ON THE APPROXIMATELY CONTINUOUS DENJOY INTEGRAL

YÔTO KUBOTA

(Received March 18, 1963)

1. Introduction. In the present paper we shall consider an integral of Denjoy's type whose indefinite integral is approximately continuous. The integral is defined descriptively by the method of S.Saks [5]. We call this integral the approximately continuous Denjoy integral or *AD*-integral. G.Sunouchi and M.Utagawa [4] have introduced the approximately continuous Perron integral or *AP*-integral which is more general than Burkill's approximately continuous Perron integral [1]. It will be proved that the *AD*-integral includes the *AP*-integral.

In §2 we shall define the *AP*-integral with the notion ACG_-(defined below) and prove its fundamental properties. In §4 the relation between the *AD*-integral and the *AP*-integral will be discussed by the method of J.Ridder [3].

The author expresses his thanks to Dr. G.Sunouchi and Dr. T.Tsuchikura for their kind suggestions and criticisms.

2. The approximately continuous Denjoy integral.

DEFINITION 2.1. The finite function $f(x)$ is said to be AC below [AC above] on a set E if to each positive number ε, there corresponds a number δ

$$\sum \{f(b_k) - f(a_k)\} > -\varepsilon \qquad \left[\sum \{f(b_k) - f(a_k)\} < \varepsilon\right]$$

such that for all finite sequence of non-overlapping intervals $\{(a_k, b_k)\}$ with end points on E and such that

$$\sum (b_k - a_k) < \delta.$$

If $f(x)$ is both AC below and AC above on E, then we say that $f(x)$ is AC on E.

DEFINITION 2.2. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is AC below [AC above], then $f(x)$ is termed ACG below [ACG above] on E. If $f(x)$ is both ACG below and ACG above on E, then we say that $f(x)$ is ACG_- on E.

The notion ACG_- is more general than that of ACG stated in [5, p.223], for the continuity is not assumed in the definition of ACG_-.

DEFINITION 2.3. Let $f(x)$ be a function defined in $[a, b]$ and suppose there exists a function $F(x)$ such that

(i) $F(x)$ is approximately continuous on $[a, b]$,

(ii) $F(x)$ is ACG_- on $[a, b]$,

(iii) $AD\ F(x) = f(x)$ a. e.,

then $f(x)$ is said to be integrable in the approximately continuous Denjoy sense or AD-integrable on $[a, b]$ and write

$$(AD) \int_a^b f(t)\, dt = F(b) - F(a).$$

The function $F(x)$ is said to be an indefinite AD-integral of $f(x)$ in $[a, b]$.

Definition 2.3 requires a uniqueness theorem, namely, that, if $F_1(x)$ and $F_2(x)$ both satisfy the conditions of Definition 2.3, then

$$F_1(b) - F_1(a) = F_2(b) - F_2(a);$$

this is supplied by the following theorem.

THEOREM 2.1. *If $F(x)$ is approximately continuous, ACG_- on $[a, b]$ and*

$$\overline{D}^+ F(x) \geqq 0 \quad \text{a. e.}, \tag{*}$$

then $F(x)$ is non-decreasing on $[a, b]$.

Suppose that Theorem 2.1 is true, then it also holds under the condition

$$AD\ F(x) \geqq 0$$

instead of the condition (*), for $AD\, F(x) \leqq \overline{D}^+ F(x)$. If we put, in this case,

$$G(x) = F_1(x) - F_2(x),$$

then $G(x)$ is approximately continuous, ACG_- and

$$AD\ G(x) = 0 \quad \text{a. e.}$$

Hence $G(x)$ is constant, that is,

$$F_1(b) - F_1(a) = F_2(b) - F_2(a).$$

For the proof of Theorem 2.1. we need some lemmas.

LEMMA 2.1. *A function $F(x)$ which is ACG_- on $[a, b]$ necessarily fulfils the condition (N), that is, $|F(H)| = 0$ for every set $H \subset [a, b]$ of measure zero where we put*

$$F(H) = \{F(x) : x \in H\}.$$

PROOF. This lemma is an extension of the theorem concering the notion ACG stated in [5, p 225], but the proof is done by the same method.

