APPROXIMATION OF FUNCTIONS BY RIESZ MEAN OF THEIR FOURIER SERIES

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Let $\varphi(u)$ be defined in $0 \leqq u \leqq 1$ and continuous at $u = 0$ and of bounded variation on (0, 1). Then we consider the mean of series $\sum a_n$ by

$$\sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}\right) a_k, \quad \varphi(0) = 1.$$

When $\varphi(u) = (1 - u^\beta)^\delta$ $(\beta, \delta > 0)$, we say this Riesz mean of the series.

Let $f(x)$ be periodic and integrable over $(0, 2\pi)$, let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x),$$

and be its Riesz mean

$$R_n(x,f) = A_0 + \varphi\left(\frac{1}{n}\right) A_1(x) + \cdots + \varphi\left(\frac{n-1}{n}\right) A_{n-1}(x), \ \varphi(u) = (1 - u^\beta)^\delta.$$

When $\delta = 1$ and β is an integer, the approximation of $f(x)$ by Riesz mean $R_n(x,f)$ was solved by Zygmund [4].

Sz. Nagy [3] treated the general case. He did not calculate completely, but if we calculate following his method, we have,

THEOREM A. (SZ. NAGY). *If $f(x)$ is r-times differentiable and $f^{(r)}(x) \in \mathrm{Lip}\,\alpha\ (0 < \alpha \leqq 1)$, then*

$$|R_n(x,f) - f(x)| = O\left(\frac{1}{n^{\alpha+r}}\right), \text{ if } \gamma > \alpha + r,$$

$$|R_n(x,f) - f(x)| = O\left(\frac{\log n}{n^{\alpha+r}}\right), \text{ if } \gamma = \alpha + r, (*)$$

$$|R_n(x,f) - f(x)| = O\left(\frac{1}{n^{\gamma}}\right), \text{ if } \gamma < \alpha + r,$$

where $\gamma = \min(\beta, r + \delta)$. In the special case $\alpha = 0$ and $r =$ an even integer of (*), the factor $\log n$ *is suppressed.*

From this, we may infer that his order of approximation depends upon δ.

However we can prove that this does not occur. That is, we can prove the following theorem.

THEOREM. *If* $f(x)$ *is r-th differentiable and* $f^{(r)}(x) \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$, *then*

$$(1)\qquad |R_n(x,f)-f(x)| = O\left(\frac{1}{n^{\alpha+r}}\right),\ \text{if } \beta>\alpha+r,$$

$$(2)\qquad |R_n(x,f)-f(x)| = O\left(\frac{\log n}{n^{\alpha+r}}\right),\ \text{if } \beta=\alpha+r, (**)$$

$$(3)\qquad |R_n(x,f)-f(x)| = O\left(\frac{1}{n^{\beta}}\right),\ \text{if } \beta<\alpha+r.$$

In the special case $\beta=r=$ *an even integer and* $\alpha=0$ *of* (**),

$$(2')\qquad |R_n(x,f)-f(x)| = O\left(\frac{1}{n^{r}}\right).$$

The proof of this theorem is easily reducible to the following two propositions.

PROPOSITION 1. *The saturation order and saturation class of Riesz mean are* $n^{-\beta}$ *and* $\sum k^{\beta}A_k(x) \in L^{\infty}(0,2\pi)$, *respectively.*

This is independent of δ. The necessity part is proved by G.Sunouchi-C.Watari [2], and sufficiency part is given by Sz. Nagy [3] implicitly. For the sake of completeness, we shall perform the calculation following Nagy's method.

LEMMA. *Let us write* $\rho_n^{[\beta]} = \sup_f \max_x |R_n(x,f)-f(x)|$, *where the supremum is taken over the class consisting of functions for which*

$$\sum_{k=1}^{\infty} k^{\beta}A_k(x) \sim f^{[\beta]}(x), \quad |f^{[\beta]}(x)| \leqq 1.$$

Then,

$$\rho_n^{[\beta]} = O\left(\frac{1}{n^{\beta}}\right).$$

In order to prove the lemma, we use Nagy's theorem B [3].

THEOREM B. (Sz. Nagy). *For given* $\beta>0$, *we set* $\psi_\beta(u) = u^{-\beta}(1-\varphi(u))$ *in* $0<u\leqq 1$. *Furthermore, we assume that* $\psi_\beta(0)=\psi_\beta(+0)$ *exists and that* $\psi_\beta(u)$ *satisfies the following conditions.*

[i] $\psi'_\beta(u)$ *is of bounded variation except at least finite points*: $0, a_1, \cdots, a_p, 1$.
[ii] *The next integrals converge.*

$$\int_{+0} u\,|d\psi'_\beta(u)|, \quad \int' |u - a_i| \log \frac{1}{|u - a_i|}\, |d\psi'_\beta(u)|,$$

and

$$\int^{1-0} (1 - u) \log \frac{1}{1 - u}\, |d\psi'_\beta(u)|,$$

where $\int'$ *means* $\int_b^{a-0} + \int_{a+}^{c}$, *if an interval* (b, c) *contains an exceptional point* a.

Then,

$$\rho_n^{[\beta]} = O\,(n^{-\beta}).$$

The points that do not satisfy the conditions of theorem B are called (N)-singular points.

PROOF OF LEMMA. We have only to verify that the points $u = 0$, $u = 1$ are not (N)-singular.

(I) In a neighbourhood of $u = 0$, we have

$$\psi_\beta(n) = u^{-\beta}\{1 - (1 - u^\beta)^\delta\} \fallingdotseq \delta,$$

and the point $u = 0$ is not (N)-singular.

