overset{*}{\Gamma}$ *be a* ϕ*-connection. Then a necessary and sufficient condition for a linear connection* Γ *to be a* ϕ*-connection is that there exists a tensor field* U *of type* (1,2) *such as*

$$\Gamma = \Phi\overset{*}{\Gamma} + U, \qquad \Phi_2 U = 0.$$

Next, we shall give a pro of of Theorem 1 other than that given in [3].

PROOF OF THEOREM 1. Let $\overset{*}{\Gamma}$ be an arbitrary linear connection. Then a linear connection Γ is a ϕ-connection if and only if

$$T_{ji}{}^h - \phi_i{}^b T_{jb}{}^a \psi_a{}^h = (\overset{*}{\nabla}_j \phi_i{}^a)\psi_a{}^h, \tag{2.2}$$

$\overset{*}{\nabla}$ denoting the covariant differentiation with respect to $\overset{*}{\Gamma}$, where we have put

$$T_{ji}{}^h = \Gamma_{ji}^h - \overset{*}{\Gamma}{}_{ji}^h.$$

Taking account of the identities (1.7) and (1.8), we see easily that the equation (2.2) is equivalent to

$$\Phi_2 T = A, \tag{2.3}$$

where the unknown tensor field $T_{ji}{}^h$ is denoted by T and $A = (A_{ji}{}^h)$ is the tensor field given by

$$A_{ji}{}^h = \frac{1}{p}\sum_{r=1}^{p-1} (\overset{*}{\nabla}_j \overset{r}{\phi}_i{}^a)\overset{r}{\psi}_a{}^h.$$

Here, if we take account of (1.7), we have $\Phi_2 A = A$. This means that $T = A$ is a solution of (2.3). Therefore, Lemma 2 implies that the general solution of (2. 3) is given by

$$T = A + \Phi_1 U,$$

where U is a certain tensor field of type (1, 2). Thus, a linear connection Γ is a ϕ-connection if and only if

$$\begin{aligned}\Gamma &= \overset{*}{\Gamma} + A + \Phi_1 U,\\ &= \Phi\overset{*}{\Gamma} + \Phi_1 U,\\ &= \Phi(\overset{*}{\Gamma} + U).\end{aligned}$$

This proves Theorem 1.

3. The tensor $L_{ji}{}^h$. Let $\overset{\circ}{\Gamma}$ be a symmetric linear connection and put $\Gamma = \Phi\overset{\circ}{\Gamma}$, which is a ϕ-connection. Denoting by $S = (S_{ji}{}^h)$ the torsion tensor of the ϕ-connection Γ, we have

$$S_{ji}{}^h = \frac{1}{2}\sum_{r=1}^{p-1} (\overset{\circ}{\nabla}_{[j}\overset{r}{\phi}_{i]}{}^a)\overset{r}{\psi}_a{}^h, \tag{3.1}$$

where $\overset{\circ}{\nabla}$ means the covariant differentiation with respect to $\overset{\circ}{\Gamma}$. Since $\overset{\circ}{\Gamma}$ is symmetric, it follows from (3.1)

$$S_{ji}{}^h = \frac{1}{p}\left\{\sum_{r=1}^{p-1} (\partial_{[j}\phi_{i]}{}^a)\overset{r}{\psi}_a{}^h - \sum_{r=1}^{p-1} \phi_{[j}{}^b\overset{\circ}{\Gamma}_{i]}{}^a{}_b\overset{r}{\psi}_a{}^h\right\}, \tag{3.2}$$

where $\overset{\circ}{\Gamma}_{ji}{}^h$ denote the coefficients of $\overset{\circ}{\Gamma}$. Now, taking account of (3. 2) and the symmetry of $\overset{\circ}{\Gamma}$, we see that the tensor field

$$L_{ji}{}^h = S_{ji}{}^h - (\Phi_1 S_{ji}{}^h - \Phi_1 S_{ij}{}^h) \tag{3.3}$$

is independent of the connection $\overset{\circ}{\Gamma}$, i.e. $L_{ji}{}^h$ is a tensor field completely determined by the given structure ϕ. The tensor $L_{ji}{}^h$ can be explicitly written down as

$$\begin{aligned}p^2 L_{ji}{}^h &= (p-1)\sum_{r=1}^{p-1} (\partial_{[j}\phi_{i]}{}^a\overset{r}{\psi})_a{}^h\\ &- \sum_{s=1}^{p-1}\sum_{r=1}^{p-1} \overset{s}{\phi}_{[j}{}^b(\partial_{|b|}\overset{r}{\phi}_{i]}{}^a)\overset{r+s}{\psi}{}_a{}^h.\end{aligned}$$