Since $[a, b]$ is expressible as the sum of a sequence of sets E_k on each of which $F(x)$ is AC, it is sufficient to prove that $|F(H)| = 0$ for any set H of measure zero and F a function AC on H.

We denote by $M(E)$ and $m(E)$ respectively the upper and lower bounds of F on E, when E is any subset of H, and we write $M(E) = m(E) = 0$ in the case in which E is empty set.

Since $F(x)$ is AC on H, for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\left| \sum \{F(b_k) - F(a_k)\} \right| < \varepsilon$$

for every sequence of non-overlapping intervals $\{I_k\}$ $(I_k = (a_k, b_k))$ with end points on H and

$$\sum |I_k| < \delta.$$

By the definition of M, m, we can find α_k, $\beta_k \in H \cdot I_k$ $(k = 1, 2, \cdots)$ such that

$$M(H \cdot I_k) - \frac{\varepsilon}{2^k} < F(\beta_k),$$

$$m(H \cdot I_k) + \frac{\varepsilon}{2^k} > F(\alpha_k).$$

Hence we obtain

$$\sum \{M(H \cdot I_k) - m(H \cdot I_k)\} < \sum \left\{ F(\beta_k) - F(\alpha_k) + \frac{\varepsilon}{2^{k-1}} \right\} < 3\varepsilon.$$

Since $|H| = 0$, we can determine a sequence of non-overlapping intervals $\{I_k\}$ with end points on H which satisfies

$$\sum |I_k| < \delta$$

and $H \subset \bigcup I_k$.
Therefore, since

$$|F(H \cdot I_k)| \leqq M(H \cdot I_k) - m(H \cdot I_k)$$

it follows that

$$|F(H)| \leqq \sum |F(H \cdot I_k)| < 3\varepsilon.$$

Hence $|F(H)| = 0$.

LEMMA 2. 2. *If $F(x)$ satisfies the following conditions;*
(i) *$F(x)$ is approximately continuous on $[a, b]$,*

(ii) *$F(E)$ contains no interval where we put* $E = \{x : \overline{D}^{+}F(x) \leqq 0\}$, *then $F(x)$ is non-decreasing on $[a, b]$.*

PROOF. Suppose that there exist two points c and d such that $c < d$ and that $F(d) < F(c)$. Then by (ii) we can determine a value y_0 not belonging to $F(E)$ and such that $F(d) < y_0 < F(c)$.

We put

$$x_0 = \sup \{x : F(x) \geqq y_0,\ x \in [c, d]\}.$$

Then we have $c \leqq x_0 \leqq d$, but we can prove that $c < x_0 < d$. If $x_0 = c$, then it holds that for any $t > c$

$$F(t) < y_0 < F(c),$$

and hence

$$\varlimsup_{x \to c+0} F(x) < F(c).$$

It follows from the relation $\varlimsup_{x \to c+0} \text{ap } F(x) \leqq \varlimsup_{x \to c+0} F(x)$ that

$$\varlimsup_{x \to c+0} \text{ap } F(x) < F(c)$$

which is a contradiction to the fact that $F(x)$ is approximately continuous at c. If $x_0 = d$ and d is an isolated point of the set $A = \{x : F(x) \geqq y_0, x \in [c, d]\}$ then $F(d) \geqq y_0$ which contradicts the relation $F(d) < y_0$. If $x_0 = d$ and d is a limiting point of A, then there exists an increasing sequence $\{t_n\}$ which converges to d and

$$F(t_n) \geqq y_0 > F(d).$$

Let ε be an arbitrary positive number such that

$$y_0 - \varepsilon > F(d).$$

Since $F(x)$ is approximately continuous at t_n, there exists, for each t_n, a measurable set $E(t_n)$ whose density at t_n is one and $F(x) \to F(t_n)$ as x tends to t_n on $E(t_n)$. Therefore we can find a positive sequence $\{h_n\}$,

$$h_1 > h_2 > \cdots > h_n > \cdots,$$

converging to 0, such that for each n, $E(t_n)I(h_n) \ni x$ implies

$$F(x) > F(t_n) - \varepsilon \geqq y_0 - \varepsilon > F(d)$$

and such that

$$|E(t_n)\ I(h_n)| \geqq h_n/2,$$

where we denote by $I(h_n)$ the interval containing t_n in its interior and its length is h_n.

