SATURATION IN THE LOCAL APPROXIMATION

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(Received November 29, 1964)

§1. **Introduction**. Let $f(x)$ be integrable in $(-\pi, \pi)$ and periodic with period 2π, and let its Fourier series be

$$(1) \qquad S(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

The Poisson integral of $f(x)$

$$(2) \qquad f(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) P_r(t)\, dt = \sum_{n=0}^{\infty} A_n(x) r^n$$

satisfies the Laplace equation

$$\left(r \frac{\partial}{\partial r}\right)^2 f(r, x) + \frac{\partial^2}{\partial x^2} f(r, x) = 0$$

inside the unit circle. Moreover if $f(x)$ is continuous over (a, b), then $f(r, x)$ tends uniformly to $f(x)$ as $r \to 1$ over (a', b') situated inside (a, b). What can we say about $f(x)$, when the rapidity of approximation of $f(x)$ by $f(r, x)$ is given ? If (a, b) is $(-\pi, \pi)$ and

$$f(r, x) - f(x) = o(1 - r),$$

uniformly if and only if $f(x)$ is a constant and

$$f(r, x) - f(x) = O(1 - r)$$

uniformly if and only if $\widetilde{f}(x)$ satisfies the Lipschitz condition, see G. Sunouchi and C. Watari [5]. This is a saturation theorem.

We shall investigate this problem over an interval (a, b) situated inside $(-\pi, \pi)$ in §2 of this note. In §3 and §4, we consider such problem about

Cesàro means (C, α) and Riesz means (R, n^λ, k) of Fourier series (1). The cases $\alpha=1$ and $\lambda=k=$ an integer have been solved in the previous notes [7]. However the treatment of this paper is simpler and more systematic than the former notes.

For the sake of simplicity, we consider only uniform approximation and norm means uniform norm over (a, b). But another norm may be treated by the same method.

Throughout this paper, (a, b) means an interval situated in $(-\pi, \pi)$ and (a', b') is any interval totally interior in (a, b).

§2. **Poisson integral**. For the local saturation of Poisson integral, we show the following theorem.

THEOREM 1. (1°) *If*

$$\| f(r, x) - f(x) \| = o(1-r), \quad \text{as} \quad r \to 1 \tag{3}$$

over (a, b), *then* $\tilde{f}(x)$ *is a constant over* (a', b'). *Conversely, if* $\tilde{f}(x)$ *is a constant over* (a, b), *then* (3) *holds good over* (a', b').

(2°) *If*

$$\| f(r, x) - f(x) \| = O(1-r) \quad \text{as } r \to 1, \tag{4}$$

over (a, b), *then* $\tilde{f}'(x) \in L^\infty(a', b')$. *Conversely if* $\tilde{f}'(x) \in L^\infty(a, b)$, *then* (4) *holds good over* (a', b').

PROOF. Since

$$(1-r)^{-1}\{f(x) - f(r, x)\} \sim \sum_{n=0}^{\infty} A_n(x)(1-r^n)(1-r)^{-1}$$

and the Fourier coefficients

$$A_n(x)(1-r^n)(1-r)^{-1} \quad (0 < r < 1)$$

are of order $o(n)$. We have, by the localization theorem [9, p. 367], for any $\varepsilon > 0$,

$$\| \sigma_N [(1-r)^{-1}\{f(x) - f(r, x)\}] \| \leqq 2\varepsilon, \text{ for } 1 > r \geqq r_0,\ N \geqq N_0 \text{ over } (a', b'),$$

provided that (3) is true, where $\sigma_N(f)$ denotes the arithmetic means of the Fourier series of $f(x)$. Letting $r \to 1$, we get

$$\| \sigma_N \{\tilde{S}'(f)\} \| \leqq 2\varepsilon,$$

where $\widetilde{S}'(f)$ denotes the derived conjugate series of $f(x)$. Hence, $\sigma_N\{\widetilde{S}'(f)\}$ converges uniformly to zero over (a',b') and we get $\widetilde{f}(x)$ is a constant over (a',b'), see G. Sunouchi [6, Theorem 1].

