ON SOME SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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§1. Introduction. Roughly speaking the purpose of the present paper is to present a sufficient condition that a system of partial differential equations of the first order in many dependent variables and independent variables be complete.

A *complete system* is defined as follows.

Let

$$(1.\ 1)\qquad \begin{aligned} &F^A(\partial_i u^\kappa\,; u^\kappa\,; x^h) = 0 \qquad (A = 1, \cdots, N) \\ &G^K(u^\kappa\,; x^h) = 0 \qquad\qquad (K = 1, \cdots, P) \end{aligned}$$

be a system of partial differential equations in m dependent variables[1] $u^1, \cdots, u^m$ and n independent variables $x^1, \cdots, x^n$, indices being used as follows,

$$A, B, C, \cdots = 1, \cdots, N; \quad J, K, L, \cdots = 1, \cdots, P;$$

$$\kappa, \rho, \sigma, \cdots = 1, \cdots, m; \qquad h, i, j, \cdots = 1, \cdots, n.$$

Assuming that (1. 1) admits a solution

$$u^\kappa = u^\kappa\ (x^1, \cdots, x^n)$$

we obtain

$$G^K|_\kappa \partial_i u^\kappa + G^K|_i = 0 \text{ [2]}$$

where

$$G^K|_\kappa = \partial G^K/\partial u^\kappa, \quad G^K|_i = \partial G^K/\partial x^i.$$

We find that (1. 1) is equivalent to the system (1. 2) of partial differential equations composed of

$$(1.2.1)\qquad G^K(u^\kappa\,; x^h) = 0\,,$$

$$(1.2.2)\qquad F^A(\partial_i u^\kappa\,; u^\kappa\,; x^h) = 0\,,$$

$$(1.2.3)\qquad G^K|_\kappa \partial_i u^\kappa + G^K|_i = 0\,.$$

1) Some of u may eventually become independent variables.
2) We adopt the summation convention.

The system (1.1) is a complete system if and only if the following conditions are fulfilled.[3)]

(i) No equation in $x^1, \cdots, x^n$ only is obtained by eliminating u^κ from (1.2.1).

(ii) Any equation obtained by eliminating $\partial_i u^\kappa$ from (1.2.2) and (1.2.3) is not independent of (1.2.1).

(iii) Equations obtained by differentiating (1.2.1) and (1.2.2) partially ν times ($\nu = 1, 2, \cdots$) with respect to x (u being considered as functions of x) will be denoted by $(1.2.1)_\nu$ and $(1.2.2)_\nu$ respectively. Then any equation obtained by eliminating $\partial_{i_{\nu+1}} \cdots \partial_{i_1} u^\kappa$ from $(1.2.1)_{\nu+1}$ and $(1.2.2)_\nu$ is not independent of $(1.2.1), (1.2.1)_1, \cdots, (1.2.1)_\nu$ and $(1.2.2), (1.2.2)_1, \cdots, (1.2.2)_{\nu-1}$.

In general it is not possible to distinguish a system (1.1) or (1.2) to be complete or not by examining only the first derivatives $(1.2)_1$. The purpose of the present paper is to give a sufficient condition that this be possible.

Before defining a complete system of partial differential equations an equivalent exterior differential system and its prolongations are explained in §2, together with some symbols to be used. No unknown result is contained there. In §3 a complete system is defined. In §4 a tool for discerning completeness of a system is introduced. But a necessary and sufficient condition for a system to be complete contains an infinite sequence of equations.

Now, there are some systems of partial differential equations for which a necessary and sufficient condition to be complete is given by equations in which only the first or the second derivatives of the functions appearing in the given equations are concerned. We call them *E-simple systems.* These are treated in §5 and §6 which are the essential part of the present paper.

In §7 systems with superfluous equations are treated. In §8 an example of *E-simple systems* is given.

§2. A system of partial differential equations and an exterior differential system. Let

$$G^K(u^\kappa; x^h), \quad F^A(u_i^\kappa; u^\kappa; x^h)$$

be C^ω functions of $m(n+1)+n$ variables $u_i^\kappa, u^\kappa, x^h$ in some domain D containing a point $P_0 = ((u_i^\kappa)_0, (u^\kappa)_0, (x^h)_0)$ satisfying $G^K = 0$, $G^K|_\kappa u_i^\kappa + G^K|_i = 0$, $F^A = 0$.

We introduce an infinite sequence of new variables

$$u_{i_2 i_1}{}^\kappa, \; u_{i_3 i_2 i_1}{}^\kappa, \; \cdots, \; u_{i_p i_{p-1} \cdots i_1}{}^\kappa, \; \cdots$$

and an operation ∇_k, which is defined as follows.

3) A more precise definition is given in § 3.

Let

$$F(u_{i_p\dots i_1}{}^\kappa;\ u_{i_{p-1}\dots i_1}{}^\kappa;\ \cdots;\ u_{i_1}{}^\kappa;\ u^\kappa;\ x^h)$$

be a C^1 function of $m(n^p+n^{p-1}+\cdots+1)+n$ variables $u_{i_p\dots i_1}{}^\kappa, \cdots, u_{i_1}{}^\kappa, u^\kappa, x^h$. Then $\nabla_k F$ is the following C^0 function of $m(n^{p+1}+n^p+\cdots+1)+n$ variables $u_{i_{p+1}i_p\dots i_1}{}^\kappa, \cdots, u_{i_1}{}^\kappa, u^\kappa, x^h$,

$$(2.\ 1)\qquad (\nabla_k F)(u_{i_{p+1}i_p\dots i_1}{}^\kappa;\ \cdots;\ u^\kappa;\ x^h)$$
$$\equiv u_{ki_p\dots i_1}{}^\kappa F|^{i_p\dots i_1}{}_\kappa + u_{ki_{p-1}\dots i_1}{}^\kappa F|^{i_{p-1}\dots i_1}{}_\kappa + \cdots + u_k{}^\kappa F|_\kappa + F|_k$$

where

$$F|^{i_r\dots i_1}{}_\kappa = \partial F/\partial u_{i_r\dots i_1}{}^\kappa\,,$$

$$F|_\kappa = \partial F/\partial u^\kappa\,,$$

$$F|_i = \partial F/\partial x^i\,.$$

Hence, if F is a C^q function, we can define the function

$$(\nabla_{k_q}\cdots\nabla_{k_1}F)(u_{i_{p+q}\dots i_1}{}^\kappa;\cdots;u^\kappa;x^h)\,.$$

The system (1. 1) or (1. 2) of partial differential equations is equivalent to a closed exterior differential system Σ composed of

$$(2.2.1)\qquad G^K(u^\kappa;x^h)=0\,,$$

$$(2.2.2)\qquad F^A(u_i{}^\kappa;u^\kappa;x^h)=0\,,$$

$$(2.2.3)\qquad (\nabla_k G^K)(u_i{}^\kappa;u^\kappa;x^h)=0\,,$$

$$(2.2.4)\qquad dF^A=0\,,$$

$$(2.2.5)\qquad d(\nabla_k G^K)=0\,,$$

$$(2.2.6)\qquad du^\kappa - u_i{}^\kappa dx^i=0\,,$$

$$(2.2.7)\qquad du_i{}^\kappa\wedge dx^i=0\,,$$

where (2.2.2), (2.2.3) are derived from (1. 2. 2), (1. 2. 3) by substituting (2. 2. 6) or $\partial_i u^\kappa = u_i{}^\kappa$, while (2. 2. 4), (2. 2. 5), (2. 2. 7) are necessary to make (2. 2) a closed system. Σ contains n independent variables x^h and $m(n+1)$ dependent variables $u_i{}^\kappa$, u^κ.

The first prolongation of Σ is obtained by putting

$$(2.\ 3)\qquad du_i{}^\kappa = u_{ji}{}^\kappa dx^j\,.$$

Thus we obtain a closed exterior differential system Σ^1 composed of

(2.4.1) $$G^K = 0,\ \nabla_k G^K = 0,\ \nabla_{k_2}\nabla_{k_1} G^K = 0\,,$$

(2.4.2) $$F^A = 0,\ \nabla_k F^A = 0\,,$$

(2.4.3) $$du^\kappa - u_i^{\ \kappa} dx^i = 0\,,$$

(2.4.4) $$u_{i_2 i_1}{}^\kappa - u_{(i_2 i_1)}{}^\kappa = 0\,,$$

(2.4.5) $$du_{i_1}{}^\kappa - u_{k i_1}{}^\kappa dx^k = 0\,,$$

(2.4.6) $$d(\nabla_{k_2}\nabla_{k_1} G^K) = 0\,,$$

(2.4.7) $$d(\nabla_k F^A) = 0\,,$$

(2.4.8) $$du_{i_2 i_1}{}^\kappa - du_{(i_2 i_1)}{}^\kappa = 0\,,$$

(2.4.9) $$du_{k i_1}{}^\kappa \wedge dx^k = 0\,,$$

$u_{(ji)}{}^\kappa$, $u_{(kji)}{}^\kappa$ etc. standing for the symmetric parts.