We put

$$E(d) = \bigcup_{n=1}^{\infty} E(t_n)\cdot I(h_n).$$

Let h be any positive number sufficiently small. Then we can find h_n and h_{n+1} such that

$$h_{n+1} \leqq h \leqq h_n.$$

Hence we have for $I(h) = [d - h, d]$

$$\frac{|E(d)\cdot I(h)|}{h} \geqq \frac{1}{2}$$

and therefore the left-hand density of the set $E(d)$ at d is not zero. Since we have for $x \in E(d)$

$$F(x) > y_0 - \varepsilon > F(d)$$

it follows that

$$\{x : F(x) > y_0 - \varepsilon\} \supset E(d).$$

Hence the left-hand density of the set $\{x : F(x) > y_0 - \varepsilon\}$ at d is not zero, and we obtain from the definition of $\overline{\lim}_{x \to d-0}$ ap $F(x)$ that

$$\overline{\lim_{x \to d-0}} \text{ ap } F(x) \geqq y_0 - \varepsilon > F(d)$$

which is in contradiction with the approximate continuity of $F(x)$ at d. Thus we have proved that $c < x_0 < d$.

Next we shall prove by the same method described above that

$$F(x_0) = y_0.$$

Suppose that $F(x_0) > y$. Then for any $t > x_0$ we have

$$F(t) < y_0 < F(x_0)$$

and therefore

$$\overline{\lim_{x \to x_0+0}} \text{ ap } F(x) < F(x_0).$$

If $F(x_0) < y_0$ and x_0 is an isolated point of A then by the definition of x_0, $F(x_0) \geqq y$. Also if $F(x_0) < y_0$ and x_0 is a limiting point of A then we can choose a sequence $\{t_n\}$ which converges to x_0 and $t_n \in [c, d]$ such that

$$F(t_n) \geqq y_0 > F(x_0)$$

which implies

$$\overline{\lim_{x\to x_0}}\ \text{ap}\ F(x) > F(x_0).$$

Since we have arrived at a contradiction in each three cases above, we obtain that $F(x_0) = y_0$. Hence we have $x_0 \overline{\in} E$.

On the other hand, we have for $x_0 < x < d$

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{F(x)-y_0}{x-x_0} < 0$$

and hence

$$\overline{D}^+F(x_0) \leqq 0,$$

that is, $x_0 \in E$, which is a contradiction.

PROOF OF THEOREM 2.1. Let ε be any positive number and let

$$G(x) = F(x) + \varepsilon x.$$

The function $G(x)$ is approximately continuous and ACG_- on $[a,b]$. Moreover we have

$$\overline{D}^+G(x) = \overline{D}^+F(x) + \varepsilon > 0 \qquad \text{a. e.}$$

Therefore the set

$$E = \{x : \overline{D}^+G(x) \leqq 0\}$$

is of measure zero. By Lemma 2.1 we have $|G(E)| = 0$, and hence the set $G(E)$ can not contain any interval. It follows from Lemma 2.2 that $G(x)$ is non-decreasing on $[a,b]$. For any $x_1 < x_2$

$$G(x_2) - G(x_1) = F(x_1) - F(x_2) + \varepsilon(x_2 - x_1) \geqq 0.$$

By making $\varepsilon \to 0$ we have proved that the function $F(x)$ is itself non-decreasing.

THEOREM 2.2.