Conversely, if $f(x)$ is a constant in (a,b), then $\sum kA_k(x)$ is $(C,1)$-summable to zero over (a',b') uniformly by the above cited theorem. So $\sum kA_k(x)$ is Abel summable to zero over (a',b') uniformly. Taylor expansion yields

$$f(r,x)=f(x)+(1-r)f'_r(\xi,x)\,,\quad r<\xi<1$$

where

$$f'_r(\xi,x)=\sum_{k=1}^{\infty}kA_k(x)\,\xi^{k-1}\,,$$

tends to zero uniformly over (a',b') as $\xi\to 1$. Hence

$$f(r,x)-f(x)=o(1-r)\quad\text{as}\quad r\to 1\,.$$

uniformly over (a',b').

(2°) If

$$f(r,x)-f(x)=O(1-r)\quad\text{as}\quad r\to 1$$

uniformly over (a,b), then the same device to the above yields

$$\sigma_N\,\{\widetilde{S}'(f)\}\;=O(1)$$

uniformly over (a',b') and $\widetilde{f}'(x)\in L^\infty(a',b')$.

Conversely, if $\widetilde{f}'(x)\in L^\infty(a,b)$, then

$$\sigma_N\,\{\widetilde{S}'(f)\}\;=O(1)$$

over (a',b') (see G. Sunouchi [6]) and the proof is the same to (1°).

By the same method, we may prove the following more general theorem.

THEOREM 2. (1°) *If*

$$\left\|f(r,x)-f(x)-(1-r)\frac{\partial f}{\partial n}-\cdots-\frac{(1-r)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial n}\right)^{k-1}f\right\| \tag{5}$$

$$=o\left\{\frac{(1-r)^k}{k!}\right\}$$

over (a, b), *then* $\left(\frac{\partial}{\partial n}\right)^k f$ *is zero over* (a', b'), *where*

$$\frac{\partial}{\partial n} f = \left[-\frac{\partial}{\partial r} f(r, x)\right]_{r=1}.$$

Conversely, if $\left(\frac{\partial}{\partial n}\right)^k f$ *is zero over* (a, b), *then* (5) *holds good over* (a', b').

(2°) *If*

$$(6) \qquad \left\| f(r, x) - f(x) - (1-r)\frac{\partial f}{\partial n} - \cdots - \frac{(1-r)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial n}\right)^{k-1} f \right\| = O\left\{\frac{(1-r)^k}{k!}\right\}$$

over (a, b), *then* $\left(\frac{\partial}{\partial n}\right)^k f \in L^\infty(a', b')$. *Conversely if* $\left(\frac{\partial}{\partial n}\right)^k f \in L^\infty(a, b)$, *then* (6) *holds good over* (a', b').

The whole interval case has been investigated by R. Leis [4] and P. L. Butzer and G. Sunouchi [1].

§3. (C, α) **means of Fourier series.** Let

$$A_n^\alpha = \binom{n+\alpha}{n} \cong \frac{n^\alpha}{\Gamma(\alpha+1)} \qquad (\alpha > -1)$$

and denote the (C, α) sums and the (C, α) means of the series (1) by

$$S_n^\alpha(x, f) = S_n^\alpha(x) = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha A_\nu(x),$$

$$\sigma_n^\alpha(x, f) = \sigma_n^\alpha(x) = S_n^\alpha(x)/A_n^\alpha.$$

respectively. In this case

$$1 - \frac{A_{n-\nu}^\alpha}{A_n^\alpha} = 1 - \frac{(n-\nu)^\alpha}{n^\alpha} \cong \alpha \cdot \frac{\nu}{n} \quad \text{as} \quad n \to \infty$$

for fixed ν. Hence we may expect that the order of saturation is n^{-1}. However, if

$$\| \sigma_n^\alpha(x, f) - f \| = o\left(\frac{1}{n}\right)$$

over (a, b), then by the limitation theorem of (C, α)-means, we have $a_n = o(n^{\alpha-1})$. When $0 < \alpha < 1$, no local condition yields $a_n = o(n^{\alpha-1})$. So we have to consider $\alpha \geqq 1$ for local saturation problem.

THEOREM 3. (1°) *If*

$$\| \sigma_n^\alpha(x) - f(x) \| = o(n^{-1}), \quad \alpha \geqq 1, \tag{7}$$

over (a, b), *then* $\tilde{f}(x)$ *is constant over* (a', b'). *Conversely if* $\tilde{f}(x)$ *is constant over* (a, b), *then* (7) *holds good over* (a', b').

