A-GENUS AND DIFFERENTIABLE IMBEDDING

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Introduction. Atiyah and Hirzebruch provided us with a very useful mean dealing with the non-imbeddability problem and clarified the relations between the divisibility of A-genus and the differentiable imbedding of a compact orientable differentiable $4n$-manifold ([2]). Furthermore they exactly computed the index of a $4n$-manifold imbedded in the $(4n+4)$-euclidean space ([1]).

In this paper we shall improve our previous paper ([5]) by means of above theorem and we shall clarify the divisibility of the cobordism coefficients in the case of dimension 8, 12, and 16.

1. Let M_{4n} be a compact orientable differentiable $4n$-manifold and let

$$M_{4n} \sim \sum_{i_1+\dots+i_t=n} A^n_{i_1\dots i_t} P_{2i_1}(c)\cdots P_{2i_t}(c) \quad \text{mod torsion} \tag{1. 1}$$

be its cobordism decomposition, where $P_{2i}(c)$ denotes the complex projective space of complex dimension $2i$ and A's denote some rational numbers. It is known that A^2_2, A^2_{11}, A^3_3, A^3_{21}, A^3_{111}, $3A^4_4$, A^4_{31}, A^4_{22}, A^4_{211}, $3A^4_{1111}$ are integers ([4],[6]). Let p_i or $\bar{p}_i$ be the Pontryagin class or dual-Pontryagin class of dimension $4i$ respectively. Then these cobordism coefficients are expressed as follows ([5]):

$$\tau = \text{index} = \sum_{i_1+\dots+i_t=n} A^n_{i_1\dots i_t}, \tag{1. 2}$$

$$(1.\ 3)\quad \begin{cases} \text{(a)}\ \ A^2_2 = \dfrac{1}{5}(-2p_2+p_1^2)[M_8] = \dfrac{1}{5}(2\bar{p}_2-\bar{p}_1^2)[M_8], \\[2ex] \text{(b)}\ \ A^2_{11} = \dfrac{1}{9}(5p_2-2p_1^2)[M_8] = \dfrac{1}{9}(-5\bar{p}_2+3\bar{p}_1^2)[M_8], \\[2ex] \text{(c)}\ \ \tau = \dfrac{1}{45}(7p_2-p_1^2)[M_8] = \dfrac{1}{45}(-7\bar{p}_2+6\bar{p}_1^2)[M_8], \end{cases}$$

$$(1.\ 4)\quad \begin{cases} \text{(a)}\ \ A^3_3 = \dfrac{1}{7}(3p_3-3p_2p_1+p_1^3)[M_{12}] = \dfrac{1}{7}(3\bar{p}_3-3\bar{p}_2\bar{p}_1+\bar{p}_1^3)[M_{12}], \\[2ex] \text{(b)}\ \ A^3_{21} = \dfrac{1}{15}(-21p_3+19p_2p_1-6p_1^3)[M_{12}] = \dfrac{1}{15}(-21\bar{p}_3+23\bar{p}_2\bar{p}_1 \\ \qquad\qquad -8\bar{p}_1^3)[M_{12}], \\[2ex] \text{(c)}\ \ A^3_{111} = \dfrac{1}{27}(28p_3-23p_2p_1+7p_1^3)[M_{12}] = \dfrac{1}{27}(28\bar{p}_3-33\bar{p}_2\bar{p}_1 \end{cases}$$

$$
\begin{cases}
\quad + 12\bar{p}_1^3)[M_{12}], \\
(\mathrm{d}) \quad \tau = \dfrac{1}{3^3\cdot5\cdot7}(62p_3 - 13p_2p_1 + 2p_1^3)[M_{12}] \\
\qquad = \dfrac{1}{3^3\cdot5\cdot7}(62\bar{p}_3 - 111\bar{p}_2\bar{p}_1 + 51\bar{p}_1^3)[M_{12}],
\end{cases}
$$