(i) *If $f(x)$ is AD-integrable on $[a,b]$ and $f(x) = g(x)$ a.e. then $g(x)$ is also AD-integrable and*

$$(AD)\int_a^b f(t)\,dt = (AD)\int_a^b g(t)dt.$$

(ii) *If $f(x)$ and $g(x)$ are both AD-integrable on $[a,b]$, then $\alpha f(x) + \beta g(x)$ is AD-integrable and*

$$(AD)\int_a^b (\alpha f + \beta g)dt = \alpha(AD)\int_a^b f(t)dt + \beta(AD)\int_a^b g(t)dt.$$

PROOF. The proof follows immediately from Definition 2.1.

THEOREM 2.3. *A function $f(x)$ which is AD-integrable on $[a,b]$ and $f(x) \geqq 0$ is L-integrable on $[a,b]$ and*

$$(AD)\int_a^b f(t)\,dt = (L)\int_a^b f(t)dt.$$

PROOF. Since $f(x)$ is AD-integrable on $[a,b]$, there exists a function $F(x)$ which is approximately continuous and ACG_- on $[a,b]$ and such that

$$AD\ F(x) = f(x) \qquad \text{a. e.}$$

Since $f(x) \geqq 0$, we have

$$AD\,F(x) \geqq 0 \qquad \text{a. e.}$$

It follows from Theorem 2.1 that $F(x)$ is non-decreasing on $[a,b]$, and hence

$$AD\,F(x) = F'(x) = f(x) \qquad \text{a. e.}$$

Therefore $f(x)$ is L-integrable and

$$(L)\int_a^b f(t)\,dt = (AD)\int_a^b f(t)\,dt = F(b) - F(a).$$

THEOREM 2.4. *Given a non-decreasing sequence $\{f_n\}$ of functions which are AD-integrable on $[a,b]$ and whose AD-integral over $[a,b]$ constitute a sequence bounded above, the function*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

is itself AD-integrable on $[a,b]$ and we have

$$(AD)\int_a^b f(t)\,dt = \lim_{n\to\infty} (AD)\int_a^b f_n(t)\,dt.$$

PROOF. The sequence of functions $f_n - f_1$ is non-decreasing, bounded above and

$$\lim_{n\to\infty} (f_n - f_1) = f - f_1.$$

Since $f_n - f_1 \geqq 0$, it follows from Theorem 2.3 that $f_n - f_1$ is L-integrable for each n. Therefore by Lebesgue's theorem, the limit function $f - f_1$ is L-integrable and

$$\lim_{n\to\infty} (L)\int_a^b (f_n - f_1)\,dt = (L)\int_a^b (f - f_1)\,dt,$$

that is,

$$\lim_{n\to\infty} (AD)\int_a^b f_n(t)\,dt = (AD)\int_a^b f(t)\,dt.$$

3. The approximately continuous Perron integral. G.Sunouchi and M.Utagawa [4] have introduced the approximately continuous Perron integral or AP-integral using the following upper and lower functions.

DEFINITION 3.1. $U(x)$ $[L(x)]$ is termed upper [lower] function of a measurable function $f(x)$ in $[a, b]$, provided that

(i) $U(a) = 0 \quad [L(a) = 0]$,

(ii) $\underline{AD}\, U(x) > -\infty \quad [\overline{AD}\, L(x) < +\infty]$ at each point x,

(iii) $\underline{AD}\, U(x) \geqq f(x) \quad [\overline{AD}\, L(x) \leqq f(x)]$ at each point x.

DEFINITION 3.2. If $f(x)$ has upper and lower functions in $[a, b]$ and

$$\inf_U U(b) = \sup_L L(b),$$

then $f(x)$ is termed integrable in AP-sense or AP-integrable. The common value of the two bounds is called the definite AP-integral of $f(x)$ and is denoted by

$$(AP) \int_a^b f(t)\, dt.$$

The following theorems have been proved by G.Sunouchi and M.Utagawa [4].

THEOREM 3.1. *The function* $U(x) - L(x)$ *is non-decreasing on* $[a, b]$.

THEOREM 3.2. *If* $f(x)$ *is AP-integrable on* $[a, b]$ *then* $f(x)$ *is also so in every interval* $[a, x]$ *for* $a < x < b$.