(2°) *If*

$$\| \sigma_n^\alpha(x) - f(x) \| = O(n^{-1}), \quad \alpha \geqq 1 \tag{8}$$

over (a, b), *then* $\tilde{f}'(x) \in L^\infty(a', b')$. *Conversely if* $\tilde{f}'(x) \in L^\infty(a, b)$, *then* (8) *holds good over* (a', b').

PROOF. (1°) If (7) holds then

$$\left\| \sum_{\nu=0}^{N-1} \left(1 - \frac{\nu}{N}\right) n \left(1 - \frac{A_{n-\nu}^\alpha}{A_n^\alpha}\right) A_\nu(x) \right\| \leqq \varepsilon$$

over (a', b'), for large n and $N \geqq N_0$. Letting $n \to \infty$, we have

$$\left\| \sum_{\nu=0}^{N-1} \left(1 - \frac{\nu}{N}\right) \nu A_\nu(x) \right\| \leqq \varepsilon/\alpha \quad \text{over} \quad (a', b').$$

Hence $\| \tilde{f}'(x) \| \leqq \varepsilon$, and ε is arbitrary, we get

$$\tilde{f}'(x) = 0 \quad \text{over} \quad (a', b').$$

Conversely $\tilde{f}'(x) = 0$ over (a, b), then the derived conjugate series

$$\sum_{\nu=1}^{\infty} \nu A_\nu(x)$$

is (C, α) summable $(\alpha \geqq 1)$ to zero over (a', b'), by the localization theorem. By a theorem of L. Jésmanowicz [3], we get

$$\| \sigma_n^\alpha(x) - f(x) \| = o(n^{-1})$$

over (a', b').

The proof of part (2°) is almost the same to the case (1°).

§4. Riesz means of Fourier series. When infinite series $\sum a_n$ is given, we set the following definition and notation, see K. Chandrasekharan and S. Minakshisundaram [2].

We set

$$A(t) = \sum_{\lambda_\nu \leqq t} a_\nu \tag{9}$$

$$A^k(t) = \sum_{\lambda_\nu \leqq t} (t-\lambda_\nu)^k a_\nu = k\int_0^t (t-\tau)^{k-1} A(\tau)\, d\tau\,, \quad (k>0) \tag{10}$$

then

$$\frac{d}{dt}[A^k(t)] = kA^{k-1}(t)\,, \quad k>1\,. \tag{11}$$

If

$$C^k(t) = t^{-k} A^k(t)$$

tends to s as $t \to \infty$, then we say that $\sum a_n$ is (R, λ_n, k) summable to s. Further we set

$$b_n = \lambda_n a_n,\ B(t) = \sum_{\lambda_\nu \leqq t} \lambda_\nu a_\nu \tag{12}$$

$$B^k(t) = \sum_{\lambda_\nu \leqq t} (t-\lambda_\nu)^k \lambda_\nu a_\nu \quad (k>0), \tag{13}$$

then we have the relation

$$tA^k(t) - A^{k+1}(t) = B^k(t)\,. \tag{14}$$

For the later use, we need the following lemma.

LEMMA. *If* $k>0$ *and*

$$t^{-k} B^k(t) = o(1) \quad as \quad t\to\infty, \quad then$$
$$C^k(t) - s = o(t^{-1}) \quad as \quad t\to\infty\,,$$

and if hypothesis is $O(1)$, *then conclusion is also* $O(t^{-1})$.

PROOF. From (11) and (14) we have

$$\frac{t}{k+1} \frac{d}{dt}\left\{A^{k+1}(t)\right\} - A^{k+1}(t) = B^{k}(t)\,,$$

and from this

$$A^{k+1}(t) = (k+1)\,t^{k+1}\int_0^t \frac{B^k(x)}{x^{k+2}}\,dx.$$

If we set

$$C^{k+1}(t) = t^{-(k+1)}A^{k+1}(t) \quad\text{and}\quad D^k(t) = t^{-k}\,B^k(t)\,,$$

then

$$C^{k+1}(t) = (k+1)\int_0^t \frac{D^k(t)}{t^2}\,dt\,.$$

The relation

$$C^k(t) - C^{k+1}(t) = \frac{D^k(t)}{t}$$

becomes

$$C^k(t) = (k+1)\int_0^t \frac{D^k(t)}{t^2}\,dt + \frac{D^k(t)}{t}\,.$$

From the hypothesis, we have

$$D^k(t) = o(1)$$

as $t \to \infty$, and

$$\int_t^\infty \frac{D^k(t)}{t^2}\,dt = o\left(\frac{1}{t}\right)$$

as $t \to \infty$. Hence

$$C^k(t) = (k+1)\int_0^\infty \frac{D^k(t)}{t^2}\,dt + \frac{D^k(t)}{t} + o\left(\frac{1}{t}\right).$$

If we put

$$(k+1)\int_0^\infty \frac{D^k(t)}{t^2}\,dt = s,$$

then

$$C^k(t) - s = o\left(\frac{1}{t}\right)$$

as $t \to \infty$.