Now, we consider a linear connection $\hat{\Gamma}$ defined by

$$\hat{\Gamma} = \Gamma - 2\Phi_1 S.$$

Then, by Theorem 2, $\hat{\Gamma}$ is a ϕ-connection. Its torsion tensor is obviously equal to $L_{ji}{}^h$. Thus, we have

LEMMA 4. *In any manifold admitting a (p, ε)-structure ϕ, there always exists a ϕ-connection whose torsion tensor is equal to $L_{ji}{}^{h}$.*

Taking account of the definition (3.3) of the tensor field $L_{ji}{}^{h}$, we see that $L_{ji}{}^{h}$ vanishes if the manifold admits a symmetric ϕ-connection. Then Lemma 4 implies

THEOREM 3. *In a manifold admittinga (p, ε)-structure, a necessary and sufficient condition for the corresponding tensor field $L_{ji}{}^{h}$ to vanish identically is that there exists a symmetric ϕ-connection.*

We shall next give simple forms of $L_{ji}{}^{h}$ for some smaller values of p.

When $p = 2$, (3.4) is reduced to

$$4\varepsilon L_{ji}{}^{h} = (\partial_{[j}\phi_{i]}{}^{a})\phi_{a}{}^{h} - \phi_{[j}{}^{b}\partial_{|b|}\phi_{i]}{}^{h}, \qquad (p = 2),$$

which is nothing but the Nijenhuis tensor of the almost complex structure ϕ (if $\varepsilon = -1$) or that of the almost product structure ϕ (if $\varepsilon = +1$).

When $p = 3$, (3.4) is reduced to

$$\begin{aligned} 9\varepsilon L_{ji}{}^{h} = {} & 2\{(\partial_{[j}\overset{1}{\phi}_{i]}{}^{a})\overset{2}{\phi}_{a} + (\partial_{[j}\overset{2}{\phi}_{i]}{}^{a})\overset{1}{\phi}_{a}{}^{h}\} \\ & - \{(\overset{1}{\phi}_{[j}{}^{b}\partial_{|b|}\overset{1}{\phi}_{i]}{}^{a})\overset{1}{\phi}_{a}{}^{h} + (\overset{2}{\phi}_{[j}{}^{b}\partial_{|b|}\overset{2}{\phi}_{i]}{}^{a})\overset{2}{\phi}_{a}{}^{h}\} \\ & - \{\overset{1}{\phi}_{[j}{}^{b}\partial_{|b|}\overset{2}{\phi}_{i]}{}^{h} + \overset{2}{\phi}_{[j}{}^{b}\partial_{|b|}\overset{1}{\phi}_{i]}{}^{h}\} \qquad (p = 3). \end{aligned}$$

4. $(3, +1)$-structures. Let ϕ be a $(p, -1)$-structure. If p is odd, $-\phi$ is obviously a $(p, +1)$-structure. Then, in the case where p is odd, it is sufficient for us to consider only $(p, +1)$-structures.

Let ϕ be a $(3, +1)$-structure. First, putting

$$Q = \frac{1}{3}(I + \phi + \phi^{2}),$$

we have

$$Q^{2} = Q.$$

Hence, defining P by

$$P = I - Q,$$

we see easily that

$$P^{2} = P, \ P \cdot Q = Q \cdot P = 0,$$

i.e. that the pair (P, Q) defines an almost product structure if $Q \neq 0$. When $Q = 0$, if we put

$$F = \frac{1}{\sqrt{3}}(I + 2\phi),$$

we have

$$F^2 = -I.$$

This implies that the manifold admits an almost complex structure if $Q = 0$.

Next, putting in general case

$$F = \frac{1}{\sqrt{3}}(\phi - \phi^2),$$

we obtain easily

$$F^2 = -P, \qquad F\cdot P = P\cdot F = F, \tag{4.1}$$

which implies

$$F\cdot Q = Q\cdot F = 0, \tag{4.2}$$

$$F^3 = -F. \tag{4.3}$$

Conversely, we assume the existence of a non-zero tensor field F of type (1, 1) which satisfies (4. 3). On putting

$$\phi = \frac{3}{2}F^2 + \frac{\sqrt{3}}{2}F + I,$$

it is easily verified

$$\phi^3 = I.$$

Summing up, we obtain

LEMMA 5. *A necessary and sufficient condition for a manifold to admit a* $(3, +1)$*-structure* ϕ *is that it admits a non-zero tensor field* F *of type* (1,1) *satisfying* $F^3 = -F$.