The first four sets of equations ((2.4.1)–(2.4.4)) are derived by substituting (2.3) into Σ. The last four sets ((2.4.6)–(2.4.9)) are necessary to make the system Σ^1 be closed.

We can proceed in such a way to the pth prolongation Σ^p which is composed of

$(2.5.1)_0$ $$G^K = 0\,,$$

$(2.5.1)_1$ $$\nabla_k G^K = 0\,,$$

…… ……………

$(2.5.1)_{p+1}$ $$\nabla_{k_{p+1}} \cdots \nabla_{k_1} G^K = 0\,,$$

$(2.5.2)_0$ $$F^A = 0\,,$$

$(2.5.2)_1$ $$\nabla_k F^A = 0\,,$$

…… ……………

$(2.5.2)_p$ $$\nabla_{k_p} \cdots \nabla_{k_1} F^A = 0\,,$$

$(2.5.3)_1$ $$u_{i_2 i_1}{}^\kappa - u_{(i_2 i_1)}{}^\kappa = 0\,,$$

…… ……………

$(2.5.3)_p$ $$u_{i_{p+1} i_p \dots i_1}{}^\kappa - u_{(i_{p+1} i_p \dots i_1)}{}^\kappa = 0\,,$$

$(2.5.4)_0$ $$du^\kappa - u_k^{\ \kappa} dx^k = 0\,,$$

$$(2.5.4)_1 \qquad du_i^\kappa - u_{ki}{}^\kappa dx^k = 0\,,$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$

$$(2.5.4)_p \qquad du_{i_p\ldots i_1}{}^\kappa - u_{ki_p\ldots i_1}{}^\kappa dx^k = 0\,,$$

$$(2.5.5)_p \qquad d(\nabla_{k_{p+1}}\cdots\nabla_{k_1}G^K) = 0\,,$$

$$(2.5.6)_p \qquad d(\nabla_{k_p}\cdots\nabla_{k_1}F^A) = 0\,,$$

$$(2.5.7)_p \qquad du_{i_{p+1}i_p\ldots i_1}{}^\kappa - du_{(i_{p+1}i_p\ldots i_1)}{}^\kappa = 0\,,$$

$$(2.5.8)_p \qquad du_{ki_p\ldots i_1}{}^\kappa \wedge dx^k = 0\,.$$

(1.1) or Σ is solved by finding a solution of Σ^p or the infinite prolongation Σ^∞ composed of

$$(2.5.1)_p \qquad (p = 0, 1, 2, \cdots)\,,$$

$$(2.5.2)_p \qquad (p = 0, 1, 2, \cdots)\,,$$

$$(2.5.3)_p \qquad (p = 1, 2, 3, \cdots)\,,$$

$$(2.5.4)_p \qquad (p = 0, 1, 2, \cdots)$$

(see H. H. Johnson [1]). Since G^K and F^A are of class C^ω and $\nabla_{k_{r+1}}\cdots\nabla_{k_1}G^K$ and $\nabla_{k_r}\cdots\nabla_{k_1}F^A$ are polynomials in $u_{i_2i_1}{}^\kappa, \cdots, u_{i_{r+1}\ldots i_1}{}^\kappa$, $\nabla_{k_{p+1}}\cdots\nabla_{k_1}G^K$ and $\nabla_{k_p}\cdots\nabla_{k_1}F^A$ are C^ω functions of $u_{i_{p+1}\ldots i_1}{}^\kappa, \cdots, u^\kappa$, x^h for any point $(u_i^\kappa, u^\kappa, x^h)$ in D and infinite prolongation is possible.

Let us assume that an infinite sequence of numbers

$$(2.6) \qquad (x^h)_1\,, (u^\kappa)_1\,, (u_i^\kappa)_1\,, \cdots, (u_{i_r\ldots i_1}{}^\kappa)_1\,, \cdots$$

satisfies $(2.5.1)_p$, $(2.5.2)_p$, $(2.5.3)_{p+1}$ for $p = 0, 1, 2, \cdots$ simultaneously. We assume moreover that $((u_i^\kappa)_1, (u^\kappa)_1, (x^h)_1) \in D$.

For each sequence $(i_1, \cdots, i_p)$ of numbers $1 \leqq i_r \leqq n$ let ρ_1 be the number of numbers $i_1, \cdots, i_p$ which are equal to 1 and so on, and put

$$\rho(i_1, \cdots, i_p) = \rho_1!\,\rho_2! \cdots \rho_n!\,.$$

Then the theorem of H. H. Johnson [1] states that

If there exist positive numbers $M, R_1, \cdots, R_n$ such that for every natural numbers p and $i_1, \cdots, i_p$ $(1 \leqq i_1, \cdots, i_p \leqq n)$ the inequalities

$$\frac{|u_{i_p\ldots i_1}{}^\kappa|\, R_1{}^{\rho_1}\cdots R_n{}^{\rho_n}}{\rho(i_1\,, \cdots, i_p)} < M$$

are simultaneously satisfied, then there exists a unique solution of (1.1) defined in some domain D_x'[4] in D and satisfying the initial conditions

$$u^\kappa(x_1) = (u^\kappa)_1\,,$$

$$(\partial_i u^\kappa)(x_1) = (u_i{}^\kappa)_1\,,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$(\partial_{i_p}\cdots\partial_{i_1} u^\kappa)(x_1) = (u_{i_p\ldots i_1}{}^\kappa)_1\,,$$

$$\cdots\cdots\cdots\cdots\cdots.$$

But it is not a simple task in general to find a sequence (2.6). For example we can not obtain $(u^\kappa)_1$ and $(u_i{}^\kappa)_1$ from $(2.5.1)_0$, $(2.5.1)_1$, and $(2.5.2)_0$ alone in general, as it may happen that we get some other equations in $u_i{}^\kappa$, u^κ and x^h by eliminating $u_{i_2 i_1}{}^\kappa$, $u_{i_3 i_2 i_1}{}^\kappa, \cdots$ from $(2.5.1)_2$, $(2.5.1)_3, \cdots, (2.5.2)_1$, $(2.5.2)_2$, $\cdots$ and $(2.5.3)_1$, $(2.5.3)_2, \cdots$. In following paragraphs we study equations where no such difficulty occurs.

§3. A complete system of partial differential equations. In (2.2) we have three sets G^K, $\nabla_k G^K$, F^A of functions of $x^h, u^\kappa, u_i{}^\kappa$. Let us consider a matrix $(G^K|_\kappa)$ where

$$G^k|_\kappa = \partial G^K/\partial u^\kappa\,,$$

K indicating the rows and κ the columns, and a matrix

$$(G^K\,|_\kappa \delta^i_k,\ F^A\,|^i_\kappa)$$

with $nP+N$ rows and mn columns obtained from

$$\partial(\nabla_k G^K)/\partial u_i{}^\kappa \quad\text{and}\quad \partial F^A/\partial u_i{}^\kappa\,,$$

$\genfrac{}{}{0pt}{}{K}{k}$ and A indicating the rows and $\genfrac{}{}{0pt}{}{i}{\kappa}$ the columns.

Before defining a complete system precisely, we first set aside all superfluous equations from (1.2) and assume that

(A. 1) $$\operatorname{rank}(G^K|_\kappa) = P\,,$$

(A. 2) $$\operatorname{rank}(G^K|_\kappa\delta^i_k,\ F^A|^i_\kappa) = nP + N$$

in some domain D.

4) We define D_x as the set of $(x^1, \cdots, x^n)$ such that $(u_i{}^\kappa;\, u^\kappa;\, x^h) \in D$ for some $u_i{}^\kappa$ and u^κ. D_x' is some domain in D_x such that $(x_1) \in D_x'$ and $((u_i{}^\kappa)_1, (u^\kappa)_1, (x^h)_1) \in D$.

This assumption (A) stands for (i) and (ii) of §1, for, it is easily seen that in a suitable domain D' containing the point P_0 (i) is satisfied because of (A. 1), while the simultaneous equations (2. 2. 2) and (2. 2. 3) are by virtue of the assumption (A. 2) reduced to some equations which determine $nP+N$ of the u_i^κ's as functions of x^h, u^κ and the remaining $mn-(nP+N)$ of the u_i^κ's, and consequently no equation is obtained by eliminating u_i^κ from (2. 2. 2) and (2. 2. 3). We remark in passing that

$$\operatorname{rank}(G^K|_\kappa \delta_k^i) = nP,$$

for any $L^k{}_K$ satisfying $L^k{}_K G^K|_\kappa \delta_k^i = 0$ satisfies $L^i{}_K G^K|_\kappa = 0$, hence $L^i{}_K = 0$ by virtue of (A. 1). This shows that assumption (A) does not restrict a system of partial differential equations unduely as long as we are considering only in a domain D'.

We shall consider only u_i^κ and u^κ such that $(u_i^\kappa, u^\kappa, x^h) \in D'$.