THEOREM 3.3. *The indefinite AP-integral*

$$F(x) = (AP) \int_a^x f(t)\, dt$$

is approximately continuous on $[a, b]$ *and the functions* $U(x) - F(x)$ *and* $F(x) - L(x)$ *are non-decreasing.*

THEOREM 3.4. *The indefinite AP-integral* $F(x)$ *is approximately differentiable almost everywhere and*

$$AD\ F(x) = f(x) \qquad \text{a. e.}$$

4. The relation between the AD-integral and the AP-integral. In this section we shall prove that the AD-integral includes the AP-integral. For the proof we need a lemma.

LEMMA 4.1. *If* $\underline{AD}\, F(x) > -\infty$ $[\overline{AD}\, F(x) < +\infty]$ *at each point* x *of* $[a,b]$, *then* $F(x)$ *is ACG below* [*ACG above*] *on* $[a,b]$.

PROOF. We prove the first case, the other case being similar. Since $\underline{AD}\, F(x) > -\infty$, to each point x we can make correspond a positive integer n such that the set

$$\{ t : (F(t) - F(x))/(t - x) \leqq -n \}$$

has the point x as a point of dispersion. Therefore, denoting by A_n the set of the points x such that the inequality

$$0 \leqq h \leqq 1/n$$

implies both the inequalities,

$$(1) \qquad |\{ t : F(t) - F(x) \leqq -n(t - x),\ x \leqq t \leqq x + h \}| \leqq h/3,$$

and

$$(2) \qquad |\{ t : F(x) - F(t) \leqq -n(x - t),\ x - h \leqq t \leqq x \}| \leqq h/3,$$

we have

$$[a,b] = \sum A_n.$$

If we put $A_n^i = A \cap [i/n, (i+1)/n]$ for each integer i, then

$$[a,b] = \sum_{i=-\infty}^{\infty} \sum_{n=1}^{\infty} A_n^i.$$

To prove the lemma it is sufficient to show that $F(x)$ is AC below on A_n^i. For this purpose, let x_1, x_2 be any pair of points of A_n^i, and let $x_1 < x_2$. We have $0 < x_2 - x_1 \leqq 1/n$, so that by writing $x = x_1$, $h = x_2 - x_1$ in (1), we obtain

$$(3) \qquad |\{ t : F(t) - F(x_1) \leqq -n(t - x_1),\ x_1 \leqq t \leqq x_2 \}| \leqq (x_2 - x_1)/3.$$

Similarly, from (2) with $x = x_2$, and $h = x_2 - x_1$, we have

$$(4) \qquad |\{ t : F(x_2) - F(t) \leqq -n(x_2 - t),\ x_1 \leqq t \leqq x_2 \}| \leqq (x_2 - x_1)/3.$$

It follows from (3) and (4) that there exists a point $t_0 \in [x_1, x_2]$ such that

$$F(t_0) - F(x_1) > -n(t_0 - x_1),$$

and

$$F(x_2) - F(t_0) > -n(x_2 - t_0).$$

Adding these we have

$$(5) \qquad F(x_2) - F(x_1) > -n(x_2 - x_1).$$

Let $\{(a_k, b_k)\}$ be a sequence of non-overlapping intervals with end points on A_n^i. Then we have from (5)

$$\sum \{F(b_k) - F(a_k)\} > -n \sum (b_k - a_k).$$

If

$$\sum (b_k - a_k) < \varepsilon/n,$$

then we have

$$\sum \{F(b_k) - F(a_k)\} > -\varepsilon.$$

This completes the proof.