(II) In a neighbourhood of $u = 1$, we set $\nu = 1 - u$.
Since

$$\varphi(u) = [1 - (1 - \nu)^\beta]^\delta = [\beta\nu - \binom{\beta}{2}\nu^2 + \cdots]^\delta \fallingdotseq \nu^\delta q(\nu),$$
$$\psi_\beta(u) = u^{-\beta}\{1 - (1 - u)^\delta q(1 - u)\} \fallingdotseq 1 - (1 - u)^\delta q(1 - u),$$
$$\varphi'(u) = \nu^{\delta-1} q_1(\nu), \text{ and } \varphi''(u) = \nu^{\delta-2} q_2(\nu).$$

where $q(\nu)$, $q_1(\nu)$ and $q_2(\nu)$ are analytic in a neighbourhood of $\nu = 0$.
On the other hand,

$$\int_{1/2}^{1-0} (1 - u) \log \frac{1}{1 - u}\, |d\psi'_\beta(u)| = \int_{1/2}^{1-0} (1 - u) \log \frac{1}{1 - u}\, |\varphi'(u)|\, du$$
$$= \int_{+0}^{1/2} \nu^{\delta-1} \log \frac{1}{\nu}\, |q_2(\nu)|\, d\nu < \infty.$$

From these facts, we conclude that the point $u = 1$ is not (N)-singular.
Therefore, (I), (II) and theorem B yield the fact that

$$\rho_n^{[\beta]} = O(n^{-\beta}).$$

Hence, we verified Proposition 1 completely.

PROPOSITION 2 (G.SUNOUCHI [1]). *Let* $r = 0, 1, 2, \cdots$ *and* $0 < \alpha \leqq 1$. *Suppose that for linear approximation processes* $T_n(f)$

(1) $$|f(x)| \leqq M_1 \text{ implies } |T_n(f)(x)| \leqq K_1 M_1,$$

and

(2) $$|f^{[\beta]}(x)| \leqq M_2 \text{ implies } |f(x) - T_n(f)(x)| \leqq K_2 M_2 n^{-\beta},$$

where $n^{-\beta}$ *is the best approximation of the class of functions*

$$f^{(r)}(x) \in {}^2\Lambda_\alpha : r + \alpha = \beta, \ r \text{ is an integer}, \ 0 < \alpha \leqq 1.$$

Then,

$$f^{(r)}(x) \in {}^2\Lambda_\alpha, \ r + \alpha < \beta \Leftrightarrow f(x) - T_n(f)(x) = O(n^{-r-\alpha}),$$

where $f^{(r)}(x) \in {}^2\Lambda_\alpha$ *means*

$$f^{(r)}(x+h) + f^{(r)}(x-h) - 2f^{(r)}(x) = O(|h|^\alpha).$$

PROOF OF THEOREM. (1) can be proved from Propositions 1 and 2. (3) can be verified from Propositions 1 or 2. Thus it remains only to show that (2) holds. For simplicity we consider $r = 0$ and $0 < \alpha < 1$. The proof of the remaining cases is entirely the same.

We set $f_\mu(x)$ the moving average of $f(x)$, that is

$$f_\mu(x) = \frac{1}{2\mu}\int_{-\mu}^{\mu} f(x+t)dt = \frac{1}{2\mu}\{F(x+\mu) - F(x-\mu)\},$$

then

$$f_\mu(x) - f(x) = \frac{1}{2\mu}\int_{-\mu}^{\mu}\{f(x+t) - f(x)\}dt = O(\mu^\alpha).$$

Moreover we set $g(x) = f(x) - f_\mu(x)$,

$$f(x) - R_n(x,f) = f_\mu(x) - R_n(x,f_\mu) + g(x) - R_n(x,g).$$

Since $g = O(\mu^\alpha)$,

$$|g - R_n(x,g)| = O(\mu^\alpha).$$

Thus it remains to estimate $|f_\mu(x) - R_n(x,f_\mu)|$. We note that

$$\frac{d^\lambda}{dx^\lambda} f_\mu(x) = \frac{1}{2\mu}\{F^\lambda(x+\mu) - F^\lambda(x-\mu)\}.$$

Since $F^\lambda(x)$ is $(1-\lambda)$-th fractional integral of $f(x)$ and now we consider the case $\lambda = \alpha$, by the well-known theorem [5],

$$
\begin{aligned}
F^{\alpha}(x+\mu)-F^{\alpha}(x-\mu) &= f_{1-\alpha}(x+\mu)-f_{1-\alpha}(x-\mu) \\
&= f_{1-\alpha}(x+\mu)-f_{1-\alpha}(x)+f_{1-\alpha}(x)-f_{1-\alpha}(x-\mu) \\
&= O\left(\mu \log\frac{1}{\mu}\right).
\end{aligned}
$$

Consequently

$$\frac{d^{\alpha}}{dx^{\alpha}} f_{\mu}(x) = O\left(\frac{1}{2\mu}\,\mu \log\frac{1}{\mu}\right) = O\left(\log\frac{1}{\mu}\right).$$

On the same way, since $\tilde{f}(x) \in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$,

$$\frac{d^{\alpha}}{dx^{\alpha}} \tilde{f}_{\mu}(x) = O\left(\log\frac{1}{\mu}\right).$$

Therefore,

$$f_{\mu}(x) \in W^{\alpha},$$

where W^{α} means the class of functions which

$$\sum_{k=1}^{\infty} k^{\alpha} A_k(x) \sim f^{[\alpha]} \in L^{\infty}(0, 2\pi).$$

By the saturation theorem, we get $|f_{\mu}(x) - R_n(x, f_{\mu})| = O\left(\frac{1}{n^{\alpha}} \log\frac{1}{\mu}\right)$.

Set $\mu = \dfrac{\pi}{n}$, then we have

$$|f(x) - R_n(x, f)| = O\left(\frac{\log n}{n^{\alpha}}\right).$$

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