Now, let g_{ji} be a positive definite Riemann metric satisfying (1. 9). Then it is easily verified

$$P_j{}^c g_{cb} P_i{}^b + Q_j{}^c g_{cb} Q_i{}^b = g_{ji},$$
$$F_j{}^c g_{cb} F_i{}^b = P_j{}^c g_{cb} P_i{}^b.$$

These two relations imply together with Lemma 5

THEOREM 4. *A necessary and sufficient condition for an* n*-dimensional manifold to admit a* $(3, +1)$*-structure is that the structure group of its tangent bundle is reducible to the group* $O(m) \times U(r)$, *where* $m \geqq 0$ *and* $r > 0$ *are certain integers such that* $m + 2r = n$.

In this theorem, we have denoted by $O(m)$ and $U(r)$ respectively the orthogonal group of the n-dimensinoal Euclidean space and the unitary group of the unitary space of r complex dimensions.

In Theorem 4, if $m = 1$, the structure ϕ is closely related to the almost contact structure and the so-called (F, ξ, η)-structure introduced by S.Sasaki [2]. That is, *any orientable manifold with a* $(3, +1)$*-structure admits a* (F, ξ, η)*-structure* (or *equivalently an almost contact structure*) *if the tensor* F *corresponding to* ϕ *is of rank* $n-1$. In fact, at any point x of the manifold the set of all vectors X^h satisfying $X^a F_a{}^h = 0$ forms a 1-dimensional subspace L_x in each tangent space T_x. The set of all L_x forms obviously a differentiable distribution of 1 dimension throughout the manifold. Let g_{ji} be a positive difinite Riemann metric satisfying (1. 9). Then, it is easily verified that the tensor $F_{ji} = F_j{}^a g_{ai}$ is skew-symmetric.

We now assume the manifold to be orientable. There exists obviously the skew-symmetric tensor field

$$\sigma^{i_1 i_2 \cdots i_n} = \frac{1}{\sqrt{g}} \varepsilon^{i_1 i_2 \cdots i_n}$$

of type $(n, 0)$, where $g = |g_{ji}|$ and $\varepsilon^{i_1 i_2 \cdots i_n}$ is equal to $+1$ if $(i_1, i_2, \cdots, i_n)$ is an even permutation; equal to -1 if $(i_1, i_2, \cdots i_n)$ is an odd permutation; equal to zero otherwise. Because the rank of F_{ji} is $2r$ $(= n-1)$, the vector

$$w^h = F_{i_1 i_2} \cdots\cdots F_{i_{n-2} i_{n-1}} \sigma^{i_1 i_2 \cdots i_{n-2} i_{n-1} h}$$

is everywhere non-zero and $w^a F_a{}^h = 0$. This means that the vector field w^h is everywhere non-zero and lying on L_x. On putting

$$\xi^h = w^h / \sqrt{g_{ji} w^j w^i},$$

ξ^h is a field of unit vectors lying on L_x at each point. Then if we put $\eta_i = \xi^a g_{ia}$, we see

$$Q_i{}^h = \eta_i \xi^h.$$

This implies together with (4. 1)

$$F_i{}^a F_a{}^h = -\delta_i^h + \eta_i \xi^h.$$

A triple $(F_i{}^h, \xi^h, \eta_i)$ satisfying this relation is called a (F, ξ, η)-structure introduced by S.Sasaki [2].

We suppose now that for the given $(3, +1)$-structure ϕ the tensor $L_{ji}{}^h$ vanishes identically. Then, by virtue of Theorem 3, the manifold admits a symmetric ϕ-connection Γ. Keeping the notations for tensor fields $P = (P_j{}^h)$, $Q = (Q_i{}^h)$ and $F = (F_i{}^h)$ as above, we obtain

$$\nabla_j P_i{}^h = 0, \ \nabla_j Q_i{}^h = 0, \ \nabla_j F_i{}^h = 0$$

as a consequence of $\nabla_j \phi_i{}^h = 0$. The first two equations show that the almost product structure (P, Q) is integrable, i.e. for any point of the manifold there exists a coordinate neighbourhood (U, x^h) of this point in which P and Q have respectively the following numerical components:[4]

$$P = \begin{pmatrix} \delta_\beta^\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & \delta_\mu^\lambda \end{pmatrix},$$

where we have assumed that P is of rank $n-m$ $(0 \leqq m < n)$.