DEFINITION 3. 1. Let us assume (A). A system (1. 1) of partial differential equations is said to be a *complete system within the qth prolongation* $(q=0, 1, 2, \cdots)$ when the corresponding exterior differential system Σ and its qth prolongation Σ^q $(\Sigma^0=\Sigma)$ have the following properties:

(i) No equation in $x^1, \cdots, x^n$ only is obtained by eliminating u^κ from $(2. 5. 1)_0$.[5]

(ii) Any equation in only $m+n$ unknowns u^κ and x^h obtained by eliminating u_i^κ from $(2. 5. 1)_1$ and $(2. 5. 2)_0$ is not independent of $(2. 5. 1)_0$, that is, any such equation is satisfied by any set of numbers (u^κ, x^h) satisfying $(2. 5. 1)_0$.

(iii) For every natural number $p \leqq q$ any equation in only $m(n^p+n^{p-1}+\cdots+1)+n$ unknowns $u_{i_p\ldots i_1}{}^\kappa, \cdots, u_{i_1}{}^\kappa$, u^κ, x^h obtained by eliminating $u_{i_{p+1}\ldots i_1}{}^\kappa$ from $(2. 5. 1)_{p+1}$, $(2. 5. 2)_p$ and $(2. 5. 3)_p$ is satisfied by any sequence of numbers $u_{i_p\ldots i_1}{}^\kappa, \cdots, u_{i_1}{}^\kappa, u^\kappa, x^h$ satisfying $(2. 5. 1)_0, \cdots, (2. 5. 1)_p$, $(2. 5. 2)_0, \cdots, (2. 5. 2)_{p-1}$ and $(2. 5. 3)_1, \cdots, (2. 5. 3)_{p-1}$.

DEFINITION 3. 2. When (1. 1) is a complete system within the qth prolongation for every natural number q, (1. 1) is said to be a *complete system.*

If (1. 1) is a complete system, the corresponding system Σ is also called a complete system.

Since the qth prolongation of Σ^p is the $(p+q)$th prolongation Σ^{p+q}, it is easily seen that, if Σ is a complete system, so is also any prolongation Σ^p.

As we have prolongations of Σ, so we have prolongations of the system (1. 1) or (1. 2).

Let S denote the system of partial differential equations composed of

5) (i) and (ii) are already satisfied because of (A).

$$G^K(u^\kappa; x^h) = 0, \tag{3.1.1}$$

$$(\nabla_k G^K)(u_i{}^\kappa; u^\kappa; x^h) = 0, \tag{3.1.2}$$

$$F^A(u_i{}^\kappa; u^\kappa; x^h) = 0, \tag{3.1.3}$$

$$\partial_i u^\kappa - u_i{}^\kappa = 0 \tag{3.1.4}$$

in $m(n+1)$ dependent variables $u_i{}^\kappa$, u^κ and n independent variables x^h. S is equivalent to (1.1).

The pth prolongation S^p of S or of (1.1) is defined by

$$G^K(u^\kappa; x^h) = 0, \tag{3.2.1$_0$}$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$

$$(\nabla_{k_{p+1}} \cdots \nabla_{k_1} G^K)(u_{i_{p+1}\dots i_1}{}^\kappa; \cdots; u^\kappa; x^h) = 0, \tag{3.2.1$_{p+1}$}$$

$$F^A(u_i{}^\kappa; u^\kappa; x^h) = 0, \tag{3.2.2$_0$}$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$

$$(\nabla_{k_p} \cdots \nabla_{k_1} F^A)(u_{i_{p+1}\dots i_1}{}^\kappa; \cdots; u^\kappa; x^h) = 0, \tag{3.2.2$_p$}$$

$$u_{i_2 i_1}{}^\kappa - u_{(i_2 i_1)}{}^\kappa = 0, \tag{3.2.3$_1$}$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$

$$u_{i_{p+1}\dots i_1}{}^\kappa - u_{(i_{p+1}\dots i_1)}{}^\kappa = 0, \tag{3.2.3$_p$}$$

$$\partial_j u^\kappa - u_j{}^\kappa = 0, \tag{3.2.4$_0$}$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$

$$\partial_j u_{i_p\dots i_1}{}^\kappa - u_{j i_p \dots i_1}{}^\kappa = 0. \tag{3.2.4$_p$}$$

Σ^p is obtained from S^p when (3.2.4) are replaced by equations which are transvections with dx^j and to the resulting system are added some equations to make the system closed.

Let us assume that (1.1) is a complete system within the zeroth prolongation and that (1.1) is completely integrable. A system (1.1) is by definition completely integrable when the system composed of (2.2.1), (2.2.2), (2.2.3) and (2.2.6) satisfies the condition that $du_i{}^\kappa \wedge dx^i = 0$ modulo dG^K, $d(\nabla_k G^K)$, dF^A and $du^\kappa - u_i{}^\kappa dx^i$, hence $du_i{}^\kappa \wedge dx^i = 0$ modulo $d(\nabla_k G^K)$, dF^A and $du^\kappa - u_i{}^\kappa dx^i$.

Since by virtue of (A) we can express $nP+N$ of the $u_i{}^\kappa$'s as functions of x^h, u^κ and the remaining $mn-(nP+N)$ of the $u_i{}^\kappa$'s, $du_i{}^\kappa \wedge dx^i$ can not vanish modulo $d(\nabla_k G^K)$, dF^A and $du^\kappa - u_i{}^\kappa dx^i$ if $nP+N<mn$. Hence we see that, if (1.1) is completely integrable, (1.2) can be solved in the form

$$
(3.3)\qquad \begin{cases} u^{\varphi} = f^{\varphi}(u^1, \cdots, u^R; x^1, \cdots, x^n) & (\varphi = R+1, \cdots, m), \\ \partial_i u^{\xi} = f_i^{\xi}(u^1, \cdots, u^R; x^1, \cdots, x^n) & (\xi, \eta = 1, \cdots, R) \end{cases}
$$

where $R = m - P$ and f_i^{ξ} satisfy

$$
\partial_j f_i^{\xi} + f_j^{\eta} \partial_{\eta} f_i^{\xi} = \partial_i f_j^{\xi} + f_i^{\eta} \partial_{\eta} f_j^{\xi}.
$$

As we get from (3.3) $\partial_i u^{\kappa}$ and the higher derivatives step by step by differentiating partially with respect to x, (1.1) is then a complete system.

The same deduction is possible also starting from any prolongation S^p.

Thus we obtain the

THEOREM 3.1. *Let us assume* (A). *A system* (1.1) *of partial differential equations is complete if* (1.1) *is completely integrable. A system* (1.1) *of partial differential equations is complete if there exists a natural number* p *such that* (1.1) *is complete within the* p*th prolongation and the* p*th prolongation* S^p *is completely integrable.*

We assumed that G^K and F^A are C^{ω} functions in D. But such strong restriction is not necessary when some prolongation is completely integrable and we define completeness in broader sense as follows.

DEFINITION 3.3. Let G^K and F^A be C^{ν} functions in D and let (1.1) be complete within the qth step, that is, let (1.1) satisfy (A) and the conditions (i), (ii), (iii) of definition 3.1 in $D' \subset D$. When the qth prolongation S^q or Σ^q of (1.1) is completely integrable, (1.1) is said to be a complete system.

§4. **Eliminators**. Let $((x^i)_1, (u^{\kappa})_1, (u_i^{\kappa})_1)$ be a point in D' satisfying $(2.5.1)_0$, $(2.5.1)_1$ and $(2.5.2)_0$. If a system (1.1) of partial differential equations is found to be complete in the narrow sense, a formal solution is obtained by taking this point and finding $(u_{i_2 i_1}{}^{\kappa})_1$ which satisfy $(2.5.1)_2$, $(2.5.2)_1$ and $(2.5.3)_1$, and so on. If (1.1) is complete in the broader sense, a solution is obtained by continuing such process only to some step and then solving a completely integrable system of equations which is obtained as a prolongation.

We shall present here a necessary and sufficient condition that a system be complete.

Let (1.1) be a complete system within the zeroth prolongation. Then (1.1) is a complete system within the first prolongation if and only if any equation obtained by eliminating $u_{ji}{}^{\kappa}$ from $(2.5.1)_2$, $(2.5.2)_1$, $(2.5.3)_1$ is satisfied identically by any set of numbers $u_i{}^{\kappa}$, u^{κ}, x^h satisfying $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.2)_0$.[6)]

6) We consider only $u_i{}^{\kappa}$, u^{κ}, x^h such that $(u_i{}^{\kappa}, u^{\kappa}, x^h) \in D'$.