The $O(1)$ case is proved similarly.

We denote (R, n^λ, k) means of $S(f)$ by

$$R_n^{\lambda,k}(x,f) = \sum_{\nu=0}^{n-1}\left(1-\frac{\nu^\lambda}{n^\lambda}\right)^k A_\nu(x).$$

If we have

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = o\left(\frac{1}{n^\lambda}\right)$$

over (a,b), then by the limitation theorem

$$a_n = o(n^{-\lambda}n^k) = o(n^{k-\lambda}).$$

Hence we have to restrict $k \geqq \lambda$.

THEOREM 4. (1°) *If λ is a positive integer and*

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = o(n^{-\lambda}), \quad k \geqq \lambda \tag{14}$$

over (a,b), then

$$\begin{cases} f(x) \ (\lambda, \textit{even}) \\ \tilde{f}(x) \ (\lambda, \textit{odd}), \end{cases} \tag{15}$$

is at most a $(\lambda-1)$-th algebraic polynomial over (a', b'). Conversely if (15) is true over (a,b), then (14) holds good over (a', b').

(2°) *If λ is a positive integer and*

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = O(n^{-\lambda}), \quad k \geqq \lambda \tag{16}$$

over (a,b), then

$$\begin{cases} f^{(\lambda)}(x) \ (\lambda, \textit{even}) \\ \tilde{f}^{(\lambda)}(x) \ (\lambda, \textit{odd}), \end{cases} \tag{17}$$

belongs to the class L^∞ over (a', b'), and conversely if (17) is true over (a,b), then (16) holds good over (a', b').

PROOF. (1°) We suppose

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = o(n^{-\lambda})$$

over (a, b). The Fourier series of

$$n^{\lambda}\{f(x) - R_n^{\lambda,k}(x,f)\} \sim n^{\lambda}\sum_{\nu=0}^{n-1}\left\{1-\left(1-\frac{\nu^{\lambda}}{n^{\lambda}}\right)^{k}\right\} A_\nu(x) + n^{\lambda}\sum_{\nu=n}^{\infty} A_\nu(x)$$

and its Fourier coefficient is $o(n^{\lambda})$. By the above method

$$\| \sigma_N^{\lambda}[n^{\lambda}\{f(x) - R_n^{\lambda,k}(x,f)\}]\| \leqq 2\varepsilon \quad \text{for} \quad n \geqq n_0, \quad N \geqq N_0,$$

over (a', b'). Letting $n \to \infty$, we have

$$\| \sigma_N^{\lambda}\{S^{(\lambda)}(f)\}\| \leqq 2\varepsilon \quad (N \geqq N_0) \quad (\lambda,\ even)$$

or

$$\| \sigma_N\{\widetilde{S}^{(\lambda)}(f)\}\| \leqq 2\varepsilon \quad (N \geqq N_0) \quad (\lambda,\ odd).$$

Hence we get the conclusion, see Sunouchi [6]. Conversely if we suppose that (15) is true, then by the localization theorem,

$$\sum k^{\lambda} A_k(x) \tag{18}$$

is (C, k)-summable to zero over (a', b') uniformly. From the second theorem of consistency, (18) is (R, n^{λ}, k)-summable to zero over (a', b') uniformly also. Hence by the lemma,

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = o(n^{\lambda})$$

over (a', b').

(2°) This case is proved almost the same. So we omit the detail.

By the same method, we get the following theorem for fractional λ.