It is easily seen that in (U, x^h) the ϕ-connection Γ has zero components except $\Gamma_{\beta\gamma}^\alpha$ and $\Gamma_{\nu\mu}^\lambda$. Taking account of (4. 1) and (4. 2), we see that F has the components

$$F = \begin{pmatrix} F_\alpha{}^\beta & 0 \\ 0 & 0 \end{pmatrix} \tag{4. 4}$$

in (U, x^h), where (4. 1) implies

$$F_\beta{}^\gamma F_\gamma{}^\alpha = -\delta_\beta^\alpha. \tag{4. 5}$$

In the neighbourhood (U, x^h) any submanifold V defined by $x^\lambda =$ const. is an integral manifold of the $(n-m)$-dimensional distribution determined by P. Then (4.4) and (4.5) mean that F induces an almost complex structure $\widetilde{F} = (F_\alpha^\beta)$ in each V. On the other hand, $\Gamma_{\gamma\beta}^\alpha$ define a symmetric linear connection $\widetilde{\Gamma}$ in each V. Moreover, $\nabla_j F_i{}^h = 0$ implies $\widetilde{\nabla}_\gamma \widetilde{F}_\beta{}^\alpha = 0$ and $\partial_\lambda F_\beta{}^\alpha = 0$, where $\widetilde{\nabla}$ denotes the covariant differentiation with respect to $\widetilde{\Gamma}$. Therefore, the almost complex structure $\widetilde{F}$ is integrable in each V. This means that for any point of V there exists in V a coordinated neighbourhood $(\widetilde{U}, \widetilde{x}^\alpha)$ of this point, in which $\widetilde{F} = (F_\beta{}^\alpha)$ has the following numerical components

$$\widetilde{F} = (F_\beta{}^\alpha) = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix},$$

where $n - m = 2r$ and I_r is the unit (r, r)-matrix. This fact implies together with $\partial_\lambda F_\beta{}^\alpha = 0$ that for any point of the manifold there exists a coordinated neighbourhood (U, x^h) of this point, in which the tensor field F has the numerical components

4) $\alpha, \beta, \gamma = 1, 2, \cdots, n-m$; $\lambda, \mu, \nu = n-m+1, n-m+2, \cdots, n$.

$$F = \begin{pmatrix} 0 & -I_r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Summing up, we have proved that if $L_{ji}{}^h = 0$, ϕ is integrable, i.e. for any point of the manifold there exists a coordinated neighbourhood (U, x^h) of this point, in which the tensor field ϕ has the numerical components

$$\phi = \begin{pmatrix} -\frac{1}{2} I_r & -\frac{\sqrt{3}}{2} I_r & 0 \\ \frac{\sqrt{3}}{2} I_r & -\frac{1}{2} I_r & 0 \\ 0 & 0 & I_m \end{pmatrix}.$$

Conversely, it is obvious that when ϕ is integrable, the tensor field $L_{ji}{}^h$ vanishes identically. Then we have

THEOREM 5. *In a manifold admitting a* $(3, +1)$*-structure* ϕ, *a necessary and sufficient condition for* ϕ *to be integrable is that the tensor field* $L_{ji}{}^h$ *vanishes identically.*

As is proved above, if ϕ is integrable, there exist two systems of integral submanifolds in the manifold, corresponding respectively to P and to Q, and each integral submanifold corresponding to P admits an integrable almose complex structure defined by F.

5. The integrability conditions of (p, ε)-structures. Corresponding to Theorem 5, we shall give without proof a theorem explaining the integrability condition of (p, ε)-structures. The (p, ε)-structure is by definition integrable when for any point of the manifold there exists a coordinated neighbourhood (U, x^h) of this point, in which the structure has numerical components.