We can write $(2.5.1)_2$ and $(2.5.2)_1$ in the form

$$G^K|_\kappa u_{ji}{}^\kappa + G_{ji}{}^K = 0\,, \tag{4. 1}$$

$$F^A|^i{}_\kappa u_{ji}{}^\kappa + F_j{}^A = 0 \tag{4. 2}$$

where $G_{ji}{}^K$ and $F_j{}^A$ are functions of $u_i{}^\kappa$, u^κ, x^h defined by

$$G_{ji}{}^K = \nabla_j \nabla_i G^K - G^K|_\kappa u_{ji}{}^\kappa\,, \tag{4. 3}$$

$$F_j{}^A = \nabla_j F^A - F^A|^i{}_\kappa u_{ji}{}^\kappa\,. \tag{4. 4}$$

Since every equation of (4. 1), (4. 2) and $(2.5.3)_1$ is a linear equation in $u_{ji}{}^\kappa$, we can eliminate $u_{ji}{}^\kappa$ by taking linear combinations of these equations, which we can write in the form

$$\begin{aligned} &L^{ji}{}_K(G^K|_\kappa u_{ji}{}^\kappa + G_{ji}{}^K) \\ &\quad + M^j{}_A(F^A|^i{}_\kappa u_{ji}{}^\kappa + F_j{}^A) \\ &\quad + N^{ji}{}_\kappa(u_{ji}{}^\kappa - u_{(ji)}{}^\kappa) = 0\,. \end{aligned} \tag{4. 5}$$

We can also consider that every equation obtained from (4.1), (4.2), $(2.5.3)_1$ by eliminating $u_{ji}{}^\kappa$ has the form (4.5) with $L^{ji}{}_K$, $M^j{}_A$, $N^{ji}{}_\kappa$ satisfying

$$(L^{ji}{}_K G^K|_\kappa + M^j{}_A F^A|^i{}_\kappa + N^{ji}{}_\kappa)\, u_{ji}{}^\kappa - N^{ji}{}_\kappa u_{(ji)}{}^\kappa = 0$$

identically, hence

$$L^{ji}{}_K G^K|_\kappa + M^j{}_A F^A|^i{}_\kappa + N^{ji}{}_\kappa - N^{(ji)}{}_\kappa = 0\,. \tag{4. 6}$$

From (4. 6) we can eliminate $N^{ji}{}_\kappa$ and obtain

$$(L^{ji}{}_K + L^{ij}{}_K) G^K|_\kappa + M^j{}_A F^A|^i{}_\kappa + M^i{}_A F^A|^j{}_\kappa = 0\,. \tag{4. 7}$$

Thus we find that (1. 1) is complete within the first prolongation if and only if every equation

$$L^{ji}{}_K G_{ji}{}^K + M^j{}_A F_j{}^A = 0\,, \tag{4. 8}$$

where $L^{ji}{}_K$, $M^j{}_A$ are restricted by (4. 7), is satisfied identically by $u_i{}^\kappa$, u^κ, x^h satisfying $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.2)_0$.

DEFINITION 4. 1. A set of numbers $(L^{ji}{}_K, M^j{}_A)$ satisfying (4. 7) is called an *eliminator of the first order.*

Let us eliminate $u_{i_{p+1}\dots i_1}{}^\kappa$ from $(2.5.1)_{p+1}$, $(2.5.2)_p$ and $(2.5.3)_p$. This is accomplished by making a linear combination

$$
\begin{aligned}
(4.\,9) \qquad & L^{i_{p+1}\dots i_1}{}_K \nabla_{i_{p+1}} \cdots \nabla_{i_1} G^K \\
& + M^{i_{p+1}\dots i_2}{}_A \nabla_{i_{p+1}} \cdots \nabla_{i_2} F^A \\
& + (N^{i_{p+1}\dots i_1}{}_\kappa - N^{(i_{p+1}\dots i_1)}{}_\kappa)\, u_{i_{p+1}\dots i_1}{}^\kappa = 0
\end{aligned}
$$

where $L^{i_{p+1}\dots i_1}{}_K$, $M^{i_p\dots i_1}{}_A$,$N^{i_{p+1}\dots i_1}{}_\kappa$ (which we shall write simply L, M, N if there is no possibility of confusion) satisfy

$$
\begin{aligned}
(4.\,10) \qquad & L^{i_{p+1}\dots i_1}{}_K G^K|_\kappa + M^{i_{p+1}\dots i_2}{}_A F^A|^{i_1}{}_\kappa \\
& + N^{i_{p+1}\dots i_1}{}_\kappa - N^{(i_{p+1}\dots i_1)}{}_\kappa = 0\,.
\end{aligned}
$$

Such L, M, N are obtained by taking L, M such that

$$
(4.\,11)_p \qquad L^{(i_{p+1}\dots i_1)}{}_K G^K|_\kappa + M^{(i_{p+1}\dots i_2}{}_A F^{|A|}|^{i_1)}{}_\kappa = 0
$$

and then taking N which satisfies (4. 10).

DEFINITION 4. 2. A set of numbers (L, M) satisfying $(4.\,11)_p$ is called an *eliminator of the pth order.*

An eliminator is considered at each point $(u_i{}^\kappa,\ u^\kappa,\ x^h)$ in D'. Generally it is not a definite function of $u_i{}^\kappa$, u^κ, x^h, but we can consider its differential $(dL,\ dM)$.

DEFINITION 4. 3. $G_{i_{p+1}\dots i_1}{}^K$ and $F_{i_p\dots i_1}{}^A$ are functions of $u_{i_p\dots i_1}{}^\kappa, \cdots, u^\kappa,\ x^h$ defined by

$$
(4.\,12)_p \qquad \begin{cases} G_{i_{p+1}\dots i_1}{}^K = \nabla_{i_{p+1}} \cdots \nabla_{i_1} G^K - G^K|_\kappa u_{i_{p+1}\dots i_1}{}^\kappa\,, \\ F_{i_{p+1}\dots i_2}{}^A = \nabla_{i_{p+1}} \cdots \nabla_{i_2} F^A - F^A|^{i_1}{}_\kappa u_{i_{p+1}\dots i_2 i_1}{}^\kappa\,. \end{cases}
$$

We find immediately that, if L, M, N satisfy (4. 10), then (4. 9) becomes

$$
(4.\,13)_p \qquad L^{i_{p+1}\dots i_1}{}_K\, G_{i_{p+1}\dots i_1}{}^K + M^{i_p\dots i_1}{}_A\, F_{i_p\dots i_1}{}^A = 0\,.
$$

Hence we find that, if (1. 1) is complete within the qth prolongation, then $(4.\,13)_p$ is satisfied identically by any sequence of numbers $u_{i_p\dots i_1}{}^\kappa, \cdots$, u^κ, x^h satisfying $(2.5.1)_0, \cdots, (2.5.1)_p$, $(2.5.2)_0, \cdots, (2.5.2)_{p-1}$ and $(2.5.3)_1, \cdots$, $(2.5.3)_{p-1}$ for $p = 1, 2, \cdots, q$.

Thus we get the following theorem.

THEOREM 4. 1. *Let* (1. 1) *be a complete system within the zeroth prolongation. Then* (1. 1) *is a complete system within the qth prolongation if and only if for each p ($p = 1, \cdots, q$) every equation of the form* $(4.\,13)_p$, *where*

(L, M) is an arbitrary eliminator of the pth order, is satisfied by any sequence of numbers $u_{i_p\ldots i_1}{}^{\kappa}, \cdots, u^{\kappa}, x^h$ satisfying $(2.5.1)_0, \cdots, (2.5.1)_p, (2.5.2)_0, \cdots, (2.5.2)_{p-1}$ and $(2.5.3)_1, \cdots, (2.5.3)_{p-1}$. If q is replaced with ∞ in this proposition we get necessary and sufficient condition that (1.1) be a complete system.

§5. **Symmetric eliminators and E-simple systems.** We first prove the

LEMMA 5.1. *Let*

$$F(v_{i_r\ldots i_1}{}^{\rho}; \cdots; v_{i_1}{}^{\rho}; v^{\rho}; x^h)$$

be any function of $v_{i_r\ldots i_1}{}^{\rho}, \cdots, v_{i_1}{}^{\rho}, v^{\rho}, x^h$ satisfying a suitable differentiability condition and where $v_{i_s\ldots i_1}{}^{\rho}$ are symmetric in $i_s, \cdots, i_1$ if $s \leqq r+2$. ($v_{i_s\ldots i_1}{}^{\rho}$ with $s = r+1, r+2$ will appear later). Let the sequence of numbers $(v_{i_r\ldots i_1}{}^{\rho}, \cdots, v^{\rho})$ be denoted by $v_X{}^{\rho}$ where X represents $i_r \cdots i_1, i_{r-1} \cdots i_1, \cdots, i_1, \phi$, $v_{\phi}{}^{\rho}$ standing for v^{ρ}. Then we have $F(v_X{}^{\rho}; x^h)$. Adopting the summation convention with respect to $X, Y, \cdots$, we define

$$(\nabla_k F)(v_{i_{r+1}\ldots i_1}{}^{\rho}; \cdots; v^{\rho}; x^h) \equiv F|^X{}_{\rho} v_{kX}{}^{\rho} + F|_k ,$$

especially

$$\nabla_k v_{i_s\ldots i_1}{}^{\rho} = v_{ki_s\ldots i_1 \rho} .$$

Then we have

$$\nabla_l \nabla_k F - \nabla_k \nabla_l F = 0 . \tag{5.1}$$

PROOF. Since we have

$$\nabla_l \nabla_k F = F|^X{}_{\rho} v_{lkX}{}^{\rho} + F|^X{}_{\rho}|^Y{}_{\sigma} v_{kX}{}^{\rho} v_{lY}{}^{\sigma}$$
$$+ F|^X{}_{\rho}|_l v_{kX}{}^{\rho} + F|_k|^Y{}_{\sigma} v_{lY}{}^{\sigma} + F|_k|_l ,$$
$$v_{lkX}{}^{\rho} = v_{klX}{}^{\rho}$$

and

$$F|^X{}_{\rho}|^Y{}_{\sigma} = F|^Y{}_{\sigma}|^X{}_{\rho} , \; F|^X{}_{\rho}|_l = F|_l|^X{}_{\rho} ,$$
$$F|_k|_l = F|_l|_k ,$$

we get (5.1) directly.