THEOREM 4.1. *The AD-integral includes the AP-integral.*

PROOF. Suppose that $f(x)$ is AP-integrable on $[a, b]$ and such that

$$F(x) = (AP) \int_a^x f(t)\, dt.$$

Then by Theorem 3.3 and Theorem 3.4, $F(x)$ is approximately continuous on $[a, b]$ and

$$AD\ F(x) = f(x) \qquad \text{a. e.}$$

Since $f(x)$ is AP-integrable, there exists a sequence of upper functions $\{U_k(x)\}$ and a sequence of lower functions $\{L_k(x)\}$ such that

$$\lim_{k\to\infty} U_k(b) = \lim_{k\to\infty} L_k(b) = F(b).$$

The function $U_k(x) - F(x)$ and $F(x) - L_k(x)$ are non-decreasing, so that we have for $x \in [a, b]$

$$(1) \qquad \lim_{k\to\infty} U_k(x) = \lim_{k\to\infty} L_k(x) = F(x).$$

Since $\underline{AD}\ U_k(x) > -\infty$ $[\overline{AD}\ L_k(x) < +\infty]$, it follows from Lemma 4.1 that U_k $[L_k]$ is ACG below $[ACG$ above$]$ on $[a, b]$. Then $[a, b]$ is expressible as the sum of a countable number of sets E_k,

$$[a, b] = \sum E_k$$

such that any U_k is AC below on any E_k and at the same time any L_k is AC

above on any E_k.

Next we shall show that $F(x)$ is ACG_- on $[a, b]$. It is sufficient to prove that $F(x)$ is AC on E_k. For this purpose we shall show that $F(x)$ is both AC below and AC above on E_k.

Suppose that $F(x)$ is not AC below on E_k. Then there exists an $\varepsilon > 0$ such that for any small $\delta>0$ we can find finite, non-overlapping intervals (a_ν, b_ν) with end points on E_k satisfying

$$\sum (b_\nu - a_\nu) < \delta$$

but

$$\sum \{F(b_\nu) - F(a_\nu)\} \leqq -\varepsilon. \tag{2}$$

Since we can find a natural number p by (1) such that

$$U_p(x) - F(x) < \varepsilon/2,$$

and since $U_p(x) - F(x)$ is non-decreasing on $[a, b]$ by Theorem 3. 3, we have

$$\begin{aligned} (3) \qquad & \sum \{U_p(b_\nu) - U_p(a_\nu)\} - \sum \{F(b_\nu) - F(a_\nu)\} \\ &= \sum [(U_p(b_\nu) - F(b_\nu)) - (U_p(a_\nu) - F(a_\nu))] \\ &\leqq U_p(b) - F(b) < \varepsilon/2. \end{aligned}$$

It follows from (2) and (3) that

$$\begin{aligned} \sum \{U_p(b_\nu) - U_p(a_\nu)\} &< \sum \{F(b_\nu) - F(a_\nu)\} + \varepsilon/2 \\ &\leqq -\varepsilon/2. \end{aligned}$$

This contradicts the fact that $U_p(x)$ is AC below on E_k, and therefore $F(x)$ is AC below on E_k.

Similarly we can prove that $F(x)$ is AC above on E_k. Thus $F(x)$ is AC on each E_k and also ACG_- on $[a, b]$. Since we have shown that $F(x)$ is approximately continuous and

$$AD\ F(x) = f(x) \qquad \text{a. e.}$$

it follows that $f(x)$ is AD-integrable on $[a, b]$ and that

$$(AD) \int_a^b f(t)\, dt = (AP) \int_a^b f(t)\, dt = F(b).$$

REFERENCES

[1] J. C. BURKILL, The approximately continuous Perron integrals, Math. Zeits., 34 (1931),

270–278.
[2] J. RIDDER, Ueber approximativ stetige Denjoy-Integrale, Fund. Math., 21 (1931), 1–10.
[3] J. RIDDER, Ueber den Perronschen Integralbegriff und seine Beziehung zu den R-L-und D-Integralen, Math. Zeits., 34 (1931), 234–269.
[4] G. SUNOUCHI and M. UTAGAWA, The generalized Perron integrals, Tôhoku Math. Journ., 1 (1949), 95–99.
[5] S. SAKS, Theory of the integral, Warsaw (1937).

HOKKAIDO GAKUGEI UNIVERSITY, HAKODATE.