$$
(1.\ 5)\quad
\begin{cases}
(\mathrm{a}) \quad A_4^4 = \dfrac{1}{9}(-4p_4 + 4p_3p_1 + 2p_2^2 - 4p_2p_1^2 + p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{9}(4\bar{p}_4 - 4\bar{p}_3\bar{p}_1 - 2\bar{p}_2^2 + 4\bar{p}_2\bar{p}_1^2 - \bar{p}_1^4)[M_{16}], \\
(\mathrm{b}) \quad A_{31}^4 = \dfrac{1}{21}(36p_4 - 33p_3p_1 - 18p_2^2 + 33p_2p_1^2 - 8p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{21}(-36\bar{p}_4 + 39\bar{p}_3\bar{p}_1 + 18\bar{p}_2^2 - 39\bar{p}_2\bar{p}_1^2 + 10\bar{p}_1^4)[M_{16}], \\
(\mathrm{c}) \quad A_{22}^4 = \dfrac{1}{25}(18p_4 - 18p_3p_1 - 7p_2^2 + 16p_2p_1^2 - 4p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{25}(-18\bar{p}_4 + 18\bar{p}_3\bar{p}_1 + 11\bar{p}_2^2 - 20\bar{p}_2\bar{p}_1^2 + 5\bar{p}_1^4)[M_{16}], \\
(\mathrm{d}) \quad A_{211}^4 = \dfrac{1}{45}(-180p_4 + 159p_3p_1 + 80p_2^2 - 150p_2p_1^2 + 36p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{45}(180\bar{p}_4 - 201\bar{p}_3\bar{p}_1 - 100\bar{p}_2^2 + 212\bar{p}_2\bar{p}_1^2 - 55\bar{p}_1^4)[M_{16}], \\
(\mathrm{e}) \quad A_{1111}^4 = \dfrac{1}{81}(165p_4 - 137p_3p_1 - 70p_2^2 + 127p_2p_1^2 - 30p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{81}(-165\bar{p}_4 + 193\bar{p}_3\bar{p}_1 + 95\bar{p}_2^2 - 208\bar{p}_2\bar{p}_1^2 + 55\bar{p}_1^4)[M_{16}], \\
(\mathrm{f}) \quad \tau = \dfrac{1}{3^4\cdot5^2\cdot7}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4)[M_{16}] \\
\qquad = \dfrac{1}{3^4\cdot5^2\cdot7}(-381\bar{p}_4 + 691\bar{p}_3\bar{p}_1 + 362\bar{p}_2^2 - 985\bar{p}_2\bar{p}_1^2 + 310\bar{p}_1^4)[M_{16}].
\end{cases}
$$

There exists a relation such that

$$(1.\ 6)\qquad \bar{p}\cdot p = 1$$

where

$$(1.\ 7)\qquad p = \sum_{k\geqq 0}(-1)^k p_k$$

and

$$(1.\ 8)\qquad \bar{p} = \sum_{k\geqq 0}\bar{p}_k.$$

We have from (1.6)

$$\text{(1. 9)} \qquad \begin{cases} p_1 = \bar{p}_1, \\ p_2 = -\bar{p}_2 + \bar{p}_1^2, \\ p_3 = \bar{p}_3 - 2\bar{p}_2\bar{p}_1 + \bar{p}_1^3, \\ p_4 = -\bar{p}_4 + 2\bar{p}_3\bar{p}_1 + \bar{p}_2^2 - 3\bar{p}_2\bar{p}_1^2 + \bar{p}_1^4. \end{cases}$$