Next we prove the

LEMMA 5.2. *$G_{i_{p+1}\ldots i_1}{}^{k}$ and $F_{i_p\ldots i_1}{}^{A}$ defined by $(4.12)_p$ are symmetric in their lower indices if $u_{i_2 i_1}{}^{\kappa}, \cdots, u_{i_p\ldots i_1}{}^{\kappa}$ are symmetric in their lower indices.*

PROOF. For small p this is proved directly. For larger p we use Lemma 5.1 in the case of $r=p-2$ and take G^K as F of this lemma. Then we find that $\nabla_{i_3}\nabla_{i_2}\nabla_{i_1}G^K$ is symmetric with respect to i_2 and i_1 as well as with respect to i_3 and i_2. Hence $\nabla_{i_3}\nabla_{i_2}\nabla_{i_1}G^K$ is symmetric in i_3, i_2, i_1. We can proceed in this way until we get $\nabla_{i_p}\cdots\nabla_{i_1}G^K$ which is symmetric in $i_p, \cdots, i_1$. If moreover $u_{i_{p+1}\dots i_1}{}^{\kappa}$ were also symmetric in the lower indices, $\nabla_{i_{p+1}}\cdots\nabla_{i_1}G^K$ would be symmetric in the lower indices. However, this is not the case now. But, since $\nabla_{i_{p+1}}\cdots\nabla_{i_1}G^K$ contains $u_{i_{p+1}\dots i_1}{}^{\kappa}$ only in the term $G^K|_{\kappa}u_{i_{p+1}\dots i_1}{}^{\kappa}$, the remaining part $G_{i_{p+1}\dots i_1}{}^{K}$ is symmetric in all lower indices. For F^A also we can proceed in the same way.

Now we find from $(4.11)_p$ that, if $(L^{i_{p+1}\dots i_1}{}_K, M^{i_p\dots i_1}{}_A)$ is an eliminator, then $(\widetilde{L}^{i_{p+1}\dots i_1}{}_K, \widetilde{M}^{i_p\dots i_1}{}_A)$, where $\widetilde{L}, \widetilde{M}$ are the symmetric parts of L, M respectively with respect to all superior indices, is also an eliminator. From Lemma 5.2 we find moreover that L, M can always be replaced with $\widetilde{L}, \widetilde{M}$ in $(4.13)_p$, for $(4.13)_p$ is used to find whether (1.1) is complete within pth prolongation or not when (1.1) is known to be complete within $(p-1)$th prolongation and $(2.5.3)_1, \cdots, (2.5.3)_{p-1}$ are assumed.

This proves that we need only to consider symmetric eliminators, a *symmetric eliminator* being defined as an eliminator (L, M) where $L=\widetilde{L}$, $M=\widetilde{M}$. In the following we consider only symmetric eliminators and the adjective "symmetric" will be dropped.

The following lemma is trivial.

LEMMA 5.3. *Let a system* (1.1) *of partial differential equations be given. At each point* $(u_i^{\kappa}, u^{\kappa}, x^h)$ *in* D' *the set of eliminators of any given order is a linear space.*

Let this linear space be denoted by V_p, where p is the order, and let us put $\dim V_p = m_p$. Then we get

LEMMA 5.4. *If* m_1 *eliminators* $(\underset{\xi}{L}, \underset{\xi}{M})$ $(\xi=1,\cdots,m_1)$ *compose a base of* V_1, (L, M) *where*

$$
\begin{aligned}
L^{i_{p+1}\dots i_1}{}_K &= \overset{\xi}{H}{}^{(i_{p+1}\dots i_3}\underset{\xi}{L}{}^{i_2 i_1)}{}_K \\
M^{i_p\dots i_1}{}_A &= \overset{\xi}{H}{}^{(i_p\dots i_2}\underset{\xi}{M}{}^{i_1)}{}_A
\end{aligned}
\tag{5.2}
$$

is an eliminator of order p.

We use indices $\xi, \eta=1,\cdots,m_1$ and adopt the summation convention with respect to these too. Being almost immediate, proof will be omitted.

DEFINITION 5.1. An eliminator (L, M) of order p of the form (5.2) is called a *simple eliminator.*

For any given system (1.1), for each order p, and at each point $(u_i^\kappa, u^\kappa, x^h)$ in D' the set of simple eliminators is also a linear space.

DEFINITION 5.2. If at every point of D' V_p is composed of simple eliminators only, (1.1) is called an *E-simple system at the order p.* If (1.1) is E-simple at every order, it is called an *E-simple system.*

Whether a system (1.1) is E-simple or not depends only upon the derivatives $G^K|_\kappa$ and $F^A|^i_\kappa$.

If a system is E-simple, every eliminator has the form (5.2) and $(4.13)_p$ becomes

$$\overset{\xi}{H}{}^{i_{p+1}\dots i_3}(\underset{\xi}{L}{}^{i_2 i_1}{}_K G_{i_{p+1}\dots i_1}{}^K + \underset{\xi}{M}{}^{i_2}{}_A F_{i_{p+1}\dots i_2}{}^A) = 0\,,$$

where we can take $\overset{\xi}{H}{}^{i_{p+1}\dots i_3}$ arbitrarily. Hence $(4.13)_p$ is equivalent to

$$L^{i_2 i_1}{}_K G_{i_{p+1}\dots i_1}{}^K + M^{i_2}{}_A F_{i_{p+1}\dots i_2}{}^A = 0\,. \tag{5.3$_p$}$$

Thus we obtain the

LEMMA 5.5. *Let a system* (1.1) *of partial differential equations be E-simple and complete within the first prolongation. A necessary and sufficient condition that it be complete within the qth prolongation is that* $(5.3)_p$ $(p=1,\cdots,q)$, *where* (L, M) *is an arbitrary eliminator of the first order, is satisfied by any sequence of numbers* $u_{i_p\dots i_1}{}^\kappa,\cdots,u^\kappa, x^h$ *satisfying* $(2.5.1)_0,\cdots,(2.5.1)_p$, $(2.5.2)_0,\cdots,(2.5.2)_{p-1}$ *and* $(2.5.3)_1,\cdots,(2.5.3)_{p-1}$. *If q is replaced with* ∞ *in this proposition we get necessary and sufficient condition that* (1.1) *be a complete system.*

§6. E-simple systems of partial differential equations. We are now going to prove the following main theorem.

THEOREM 6.1. *Let a system* (1.1) *of partial differential equations be E-simple. A necessary and sufficient condition that the system be complete is that it be complete within the first prolongation.*

We shall first prove the

LEMMA 6.2. *If* (1.1) *is complete within the first prolongation and is E-simple, then* (1.1) *is complete within the second prolongation.*

Let $(L^{ji}{}_K, M^i{}_A)$ be an eliminator of the first order satisfying (4.7). Since (1.1) is assumed to be complete within the first prolongation, (4.8) is satisfied by any $u_i{}^\kappa$, u^κ, x^h satisfying $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.2)_0$.

If $u_i{}^\kappa$, u^κ, x^h vary satisfying $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.2)_0$, and if the eliminator $(L^{ji}{}_K, M^j{}_A)$ varies continuously with $u_i{}^\kappa$, u^κ, x^h, we get

$$(dL^{ji}{}_K)G_{ji}{}^K + (dM^j{}_A)F_j{}^A + L^{ji}{}_K dG_{ji}{}^K + M^j{}_A dF_j{}^A = 0\,, \tag{6. 1}$$

$$dG^K = 0, \quad d(\nabla_k G^K) = 0, \quad dF^A = 0\,. \tag{6. 2}$$

If we take any $u_{ji}{}^\kappa$ such that the sequence of numbers $(u_{ji}{}^\kappa, u_i{}^\kappa, u^\kappa, x^h)$ satisfies $(2.5.1)_2$, $(2.5.2)_1$ and $(2.5.3)_1$ besides $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.2)_0$, then we can substitute

$$du_i{}^\kappa = u_{ji}{}^\kappa dx^j, \quad du^\kappa = u_i{}^\kappa dx^i \tag{6. 3}$$

into (6.1) and (6.2), for (6.2) becomes then equivalent to

$$\nabla_k G^K = 0, \quad \nabla_l \nabla_k G^K = 0, \quad \nabla_k F^A = 0\,, \tag{6. 4}$$

which is satisfied.