THEOREM 6. *A* (p, ε)*-structure* ϕ *is integrable if and only if the corresponding tensor field* $L_{ji}{}^h$ *vanishes identically.*

Next, corresponding to Theorem 4, we shall state without proof

THEOREM 7. *A necessary and sufficient condition for a manifold to admit a* (p,ε)*-structure is that the structue group of its tangent bundle is reducible* (i) *if* p *is odd, say* $p = 2q + 1$, *to the group*

$$O(m) \times U(r_1) \times \dots \times U(r_q),$$

where $n > m \geqq 0$, $r_1, r_2, \dots, r_q \geqq 0$,

$$m + 2(r_1 + r_2 + \dots + r_q) = n;$$

(ii) *if p is even, $p > 2$, say $p = 2q + 2$, and $\varepsilon = +1$, to the group*

$$O(m) \times O(m') \times U(r_1) \times \dots \times U(r_q),$$

where $n > m$, $m' \geqq 0$, $r_1, \cdots, r_q \geqq 0$,

$$m + m' + 2(r_1 + r_2 + \dots + r_q) = n;$$

(iii) *if p is even, say $p = 2q$, and $\varepsilon = -1$, to the group*

$$U(r_1) \times U(r_2) \times \cdots \times U(r_q),$$

where $r_1, r_2, \cdots, r_q \geqq 0$, $2(r_1 + r_2 + \dots + r_q) = n$.

Corresponding to each case given in Theorem 7, we have the following result. For each case, the integers $m, m', r_1, r_2, \cdots, r_q$ are restricted within the same ranges as in Theorem 7.

In the case (i), *where p $(= 2q + 1)$ is odd and $\varepsilon = +1$, there exist in the manifold $q + 1$ tensor fields $E, F_1, \cdots, F_q$ of type* (1, 1) *satisfying*

$$E^2 = E,\ F_u^3 = -F_u,\ E \cdot F_u = F_u \cdot E = 0,\ (u = 1, 2, \cdots, q),$$

$$F_u \cdot F_v = F_v \cdot F_u = 0, (u \neq v;\ u, v = 1, 2, \dots, q),$$

where the rank of E is m and the rank of F_u is r_u. The structure ϕ has the following decomposition:

$$\phi = E + \sum_{u=1}^{q} \left\{ \left(\cos \frac{2u\pi}{p} \right) E_u + \left(\sin \frac{2u\pi}{p} \right) F_u \right\},$$

where $E_u = -F_u^2$.

In the case (ii), *where $p (= 2q + 2)$ is even, $p > 2$ and $\varepsilon = +1$, there exist in the manifold $q + 2$ tensor fields $E, E', F_1, \cdots, F_q$ of type* (1, 1) *satisfying*

$$E^2 = E,\ E'^2 = E',\ E \cdot E' = E' \cdot E = 0,$$

$$F_u^3 = -F_u, E \cdot F_u = F_u \cdot E = 0,\ E' \cdot F_u = F_u \cdot E' = 0,\ (u = 1, 2, \cdots, q),$$

$$F_u \cdot F_v = F_v \cdot F_u = 0,\ (u \neq v;\ u, v = 1, 2, \cdots, q),$$

where the rank of E is m, the rank of E' is m' and the rank of F_u is r_u. The structuer ϕ has the following decomposition:

$$\phi = E + E' + \sum_{u=1}^{q} \left\{ \left(\cos \frac{2u\pi}{p} \right) E_u + \left(\sin \frac{2u\pi}{p} \right) F_u \right\},$$

where $E_u = -F_u^2$.

In the case (iii), *where $p (= 2q)$ is even and $\varepsilon = -1$, there exist in the* *manifold q tensor fields* $F_1, F_2, \dots F_q$ *of type* $(1,1)$ *satisfying*

$$F_u^3 = -F_u, \qquad (u = 1, 2, \dots, q),$$

$$F_u \cdot F_v = F_v \cdot F_u = 0, \ (u = v;\ u, v = 1, 2, \dots, q),$$

where the rank of F_u *is* r_u. *The structure* ϕ *has the following decomposition*:

$$\phi = \sum_{u=1}^{q} \left\{ \left(\cos \frac{2(u+1)\pi}{p} \right) E_u + \left(\sin \frac{2(u+1)\pi}{p} \right) F_u \right\},$$

where $E_u = -F_u^2$:

When the structure ϕ is integrable,in every case the projection tensor fields E, E', E_u are all integrable and F_u determines an integrable almost complex structure in each integral submanifold corresponding to the projection tensor field E_u.

BIBLIOGRAPHY

[1] M. OBATA, Affine connections on manifolds with almost complex, quaternion or Hermitian structure., Japan. Jour. Math., 26(1956), 43-77.

[2] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I., Tôhoku Math. Jour., 12(1960), 459-476.

[3] S. TACHIBANA, A remark on linear connections with some properties, Natural Science Report of Ochanomizu Univ., 10(1959), 1-6.

UNIVERSITY OF HONG KONG,
TOKYO GAKUGEI UNIVERSITY.