The A-genus is defined by

$$\text{(1.10)} \qquad A(M_{4n}) = \prod_i \frac{2\sqrt{r_i}}{\sinh 2\sqrt{r_i}}[M_{4n}]$$

where

$$\text{(1.11)} \qquad p = \prod_i (1-r_i)$$

and it is known that ([3] p.14)

$$\text{(1.12)} \qquad \begin{cases} \text{(a)} \ A(M_4) = -\dfrac{2}{3}p_1[M_4] = -\dfrac{2}{3}\bar{p}_1[M_4], \\ \text{(b)} \ A(M_8) = \dfrac{2}{45}(-4p_2 + 7p_1^2)[M_8] = \dfrac{2}{45}(4\bar{p}_2 + 3\bar{p}_1^2)[M_8], \\ \text{(c)} \ A(M_{12}) = -\dfrac{4}{3^3 \cdot 5 \cdot 7}(16p_3 - 44p_2p_1 + 31p_1^3)[M_{12}] \\ \qquad\qquad = -\dfrac{4}{3^3 \cdot 5 \cdot 7}(16\bar{p}_3 + 12\bar{p}_2\bar{p}_1 + 3\bar{p}_1^3)[M_{12}], \\ \text{(d)} \ A(M_{16}) = \dfrac{1}{3^4 \cdot 5^2 \cdot 7}(384\bar{p}_4 + 256\bar{p}_3\bar{p}_1 + 32\bar{p}_2^2 + 80\bar{p}_2\bar{p}_1^2 + 10\bar{p}_1^4)[M_{16}]. \end{cases}$$

2. Let M_{4n} be a compact orientable differentiable $4n$-manifold. If M_{4n} is differentiably imbedded in the $(4n+q)$-euclidean space E_{4n+q}, it holds that

$$\text{(2. 1)} \qquad \bar{p}_k = 0, \quad 2k \geqq q+1.$$

When $q = 2k$ we have, moreover,

$$\text{(2. 2)} \qquad \bar{p}_k = 0, \quad (2k=q),$$

because in this case

$$\text{(2. 3)} \qquad \bar{p}_k = E^2,$$

where E denotes the Euler class of the normal bundle and

(2. 4) $$E=0$$
in such a case. The following theorem is fundamental for our purpose:

THEOREM 1 (Atiyah-Hirzebruch [2])
Let M_{4n} be a compact orientable differentiable 4n-manifold differentiably imbedded in the E_{8n-2q}. Then $A(M_{4n})$ is divisible by 2^{q+1} and if moreover $q\equiv 2$ mod 4, *$A(M_{4n})$ is divisible by 2^{q+2}.*

Hereafter $M_{4n}\subset E_{4n+q}$ means the differentiable imbedding and M_{4n} denotes a compact orientable differentiable $4n$-manifold. Let us investigate the individual cases of differentiable imbedding.

$M_8\subset E_{12}$. In this case we have from (2.1) and (2.2)

(2. 5) $$\bar{p}_2 = 0.$$

Hence we have from (1.12b)

(2. 6) $$A(M_8) = \frac{2}{15}\bar{p}_1^2[M_8].$$

Meanwhile we have from Theorem 1

(2. 7) $$A(M_8) \equiv 0 \mod 16.$$

We have from (2.6) and (2.7)

(2. 8) $$\bar{p}_1^2[M_8] \equiv 0 \mod 120.$$

Hence we have from (1.3), (2.5) and (2.8)

(2. 9) $$\begin{cases} A_2^2 \equiv 0 \mod 24 \\ A_{11}^2 \equiv 0 \mod 40. \end{cases}$$

$M_8\subset E_{14}$. In this case we have from Theorem 1

(2.10) $$A(M_8) \equiv 0 \mod 4.$$

Hence we have from (1.12)

(2.11) $$\bar{p}_1^2[M_8] \equiv 0 \mod 2.$$

We have from (2.11) and (1.3a)

(2.12) $$A_2^2 \equiv 0 \mod 2.$$

Moreover we have from (1.3c)

(2.13) $$\bar{p}_2[M_8] \equiv \tau \mod 2.$$

Meanwhile we have from (1.3b) and (2.11)

(2.14) $$A_{11}^2 \equiv \bar{p}_2[M_8] \mod 2.$$

We have from (2.13) and (2.14)

(2.15) $$A_{11}^2 \equiv \tau \mod 2.$$

Thus we have the following table:

	$M_8 \subset E_{12}$	$M_8 \subset E_{14}$
A_2^2	$\equiv 0 \mod 24$	$\equiv 0 \mod 2$
A_{11}^2	$\equiv 0 \mod 40$	$\equiv \tau \mod 2$
A	$\equiv 0 \mod 16$	$\equiv 0 \mod 4$
τ	$\equiv 0 \mod 16$	

3. In this paragraph we shall deal with the case where $M_{12} \subset M_{12+q}$.

$M_{12} \subset E_{16}$. In this case we have from (2.1), (2.2) and Theorem 1

(3. 1) $$\bar{p}_2 = \bar{p}_3 = 0$$

and

(3. 2) $$A(M_{12}) \equiv 0 \mod 2^5.$$

Hence we have from (1.12c)

(3. 3) $$2\bar{p}_1^3[M_{12}] \equiv 0 \mod 7!.$$

We have from (3.1), (3.3) and (1.4)

(3. 4) $$\begin{cases} A_3^3 \equiv 0 & \mod 2^3 \cdot 3^2 \cdot 5, \\ A_{21}^3 \equiv 0 & \mod 2^6 \cdot 3 \cdot 7, \\ A_{111}^3 \equiv 0 & \mod 2^5 \cdot 5 \cdot 7. \end{cases}$$

$M_{12} \subset E_{18}$. In this case we have from (2.1), (2.2) and Theorem 1

(3. 5) $$\bar{p}_3 = 0$$

and

(3. 6) $$A(M_{12}) \equiv 0 \mod 16.$$

Hence we have from (1.12c)

(3. 7) $$\bar{p}_1^3[M_{12}] \equiv 0 \mod 4.$$

If $\tau \equiv 0 \mod 4$, we have from (3.5), (3.7) and (1.4d)

(3. 8) $$\bar{p}_2 \bar{p}_1[M_{12}] \equiv 0 \mod 4.$$

We have from (1.4), (3.7) and (3.8)

$$(3.9)\qquad \begin{cases} A_3^3 \equiv 0 & \text{mod } 4 \\ A_{21}^3 \equiv 0 & \text{mod } 4 \\ A_{111}^3 \equiv 0 & \text{mod } 4 \end{cases} \qquad (\tau \equiv 0 \quad \text{mod } 4).$$

Moreover we have from (1.4d), (3.5)and (3.7)

$$(3.10)\qquad \bar{p}_2\bar{p}_1[M_{12}] \equiv \tau \quad \text{mod } 2.$$

Hence we have from (1.4), (3.5) and (3.10)

$$(3.11)\qquad \begin{cases} A_3^3 \equiv \tau & \text{mod } 2, \\ A_{21}^3 \equiv \tau & \text{mod } 2, \\ A_{111}^3 \equiv \tau & \text{mod } 2. \end{cases}$$

$M_{12} \subset E_{20}$. In this case we have from Theorem 1

$$(3.12)\qquad A(M_{12}) \equiv 0 \quad \text{mod } 16.$$

Hence we have from (1.12c)

$$(3.13)\qquad \bar{p}_1^3[M_{12}] \equiv 0 \quad \text{mod } 4.$$

We have from (1.4d) and (3.13)

$$(3.14)\qquad \bar{p}_2\bar{p}_1[M_{12}] \equiv \tau \quad \text{mod } 2.$$

Hence we have from (1.4c)

$$(3.15)\qquad A_{111}^3 \equiv \tau \quad \text{mod } 2.$$

Thus we have the following table:

	$M_{12} \subset E_{16}$	$M_{12} \subset E_{18}$	$M_{12} \subset E_{20}$
A_3^3	$\equiv 0$ mod $2^3 \cdot 3^2 \cdot 5$	$\equiv \tau$ mod 2 $\equiv 0$ mod 4 ($\tau \equiv 0$ mod 4)	
A_{21}^3	$\equiv 0$ mod $2^6 \cdot 3 \cdot 7$	$\equiv \tau$ mod 2 $\equiv 0$ mod 4 ($\tau \equiv 0$ mod 4)	
A_{111}^3	$\equiv 0$ mod $2^5 \cdot 5 \cdot 7$	$\equiv \tau$ mod 2 $\equiv 0$ mod 4 ($\tau \equiv 0$ mod 4)	$\equiv \tau$ mod 2
A	$\equiv 0$ mod 2^5	$\equiv 0$ mod 16	$\equiv 0$ mod 16
τ	$\equiv 0$ mod $2^3 \cdot 17$		