We get from

$$\nabla_j \nabla_i G^K = G^K|_\kappa u_{ji}{}^\kappa + G_{ji}{}^K\,,$$

$$\nabla_j F^A = F^A|^i{}_\kappa u_{ji}{}^\kappa + F_j{}^A$$

the following relations,

$$\nabla_k \nabla_j \nabla_i G^K = (\nabla_k G^K|_\kappa) u_{ji}{}^\kappa + G^K|_\kappa u_{kji}{}^\kappa + \nabla_k G_{ji}{}^K\,,$$

$$\nabla_k \nabla_j F^A = (\nabla_k F^A|^i{}_\kappa) u_{ji}{}^\kappa + F^A|^i{}_\kappa u_{kji}{}^\kappa + \nabla_k F_j{}^A\,.$$

Substituting these into

$$G_{kji}{}^K = \nabla_k \nabla_j \nabla_i G^K - G^K|_\kappa u_{kji}{}^\kappa\,,$$

$$F_{kj}{}^A = \nabla_k \nabla_j F^A - F^A|^i{}_\kappa u_{kji}{}^\kappa\,,$$

we get

$$G_{kji}{}^K = (\nabla_k G^K|_\kappa)\, u_{ji}{}^\kappa + \nabla_k G_{ji}{}^K\,,$$

$$F_{kj}{}^A = (\nabla_k F^A|^i{}_\kappa)\, u_{ji}{}^\kappa + \nabla_k F_j{}^A\,.$$

Now, $G_{ji}{}^K$ and $F_j{}^A$ being functions of $u_i{}^\kappa$, u^κ, x^h, we get from (6. 1) and (6. 3)

$$(dL^{ji}{}_K)G_{ji}{}^K + (dM^j{}_A)F_j{}^A + L^{ji}{}_K(\nabla_k G_{ji}{}^K)dx^k + M^j{}_A(\nabla_k F_j{}^A)dx^k = 0$$

which is equivalent to

$$(6.5) \qquad (dL^{ji}{}_K)(\nabla_j\nabla_i G^K - G^K|_\kappa u_{ji}{}^\kappa) + (dM^j{}_A)(\nabla_j F^A - F^A|^i{}_\kappa u_{ji}{}^\kappa)$$
$$+ L^{ji}{}_K(G_{kji}{}^K - (\nabla_k G^K|_\kappa)u_{ji}{}^\kappa)\, dx^k + M^j{}_A(F_{kj}{}^A - (\nabla_k F^A|^i{}_\kappa)u_{ji}{}^\kappa)\, dx^k = 0\,.$$

On the other hand we get from (4.7)

$$d(L^{ji}{}_K G^K|_\kappa + L^{ij}{}_K G^K|_\kappa + M^j{}_A F^A|^i{}_\kappa + M^i{}_A F^A|^j{}_\kappa) = 0$$

or

$$d(L^{ji}{}_K G^K|_\kappa + M^j{}_A F^A|^i{}_\kappa)\, u_{ji}{}^\kappa = 0$$

where $du_i{}^\kappa$, du^κ, dx^h must satisfy (6.2). Hence we can substitute (6.3) and obtain

$$(6.6) \qquad \{(dL^{ji}{}_K)G^K|_\kappa + L^{ji}{}_K(\nabla_k G^K|_\kappa)\, dx^k$$
$$+ (dM^j{}_A)F^A|^i{}_\kappa + M^j{}_A(\nabla_k F^A|^i{}_\kappa)\, dx^k\}\, u_{ji}{}^\kappa = 0\,.$$

From (6.5) and (6.6) we get

$$(dL^{ji}{}_K)\nabla_j\nabla_i G^K + (dM^j{}_A)\nabla_j F^A + (L^{ji}{}_K G_{kji}{}^K + M^j{}_A F_{kj}{}^A)\, dx^k = 0\,.$$

This proves that

$$L^{ji}{}_K G_{kji}{}^K + M^j{}_A F_{kj}{}^A = 0$$

are satisfied by any eliminator (L, M) of the first order if $u_{ji}{}^\kappa$, $u_i{}^\kappa$, u^κ, x^h only satisfy $(2.5.1)_0$, $(2.5.1)_1$, $(2.5.1)_2$, $(2.5.2)_0$, $(2.5.2)_1$, $(2.5.3)_1$.

Since we have Lemma 5.5, we get Lemma 6.2.

In the same way we can prove the

LEMMA 6.3. *If* (1.1) *is E-simple and is complete within the qth prolongation, then* (1.1) *is complete within the* $(q+1)$*th prolongation.*

Then we get Theorem 6.1 by induction.

§7. Systems with redundant equations. Let the indices λ, α run for the moment as follows,

$$\lambda = 1, \cdots, P'\,; \qquad \alpha = 1, \cdots, N'\,,$$

and consider C^∞ functions $\mathfrak{G}^\lambda$ of G^K and C^∞ functions $\mathfrak{F}^\alpha$ of $\nabla_k G^K$, F^A such that

$$\mathfrak{G}^\lambda(G^K) = 0\,,$$

$$\mathfrak{F}^\alpha(\nabla_k G^K\,;F^A) = 0$$

are satisfied by $G^K=0$, $\nabla_k G^K=0$, $F^A=0$. Remember that G^K are functions of u^κ and x^h, while $\nabla_k G^K$ and F^A are functions of u_i^{κ}; u^κ; x^h. $\mathfrak{F}^\alpha$ may also contain G^K as parameters.

Replacing u_i^{κ} with $\partial_i u^\kappa$ we get a system of partial differential equations composed of

$$G^K = 0,\quad \mathfrak{G}^\lambda = 0,\quad F^A = 0,\quad \mathfrak{F}^\alpha = 0 \tag{7. 1}$$

or

$$\begin{gathered} G^K = 0,\quad \mathfrak{G}^\lambda = 0,\quad F^A = 0,\quad \mathfrak{F}^\alpha = 0\,, \\ G^K|_\kappa \partial_i u^\kappa + G^K|_i = 0\,. \end{gathered} \tag{7. 2}$$

Evidently three systems (1.1), (7.1), (7.2) are equivalent, $\mathfrak{G}^\lambda = 0$ and $\mathfrak{F}^\alpha = 0$ being superfluous equations.

It often occurs that a given system of partial differential equations contains such superfluous equations in implicate form and that we can not exclude these superfluous equations from the system without destroying its regular form. In such a case it is better to leave the system as it is.

Thus we use what are called provisional eliminators. A *provisional eliminator* of the *p*th order is a set

$$(\overset{*}{L}{}^{i_{p+1}\dots i_1}{}_K\,,\ \overset{*}{L}{}^{i_{p+1}\dots i_1}{}_\lambda\,,\ \overset{*}{M}{}^{i_p\dots i_1}{}_A\,,\ \overset{*}{M}{}^{i_p\dots i_1}{}_\alpha)$$

satisfying

$$\overset{*}{L}{}^{(i_{p+1}\dots i_1)}{}_K G^K|_\kappa + \overset{*}{L}{}^{(i_{p+1}\dots i_1)}{}_\lambda \mathfrak{G}^\lambda|_\kappa + \overset{*}{M}{}^{(i_{p+1}\dots i_2}{}_A F^{|A|}|^{i_1)}{}_\kappa + \overset{*}{M}{}^{(i_{p+1}\dots i_2}{}_\alpha \mathfrak{F}^{|\alpha|}|^{i_1)}{}_\kappa = 0 \tag{7. 3}$$

and such that $\overset{*}{L}{}^{i_{p+1}\dots i_1}{}_K$, $\overset{*}{L}{}^{i_{p+1}\dots i_1}{}_\lambda$, $\overset{*}{M}{}^{i_p\dots i_1}{}_A$, $\overset{*}{M}{}^{i_p\dots i_1}{}_\alpha$ are all symmetric in superior indices.

