ON A THEOREM OF MAILLET

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(Received September 4, 1962)

A consequence of the theorem of K.F.Roth on "Rational approximations to algebraic numbers" [1] is the

THEOREM (a). *If*

1) p_n/q_n, $n = 1,2,\cdots$, *with* $(p_n, q_n) = 1$, $q_{n+1} > q_n > 0$ *is an infinite sequence of quotients of integers, and if*

2) *there exists a sequence of real numbers* s_n, $n = 1,2,\cdots$ *such that* $\tau = \overline{\lim_{n\to\infty}} s_n > 2$, *and if*

3) *for an irrational number* ρ *the inequalities*

$$(1) \qquad |\rho - (p_n/q_n)| \leqq q_n^{-s_n},\ n = 1,2,\cdots$$

are satisfied, then ρ *is a transcendental number.*

Briefly, we shall refer to such numbers as ρ-numbers. The conditions of theorem (a) are sufficient ones. We shall give a necessary and sufficient condition for a subset of ρ-numbers, a subset containing numbers ρ_0, for which the sequences $\{p_n/q_n\}$, $q_1 > 1$, represent convergents to the simple continued fraction expansions for ρ_0-numbers.

THEOREM. *An irrational number* $[b_0, b_1, b_2\cdots]$ *with convergents* p_n/q_n, $q_1 > 1$, *is a* ρ_0*-number if and only if infinitely many partial denominators*

$$b_{n+1} > q_n^{s'_n-2},$$

where $\{s'_n\}$ *is a sequence of real numbers with* $\tau = \overline{\lim_{n\to\infty}}\, s'_n > 2$.

PROOF. We observe, since ρ, and therefore ρ_0, satisfy 1.), 2.), 3.) of theorem (a), we can extract a subsequence from $\{s_n\}$, $\{s'_n\}$ say, which also tends to τ. Now, if $s'_n \leqq s_n$, $n = 1,2,\cdots$, the inequalities (1) remain valid; and replacing s_n by $\inf_{k\geqq n} (s_k)$, we may always assume that $\{s_n\}$ is non-decreasing. Also, since $\tau = \overline{\lim_{n\to\infty}}\ s_n > 2$, we may assume without loss of generality that $s_1 > 2$.

(a) NECESSITY. The ρ_0-numbers have the property of being limits of sequences of the form $\{p_n/q_n\}$, $q_1 > 1$, $(p_n, q_n) = 1$, where according to (1)

$$(1') \qquad |\rho_0 - (p_n/q_n)| \leqq q_n^{-s_n},\ n = 1,2,\cdots,\ s_1 > 2.$$

Now, since ρ_0 is an irrational number, hence not an integer, and since (p_n, q_n)

$= 1$, for $n = 1,2,\cdots$, p_n/q_n is a convergent to ρ_0 by the approximation theorem for simple continued fractions [2], if $|\rho_0 - (p_n/q_n)| < q_n^{-2}$. But then, every p_n/q_n of (1′) is necessarily a convergent to ρ_0, since $q_n^{s_n} > q_n^2$.

Using the fact, that the $(n+1)$-st complete quotient of ρ_0, $\rho_{n+1} < b_{n+1} + 1$, we obtain according to [2] without difficulty a lower bound for the left hand side of (1′)

$$|\rho_0 - (p_n/q_n)| > [2q_n^2(b_{n+1}+1)]^{-1},\ n \geqq 1.$$

Now $$4b_{n+1} \geqq 2(b_{n+1}+1) > q_n^{s_n'-2},\ n \geqq 1\,;$$

putting $s_n' = s_n - (\log 4/\log q_n)$, $q_1 > 1$; $\tau = \overline{\lim_{n\to\infty}}\, s_n' > 2$, $s_n' > 0$, for $n = 1,2,\cdots$

and $$b_{n+1} > (1/4)\, q_n^{s_n-2} = q_n^{s_n'-2}.$$

(b) SUFFICIENCY. We assume that for infinitely many values of n an irrational number ρ_0 is such that

$$b_{n+1} > q_n^{s'_n-2}.$$

Again by the approximation theorem [2] for simple continued fractions, we have at once that

$$|\rho_0 - (p_n/q_n)| < q_n^{-2} b_{n+1}^{-1} < q_n^{-s'_n},\quad n = 1,2,\cdots,$$

and hence by theorem (a), ρ_0 is a ρ-number with $q_{n+1} > q_n > 1$, $(p_n, q_n) = 1$, $n = 1,2,\cdots$. q.e.d.

This theorem is a generalization of an earlier theorem by E.Maillet [3] for Liouville numbers and as such contains the case where ρ_0 is a Liouville number.

REFERENCES

[1] TH. SCHNEIDER, Einführung in die transzendenten Zahlen, Berlin (1957), 34.
[2] O. PERRON, Die Lehre von den Kettenbrüchen I, Stuttgart (1954), 37.
[3] E. MAILLET, Théorie des nombres transcendants, Paris (1906), 124.

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