4. In this paragraph we shall deal with the case where $M_{16} \subset E_{16+q}$.

$M_{16} \subset E_{20}$. In this case we have from (2.1),(2.2) and Theorem 1

(4. 1) $$\bar{p}_2 = \bar{p}_3 = \bar{p}_4 = 0$$
and
(4. 2) $$A(M_{16}) \equiv 0 \quad \text{mod } 2^8.$$
Hence we have from (1.12d)
(4. 3) $$\bar{p}_1^4[M_{16}] \equiv 0 \quad \text{mod } 9!.$$
We have from (1.5), (4.1) and (4.3)
(4. 4) $$\begin{cases} A_4^4 \equiv 0 \ \text{mod } 8!, \\ A_{31}^4 \equiv 0 \ \text{mod } 2^8 \cdot 3^3 \cdot 5^2, \\ A_{22}^4 \equiv 0 \ \text{mod } 2^7 \cdot 3^4 \cdot 7, \\ A_{211}^4 \equiv 0 \ \text{mod } 11 \cdot 8!, \\ A_{1111}^4 \equiv 0 \ \text{mod } 2^7 \cdot 5^2 \cdot 7 \cdot 11. \end{cases}$$

$M_{16} \subset E_{22}$. In this case we have from (2.1), (2.2) and Theorem 1
(4. 5) $$\bar{p}_3 = \bar{p}_4 = 0$$
and
(4. 6) $$A(M_{16}) \equiv 0 \quad \text{mod } 2^6.$$
Hence we have from (1.12d)
(4. 7) $$\bar{p}_1^4[M_{16}] \equiv 0 \quad \text{mod } 8.$$
Meanwhile we have from (1.5) and (4.5)
(4. 8) $$\begin{cases} \text{(a)} \ A_4^4 = \dfrac{1}{9}(-2\bar{p}_2^2 + 4\bar{p}_2\bar{p}_1^2 - \bar{p}_1^4)[M_{16}], \\ \text{(b)} \ A_{31}^4 = \dfrac{1}{21}(18\bar{p}_2^2 - 39\bar{p}_2\bar{p}_1^2 + 10\bar{p}_1^4)[M_{16}], \\ \text{(c)} \ A_{22}^4 = \dfrac{1}{25}(11\bar{p}_2^2 - 20\bar{p}_1\bar{p}_1^2 + 5\bar{p}_1^4)[M_{16}], \\ \text{(d)} \ A_{211}^4 = \dfrac{1}{45}(-100\bar{p}_2 + 212\bar{p}_2\bar{p}_1^2 - 55\bar{p}_1^4)[M_{16}], \\ \text{(e)} \ A_{1111}^4 = \dfrac{1}{81}(95\bar{p}_2^2 - 208\bar{p}_2\bar{p}_1^2 + 55\bar{p}_1^4)[M_{16}]. \end{cases}$$

We have from (4.7) and (4.8a)
(4. 9) $$3A_4^4 \equiv 0 \quad \text{mod } 2.$$
Next we have from (4.7) and (4.8d)
(4.10) $$A_{211}^4 \equiv 0 \quad \text{mod } 4.$$
Meanwhile we have from (4.8d) or (4.8c)

(4.11) $$\bar{p}_2\bar{p}_1^2[M_{16}] \equiv 0 \mod 5$$
or

(4.12) $$\bar{p}_2^2[M_{16}] \equiv 0 \mod 5$$
respectively. Hence we have from (4.8b), (4.11) and (4.12)

(4.13) $$A^4_{31} \equiv 0 \mod 5.$$
Moreover we have from (4.8e) and (4.11)

(4.14) $$3A^4_{1111} \equiv 0 \mod 5.$$
In this case we have from (1.5f) and (4.5)