Let us write the relations

$$\mathfrak{G}^\lambda|_\kappa = (\partial\mathfrak{G}^\lambda/\partial G^K)G^K|_\kappa\,,$$

$$\mathfrak{F}^\alpha|^i{}_\kappa = (\partial\mathfrak{F}^\alpha/\partial\nabla_i G^K)G^K|_\kappa + (\partial\mathfrak{F}^\alpha/\partial F^A)F^A|^i{}_\kappa$$

in the form

$$\mathfrak{G}^{\lambda}|_{\kappa} = S^{\lambda}{}_{K}G^{K}|_{\kappa}\,,$$

$$\mathfrak{F}^{\alpha}|^{i}_{\kappa} = S^{\alpha i}{}_{K}G^{K}|_{\kappa} + S^{\alpha}{}_{A}F^{A}|^{i}_{\kappa}\,.$$

Then we soon find that $(L^{i_{p+1}\cdots i_1}{}_{K},\ M^{i_p\cdots i_1}{}_{A})$ where

$$L^{i_{p+1}\cdots i_1}{}_{K} = \overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{K} + \overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{\lambda}S^{\lambda}{}_{K} + \overset{*}{M}{}^{(i_{p+1}\cdots i_2}{}_{\alpha}S^{|\alpha|i_1)}{}_{K}\,,$$

$$M^{i_p\cdots i_1}{}_{A} = \overset{*}{M}{}^{i_p\cdots i_1}{}_{A} + \overset{*}{M}{}^{i_p\cdots i_1}{}_{\alpha}S^{\alpha}{}_{A}$$

is an eliminator of the pth order of the system (1.1).

It is also found immediately that, if $(L^{i_{p+1}\cdots i_1}{}_{K},\ M^{i_p\cdots i_1}{}_{A})$ is an eliminator of (1.1), then

$$(\overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{K},\ \overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{\lambda},\ \overset{*}{M}{}^{i_p\cdots i_1}{}_{A},\ \overset{*}{M}{}^{i_p\cdots i_1}{}_{\alpha})$$

where

$$\overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{K} = L^{i_{p+1}\cdots i_1}{}_{K},\quad \overset{*}{L}{}^{i_{p+1}\cdots i_1}{}_{\lambda} = 0\,,$$

$$\overset{*}{M}{}^{i_p\cdots i_1}{}_{A} = M^{i_p\cdots i_1}{}_{A},\quad \overset{*}{M}{}^{i_p\cdots i_1}{}_{\alpha} = 0$$

is a provisional eliminator.

If every provisional eliminator

$$(\overset{*}{L}{}^{i_{p+2}\cdots i_1}{}_{K}\,,\ \overset{*}{L}{}^{i_{p+2}\cdots i_1}{}_{\lambda}\,,\ \overset{*}{M}{}^{i_{p+1}\cdots i_1}{}_{A}\,,\ \overset{*}{M}{}^{i_{p+1}\cdots i_1}{}_{\alpha})$$

of the $(p+1)$th order $(p=1,2,\cdots)$ can be written in the form

$$\begin{aligned}
\overset{*}{L}{}^{k_p\cdots k_1 ji}{}_{K} &= \overset{\xi}{H}{}^{(k_p\cdots k_1}\,\underset{\xi}{\overset{*}{L}}{}^{ji)}{}_{K}\,,\\
\overset{*}{L}{}^{k_p\cdots k_1 ji}{}_{\lambda} &= \overset{\xi}{H}{}^{(k_p\cdots k_1}\,\underset{\xi}{\overset{*}{L}}{}^{ji)}{}_{\lambda}\,,\\
\overset{*}{M}{}^{k_p\cdots k_1 i}{}_{A} &= \overset{\xi}{H}{}^{(k_p\cdots k_1}\,\underset{\xi}{\overset{*}{M}}{}^{i)}{}_{A}\,,\\
\overset{*}{M}{}^{k_p\cdots k_1 i}{}_{\alpha} &= \overset{\xi}{H}{}^{(k_p\cdots k_1}\,\underset{\xi}{\overset{*}{M}}{}^{i)}{}_{\alpha}\,,
\end{aligned}\tag{7.4}$$

where

$$(\underset{\xi}{\overset{*}{L}}{}^{ji}{}_{K}\,,\ \underset{\xi}{\overset{*}{L}}{}^{ji}{}_{\lambda}\,,\ \underset{\xi}{\overset{*}{M}}{}^{i}{}_{A},\ \underset{\xi}{\overset{*}{M}}{}^{i}{}_{\alpha})$$

is a provisional eliminator of the first order for each value of ξ in some range $\xi = 1, 2, \cdots, Z$, then the system (1.1) is an E-simple system. This is proved as follows.

Let $(L^{i_{p+2}\cdots i_1}{}_K, M^{i_{p+1}\cdots i_1}{}_A)$ be any eliminator. Then, since $(L^{i_{p+2}\cdots i_1}{}_K, 0, M^{i_{p+1}\cdots i_1}{}_A, 0)$ is a provisional eliminator, we have $\overset{\xi}{H}{}^{i_p\cdots i_1}$ such that

$$L^{k_p\cdots k_1 ji}{}_K = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{\overset{*}{L}}{}^{ji)}{}_K ,$$

$$0 = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{\overset{*}{L}}{}^{ji)}{}_\lambda ,$$

$$M^{k_p\cdots k_1 i}{}_A = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{\overset{*}{M}}{}^{i)}{}_A ,$$

$$0 = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{\overset{*}{M}}{}^{i)}{}_\alpha .$$

Taking suitable linear combinations of these equations we get

$$L^{k_p\cdots k_1 ji}{}_K = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{L}{}^{ji)}{}_K ,$$

$$M^{k_p\cdots k_1 i}{}_A = \overset{\xi}{H}{}^{(k_p\cdots k_1} \underset{\xi}{M}{}^{i)}{}_A ,$$

if we only put

$$\underset{\xi}{L}{}^{ji}{}_K = \underset{\xi}{\overset{*}{L}}{}^{ji}{}_K + \underset{\xi}{\overset{*}{L}}{}^{ji}{}_\lambda S^\lambda{}_K + \underset{\xi}{\overset{*}{M}}{}^{(j}{}_\alpha S^{|\alpha|i)}{}_K ,$$

$$\underset{\xi}{M}{}^{i}{}_A = \underset{\xi}{\overset{*}{M}}{}^{i}{}_A + \underset{\xi}{\overset{*}{M}}{}^{i}{}_\alpha S^\alpha{}_A .$$

Thus any eliminator of higher order is expressed linearly in terms of eliminators of the first order and we have proved that (1.1) is E-simple.

Let (1.1) be an E-simple system. We put

$$\begin{aligned} G_{ji}{}^\lambda &= \nabla_j \nabla_i \mathfrak{G}^\lambda - \mathfrak{G}^\lambda|_\kappa u_{ji}{}^\kappa , \\ F_j{}^\alpha &= \nabla_j \mathfrak{F}^\alpha - \mathfrak{F}^\alpha|^i{}_\kappa u_{ji}{}^\kappa . \end{aligned} \tag{7. 5}$$

Then we can prove that, if every provisional eliminator of the first order satisfies

$$\overset{*}{L}{}^{ji}{}_K G_{ji}{}^K + \overset{*}{L}{}^{ji}{}_\lambda G_{ji}{}^\lambda + \overset{*}{M}{}^j{}_A F_j{}^A + \overset{*}{M}{}^j{}_\alpha F_j{}^\alpha = 0 , \tag{7. 6}$$

(1.1) is a complete system. For this purpose we have only to put

$$\overset{*}{L}{}^{ji}{}_K = L^{ji}{}_K , \quad \overset{*}{L}{}^{ji}{}_\lambda = 0 , \quad \overset{*}{M}{}^j{}_A = M^j{}_A , \quad \overset{*}{M}{}^j{}_\alpha = 0$$

where $(L^{ji}{}_K, M^j{}_A)$ is an arbitrary eliminator.

Thus we get

THEOREM 7.1. *Let* (7.1) *be a system of partial differential equations equivalent to* (1.1), $\mathfrak{G}^\lambda = 0$ *and* $\mathfrak{F}^\alpha = 0$ *being superfluous equations. Then we can distinguish* (1.1) *to be E-simple if all provisional eliminators of higher order can be expressed in the form* (7.4). *Let* (1.1) *be E-simple. Then* (1.1) *is complete if every provisional eliminator of the first order satisfies* (7.6).

§8. An example of E-simple systems. It is not difficult to prove that any system of partial differential equations of the first order in only one dependent variable is an E-simple system. But since such systems are well-known we present another example.

Let us consider a coordinate neighborhood U of an n-dimensional C^∞ manifold M. We assume that for each point $x \in U$ and each vector $u \in M_x$ a (1, 1) tensor φ is given and that its components $\varphi_i{}^h$ are C^∞ functions of the coordinates $x^1, \cdots, x^n$ and of the components $u^1, \cdots, u^n$. We assume moreover that the matrix φ has diagonal Jordan canonical form with μ distinct eigenvalues $\sigma_1, \cdots, \sigma_\mu$ with multiplicity $m_1, \cdots, m_\mu$. Let D be a domain in U such that μ and $m_1, \cdots, m_\mu$ are constant in D.