THEOREM 5. (1°) *If*

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = o(n^{-\lambda}), \quad k \geqq \lambda \tag{19}$$

over (a, b), *then*

$$\left\| \sum_{\nu=1}^{N-1}\left(1-\frac{\nu}{N}\right)^{\lambda} \nu^{\lambda} A_\nu(x)\right\| = o(1) \tag{20}$$

over (a', b'), *and vice versa.*

(2°) *If*

$$\| R_n^{\lambda,k}(x,f) - f(x)\| = O(n^{-\lambda}), \quad k \geqq \lambda \tag{21}$$

over (a, b), *then*

$$\left\| \sum_{\nu=1}^{N-1}\left(1-\frac{\nu}{N}\right)^{\lambda} \nu^{\lambda} A_\nu(x)\right\| = O(1) \tag{22}$$

over (a', b'), *and vice versa.*

In this case, we can also characterize the class by the original function. For the whole interval case, see [8].

We write, for any positive integer s,

$$g(x,t) = \Delta^{2s} f(x, 2t) = \sum_{j=0}^{2s} (-1)^j \binom{s}{j} f\{x+(s-2j)t\} \tag{23}$$

then

$$g(x,t) = (-1)^s\, 2^{2s} \sum_{\nu=1}^{\infty} A_\nu(x) \sin^{2s} \nu t\,.$$

THEOREM 6. *If*

$$\left\| \sum_{\nu=1}^{n-1}\left\{1-\left(\frac{\nu}{n}\right)^{\lambda}\right\}^{k} \nu^{\lambda} A_\nu(x)\right\| = o(1)\,, \tag{24}$$

over (a, b), *then*

$$\left\| \int_{n^{-1}}^{\infty} \frac{\Delta^{2s} f(x, 2t)}{t^{1+\lambda}}\, dt \right\| = o(1), \quad 0 < \lambda < 2s \tag{25}$$

over (a', b') *and vice versa. If the hypothesis is* $O(1)$, *then the conclusion is also* $O(1)$.

PROOF. If (24) is true, then

$$\| R_n^{\lambda,k}(x,f) - f \| = o(n^{-\lambda}), \quad k \geqq \lambda \tag{26}$$

over (a', b'). On the other hand, by (23),

$$R_n^{\lambda,k}(t, g) = (-1)^s\, 2^{2s} \sum_{\nu=1}^{n-1} \left\{1-\left(\frac{\nu}{N}\right)^{\lambda}\right\}^{k} A_\nu(x) \sin^{2s} \nu t$$

and

$$\int_0^\infty \frac{R_n^{\lambda,k}(t,g)}{t^{1+\lambda}}dt = (-1)^s 2^{2s} \sum_{\nu=1}^{n-1}\left\{1-\left(\frac{\nu}{n}\right)^\lambda\right\}^k A_\nu(x)\int_0^\infty \frac{\sin^{2s}\nu t}{t^{1+\lambda}}dt .$$

However we get

$$(-1)^s 2^{2s}\int_0^\infty \frac{\sin^{2s}\nu t}{t^{1+\lambda}}dt = (-1)^s 2^{2s}\nu^\lambda \int_0^\infty \frac{\sin^{2s} u}{u^{1+\lambda}}du .$$

If $0<\lambda<2s$, the last integral is convergent, and we put this $c_s\nu^\lambda$. Hence

$$\int_0^\infty \frac{R_n^{\lambda,k}(t,g)}{t^{1+\lambda}}dt = c_s \sum_{\nu=1}^{n-1}\left\{1-\left(\frac{\nu}{n}\right)^\lambda\right\}^k \nu^\lambda A_\nu(x)$$

converges uniformly to zero over (a,b).

We write

$$\int_0^\infty \frac{R_n^{\lambda,k}(t,g)}{t^{1+\lambda}}dt - \int_{n^{-1}}^\infty \frac{g(t)}{t^{1+\lambda}}dt$$

$$= \int_0^{n^{-1}} \frac{R_n^{\lambda,k}(t,g)}{t^{1+\lambda}}dt + \int_{n^{-1}}^{\delta} \frac{R_n^{\lambda,k}(t,g)-g(t)}{t^{1+\lambda}} + \int_\delta^\infty \frac{R_n^{\lambda,k}(t,g)-g(t)}{t^{1+\lambda}}dt$$

$$= I_1 + I_2 + I_3 ,$$

say. If we fix a small δ, then

$$R_n^{\lambda,k}(t,g) - g(t) = o(n^{-\lambda}) \quad (k \geqq \lambda)$$

uniformly over $t\in(0,\delta)$ and $x\in(a',b')$. So

$$|I_2| \leqq o(n^{-\lambda})\int_{n^{-1}}^{\delta}\frac{dt}{t^{1+\lambda}} = o(n^{-\lambda}n^\lambda) = o(1) .$$