(4.15) $$\tau = \frac{1}{3^4 \cdot 5^2 \cdot 7}(362\bar{p}_2^2 - 985\bar{p}_2\bar{p}_1^2 + 310\bar{p}_1^4)[M_{16}].$$

Hence we have

(4.16) $$\tau \equiv \bar{p}_2\bar{p}_1^2[M_{16}] \mod 2.$$
We have from (4.8b) and (4.16)

(4.17) $$A^4_{31} \equiv \tau \mod 2.$$
Moreover we have from (4.7), (4.8c) and (4.8e)

(4.18) $$A^4_{22} \equiv \bar{p}_2^2[M_{16}] \equiv 3A^4_{1111} \mod 2.$$

$M_{16} \subset E_{30}$. In this case we have from Theorem 1

(4.19) $$A(M_{16}) \equiv 0 \mod 4.$$
Hence we have from (1.12d)

(4.20) $$\bar{p}_1^4[M_{16}] \equiv 0 \mod 2.$$
We have from (1.5a) and (4.20)

(4.21) $$3A^4_4 \equiv 0 \mod 2.$$
Thus we have the following table:

	$M_{16}\subset E_{20}$	$M_{16}\subset E_{22}$	$M_{16}\subset E_{24}$	$M_{16}\subset E_{26}$	$M_{16}\subset E_{28}$	$A_{16}\subset E_{30}$
A^4_4	$\equiv 0 \mod 8!$	$3A^4_4 \equiv 0 \mod 2$	$3A^4_4 \equiv 0 \mod 2$	$3A^4_4 \equiv 0 \mod 2$	$3A^4_4 \equiv 0 \mod 2$	$3A^4_4 \equiv 0 \mod 2$
A^4_{31}	$\equiv 0 \mod 2^8 \cdot 3^3 \cdot 5^2$	$\equiv 0 \mod 5$ $\equiv \tau \mod 2$				
A^4_{22}	$\equiv 0 \mod 2^7 \cdot 3^4 \cdot 7$	$\equiv 3A^4_{1111} \mod 2$				
A^4_{211}	$\equiv 0 \mod 11 \cdot 8!$	$\equiv 0 \mod 4$				

A^4_{1111}	$\equiv 0 \mod 2^7\cdot 5^2\cdot 7 \times 11$	$3A^4_{1111}\equiv 0 \mod 5$				
A	$\equiv 0 \mod 2^8$	$\equiv 0 \mod 2^6$	$\equiv 0 \mod 2^5$	$\equiv 0 \mod 2^4$	$\equiv 0 \mod 2^4$	$\equiv 0 \mod 2^2$
τ	$\equiv 0 \mod 2^8\cdot 31$					

5. It is known that

$$\begin{aligned}(5.\ 1)\quad \tau(M_{20}) = \frac{1}{3^5\cdot 5^2\cdot 7\cdot 11}(5110p_5 - 919p_4p_1 - 336p_3p_2 + 237p_3p_1^2 + 127p_2^2p_1 \\ - 83p_2p_1^3 + 10p_1^5)[M_{20}] \quad ([3]\text{p.}13).\end{aligned}$$

When $M_{20}\subset E_{26}$ we have

$$(5.\ 2)\qquad \bar{p}_5 = \bar{p}_4 = \bar{p}_3 = 0$$

and

$$(5.\ 3)\qquad p_5 = 3\bar{p}_2^2\bar{p}_1 - 4\bar{p}_2\bar{p}_1^3 + \bar{p}_1^5.$$

Hence we have from (5.1), (5.2) and (5.3)

$$(5.\ 4)\qquad \tau = \frac{1}{3^5\cdot 5^2\cdot 7\cdot 11}(13866\bar{p}_2^2\bar{p}_1 - 17320\bar{p}_2\bar{p}_1^3 + 4146\bar{p}_1^5)[M_{20}].$$

Therefore $\tau(M_{20})$ is even, if $M_{20}\subset E_{26}$.

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