We consider a system of partial differential equations

$$\varphi_i{}^l(x, u)\partial_m u^i - \varphi_m{}^i(x, u)\partial_i u^l + \psi_m{}^l(x, u) = 0 \tag{8.1}$$

where $\psi_m{}^l$ are also C^∞ functions of x and u, and prove that the system (8.1) is an E-simple system if ψ has the form

$$\psi_i{}^h = \varphi_i{}^k \pi_k{}^h - \pi_i{}^k \varphi_k{}^h \,. \tag{8.2}$$

First, we find immediately that, by virtue of (8.2), no equation is obtained by eliminating $u_i{}^h$ from

$$\varphi_i{}^l u_m{}^i - \varphi_m{}^i u_i{}^l + \psi_m{}^l = 0 \tag{8.3}$$

which is equivalent to

$$\varphi_i{}^l(u_m{}^i - \pi_m{}^i) - \varphi_m{}^i(u_i{}^l - \pi_i{}^l) = 0 \,.$$

We can write (8.1) in the form

$$F_m{}^l(\partial_i u^h \,; u^h \,; x^h) = 0 \tag{8.4}$$

where $F_m{}^l$ and u^h would take the place of F^A and u^κ in (1.1) respectively, but (8.3) does not contain n^2 linearly independent equations. Hence (8.4) is a system with redundant equations which we can write

$$F^A = 0, \quad \mathfrak{F}^\alpha = 0\,.$$

Since we have no equation $G^K = 0$, $\mathfrak{G}^\lambda = 0$, a provisional eliminator of the rth order is given by $(\overset{*}{M}{}^{k_r\cdots k_1}{}_A,\ \overset{*}{M}{}^{k_r\cdots k_1}{}_\alpha)$ satisfying

$$\overset{*}{M}{}^{(k_r\cdots k_1}{}_A F^{|A|}|^{i)}{}_h + \overset{*}{M}{}^{(k_r\cdots k_1}{}_\alpha \mathfrak{F}^{|\alpha|}|^{i)}{}_h = 0\,.$$

But turning to (8.4) we can also consider that a provisional eliminator of the rth order is a quantity $(M^{k_r\cdots k_1 i}{}_h)$ satisfying

$$M^{(k_r\cdots k_1|m}{}_l F_m{}^{l|}|^{i)}{}_h = 0 \tag{8.5}$$

and

$$M^{k_r\cdots k_1 i}{}_h = M^{(k_r\cdots k_1)i}{}_h\,.$$

As we get

$$F_m{}^l|^i{}_h = \varphi_h{}^l\delta^i_m - \varphi_m{}^i\delta^i_h$$

from

$$F_m{}^l(u_i{}^h\,;u^h\,;x^h) \equiv \varphi_i{}^l u_m{}^i - \varphi_m{}^i u_i{}^l + \psi_m{}^l\,,$$

(8.5) can be written in the form

$$M^{(k_r\cdots k_1 i)}{}_l\varphi_h{}^l - M^{(k_r\cdots k_1|m|}{}_h\varphi_m{}^{i)} = 0\,. \tag{8.6}$$

Since our problem is one of linear algebra, we can take a suitable frame such that the components of φ take the form

$$\varphi_i{}^h = \rho_i\delta^h_i$$

where

$$\rho_1 = \cdots = \rho_{m_1} = \sigma_1\,,\quad \rho_{m_1+1} = \cdots = \rho_{m_1+m_2} = \sigma_2\,,$$

$$\cdots\cdots\cdots,\quad \rho_{n-m_\mu+1} = \cdots = \rho_n = \sigma_\mu\,.$$

Then putting

$$A^{ji}{}_h = M^{ji}{}_h\,,$$

$$C^{k_1\cdots k_p ji}{}_h = M^{k_1\cdots k_p ji}{}_h$$

we can write (8.6) in the form

$$(8.\ 7)\qquad (\rho_h-\rho_i)A^{ji}{}_h+(\rho_h-\rho_j)A^{ij}{}_h=0\,,$$

$$(8.\ 8)\qquad (\rho_h-\rho_i)C^{k_1\dots k_pji}{}_h+(\rho_h-\rho_j)C^{k_1\dots k_pij}{}_h + \sum_{s=1}^{p}(\rho_h-\rho_{k_s})C^{k_1\dots k_{s-1}j\,k_{s+1}\dots k_pik_s}{}_h=0$$

and $C^{k_1\dots k_pji}{}_h$ is symmetric in $k_1,\cdots,k_p,j$.

Now let us use indices as follows for the moment,

$$a,b,c,\cdots=1,\cdots,n-m_\mu\,,$$

$$x,y,z=n-m_\mu+1,\cdots,n\,,$$

and put

$$\lambda_i=\sigma_\mu-\rho_i\,.$$

Then we get

$$\lambda_a\neq 0\,,\quad \lambda_x=0\,.$$

From (8.8) we obtain

$$\lambda_a C^{c_1\dots c_pba}{}_x+\lambda_b C^{c_1\dots c_pab}{}_x+\sum_{s=1}^{p}\lambda_{c_s}C^{c_1\dots c_{s-1}bc_{s+1}\dots c_pac_s}{}_x=0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_a C^{y_1\dots y_rc_{r+1}\dots c_pba}{}_x+\lambda_b C^{y_1\dots y_rc_{r+1}\dots c_pab}{}_x+\sum_{s=r+1}^{p}\lambda_{c_s}C^{y_1\dots y_rc_{r+1}\dots c_{s-1}bc_{s+1}\dots c_pac_s}{}_x=0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_a C^{y_1\dots y_pba}{}_x+\lambda_b C^{y_1\dots y_pab}{}_x=0\,,$$

$$C^{y_1\dots y_{p+1}a}{}_x=0\,.$$

Hence, if $A^{k_1\dots k_p|ji}{}_x$ are symmetric in $k_1,\cdots,k_p$ and satisfy

$$A^{c_1\dots c_p|ba}{}_x=\frac{1}{p+2}\left(C^{c_1\dots c_pba}{}_x-\frac{\lambda_b}{\lambda_a}C^{c_1\dots c_pab}{}_x\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A^{y_1\dots y_rc_{r+1}\dots c_p|ba}{}_x=\frac{1}{p+2-r}\left(C^{y_1\dots y_rc_{r+1}\dots c_pba}{}_x-\frac{\lambda_b}{\lambda_a}C^{y_1\dots y_rc_{r+1}\dots c_pab}{}_x\right),$$

. . . .

$$A^{y_1\cdots y_p|ba}{}_x = \frac{1}{2}\left(C^{y_1\cdots y_pba}{}_x - \frac{\lambda_b}{\lambda_a}C^{y_1\cdots y_pab}{}_x\right),$$

$$A^{k_1\cdots k_p|ya}{}_x = 0\,,$$

$$A^{k_1\cdots k_p|jy}{}_x + \sum_{s=1}^{p} A^{k_1\cdots k_{s-1}jk_{s+1}\cdots k_p|k_sy}{}_x = C^{k_1\cdots k_pjy}{}_x\,,$$

we have

$$A^{k_1\cdots k_p|ji}{}_x + \sum_{s=1}^{p} A^{k_1\cdots k_{s-1}jk_{s+1}\cdots k_p|k_si}{}_x = C^{k_1\cdots k_pji}{}_x$$

and

$$\lambda_i A^{k_1\cdots k_p|ji}{}_x + \lambda_j A^{k_1\cdots k_p|ij}{}_x = 0\,.$$

In such a way we get $A^{k_1\cdots k_p|ji}{}_x$ satisfying

$$A^{k_1\cdots k_p|ji}{}_h + \sum_{s=1}^{p} A^{k_1\cdots k_{s-1}jk_{s+1}\cdots k_p|k_si}{}_h = C^{k_1\cdots k_pji}{}_h\,, \tag{8.9}$$

$$(\rho_h-\rho_i)A^{k_1\cdots k_p|ji}{}_h + (\rho_h-\rho_j)A^{k_1\cdots k_p|ij}{}_h = 0\,. \tag{8.10}$$

(8.10) shows that for each sequence $(k_1,\cdots,k_p)$ of numbers $k_1,\cdots,k_p = 1,\cdots,n$ $A^{k_1\cdots k_p|ji}{}_h$ is a $(2,1)$ tensor satisfying (8.7). On the other hand (8.9) shows that

$$C^{k_1\cdots k_pji}{}_h = \delta^{(k_1}_{l_1}\cdots\delta^{k_p)}_{l_p}A^{l_1\cdots l_p|ji}{}_h + \sum_{s=1}^{p}\delta^{(k_1}_{l_1}\cdots\delta^{k_{s-1}}_{l_{s-1}}\delta^{j}_{l_s}\delta^{k_{s+1}}_{l_{s+1}}\cdots\delta^{k_p)}_{l_p}A^{l_1\cdots l_p|k_si}{}_h\,.$$

Hence (8.1) is an E-simple system.

Thus we have the

THEOREM 8.1. *Let $\varphi(x^1,\cdots,x^n;u^1,\cdots,u^n)$ be a matrix of degree n with diagonal Jordan canonical form and with μ distinct eigenvalues, μ and their multiplicity $m_1,\cdots,m_\mu$ being constant in D. Then any system* (8.1) *of partial differential equations is E-simple if ψ has the form* (8.2).

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