Since $g(t)$ belongs to the class $L(-\pi,\pi)$, $R_n^{\lambda,k}(t,g)$ converges in mean to $g(t)$ on $(-\pi,\pi)$. Hence,

$$|I_3| \leqq \lim_{n\to\infty}\int_{\delta}^\infty \frac{R_n(t,g)-g(t)}{t^{1+\lambda}}dt = o(1) .$$

On the other hand, by (26), $g(t)$ belongs to the class lip λ. By the definition of $R_n^{\lambda,k}(t,g)$ (see the formula following (26)),

$$\left[\frac{\partial^l}{\partial t^l}R_n^{\lambda,k}(t,g)\right]_{t=0} = 0, \quad l=0,1,2,\cdots,[\lambda], \qquad \lambda < 2s$$

and Taylor expansion yields

$$R_n^{\lambda,k}(t, g) = t^{[\lambda]+1}\left[\frac{\partial^{[\lambda]+1}}{\partial t^{[\lambda]+1}} R_n^{\lambda,k}(t, g)\right]_{t=\theta}, \quad (0 < \theta < t).$$

From the approximation order of $R_n^{\lambda,k}(t, g)$ and the Bernstein theorem

$$\left[\frac{\partial^{[\lambda]+1}}{\partial t^{[\lambda]+1}} R_n^{\lambda,k}(t, g)\right]_{t=\theta} = o\,(n^{[\lambda]+1-\lambda}).$$

Hence

$$\begin{aligned} |I_1| &= \int_0^{n^{-1}} \frac{t^{[\lambda]+1}}{t^{\lambda+1}}\, o(n^{[\lambda]+1-\lambda})\ dt \\ &= o(n^{[\lambda]+1-\lambda}) \int_0^{n^{-1}} t^{-\lambda+[\lambda]}\, dt = o(1). \end{aligned}$$

So

$$\int_{n^{-1}}^{\infty} \frac{g(t)}{t^{1+\lambda}}\, dt = o(1) \tag{27}$$

uniformly over (a', b').

Conversely, we suppose that (27) is true over (a, b). For a fixed n, the N-th (C, λ)-means of Fourier series of the left member of (27) is

$$\sum_{\nu=1}^{N-1}\left(1-\frac{\nu}{N}\right)^{\lambda} A_\nu(x)(-1)^s\, 2^{2s} \int_{n^{-1}}^{\infty} \frac{\sin^{2s} \nu t}{t^{1+\lambda}}\, dt,$$

and this is small for large N over (a', b') uniformly. Letting $n \to \infty$, then

$$\sum_{\nu=1}^{N-1}\left(1-\frac{\nu}{N}\right)^{\lambda} \nu^{\lambda} A_\nu(x)$$

converges to zero uniformly over (a', b'). Thus we can prove the theorem for $o(1)$. Since the case $O(1)$ is treated by the similar method, we omit details.

BIBLIOGRAPHY

[1] P. L. BUTZER and G. SUNOUCHI, Approximation theorems for the solution of Fourier's problem and Dirichlet's problems, Math. Annalen, 155 (1964), 316–330.

[2] K. CHANDRASEKHARAN and S. MINAKSHISUNDARAM, Typical means, Bombay, (1952).
[3] L. JÉSMANOWICZ, On the Cesàro means, Studia Math., 12(1951), 145–158.
[4] R. LEIS, Approximationsätze für Stetige Operatoren, Arch. Math., 14(1963), 120–124.
[5] G. SUNOUCHI and C. WATARI, On determination of the class of saturation in the theory of approximation, I, Proc. Japan Acad., 34(1959), 477–481. II, Tôhoku Math. Journ., 11(1959), 480–488.
[6] G. SUNOUCHI, Local operators on trigonometric series, Trans. Amer. Math. Soc., 104 (1962), 457–461.
[7] G. SUNOUCHI, On the class of saturation in the theory of approximation, II, Tôhoku Math. Journ., 13(1961), 112–118, III, ibid., 320–328.
[8] G. SUNOUCHI, Characterization of certain classes of functions, Tôhoku Math. Journ., 14(1962), 127–134.
[9] A. ZYGMUND, Trigonometric series, I, Cambridge (1956).

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