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SIMPLY INVARIANT SUBSPACES

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Our subject is a theorem on simply invariant subspaces of $L^p_{\mathfrak{h}}$, the usual L^p-space taking values in a Hilbert space $\mathfrak{h}$. Let X be a compact Hausdorff space and A a Dirichlet algebra on X. We shall fix a non-negative finite Borel measure m on X such that

$$f \longrightarrow \int f dm \qquad\qquad (f \in A)$$

defines a multiplicative linear functional on A. Define A_0 to be the set

$$A_0 = \{f \in A; \int f dm = 0\} .$$

Let $\mathfrak{h}$ be a separable Hilbert space and let $L^p_{\mathfrak{h}}$ $(1 \leqq p \leqq \infty)$ denote the space of $\mathfrak{h}$-valued functions on X which are weakly measurable and whose norms are in scalar $L^p(dm)$. $L^2_{\mathfrak{h}}$ is a Hilbert space for the inner product

$$(f, g) = \int (f(x), g(x))_{\mathfrak{h}} dm$$

where the inner product on the right is the one in $\mathfrak{h}$. We define $A_{\mathfrak{h}}$ by $A \otimes_\lambda \mathfrak{h}$, the completion of the algebraic tensor product $A \otimes \mathfrak{h}$ under the uniform norm in $C(X, \mathfrak{h})$ (the space of all $\mathfrak{h}$-valued continuous functions on X). For $1 \leqq p < \infty$ we define $H^p_{\mathfrak{h}}$ by

$$H^p_{\mathfrak{h}} = [A_{\mathfrak{h}}]_p$$

the closure of $A_{\mathfrak{h}}$ in $L^p_{\mathfrak{h}}$ and we define $H^\infty_{\mathfrak{h}}$ by

$$H^\infty_{\mathfrak{h}} = H^1_{\mathfrak{h}} \cap L^\infty_{\mathfrak{h}} .$$

We write H^p instead of $H^p_{\mathfrak{h}}$ in the case of $\mathfrak{h}=\boldsymbol{C}$. Call $\mathfrak{G}$ a range function if $\mathfrak{G}$ is a function on X a.e.(dm) to the family of closed subspaces of $\mathfrak{h}$. Two range functions which agree a.e. are regarded as the same function. $\mathfrak{G}$ is measurable if the orthogonal projection $G(x)$ on $\mathfrak{G}(x)$ is weakly measurable in the operator sense. We shall denote by $\widehat{\mathfrak{G}}$ the operator on $L^p_{\mathfrak{h}}$ defined by $(\widehat{\mathfrak{G}}f)(x) = G(x)f(x)$ *a.e.* Say that a subspace $\mathfrak{M}$ of $L^p_{\mathfrak{h}}$ is doubly invariant if

(i) $\mathfrak{M}$ is closed in $L^p_{\mathfrak{h}}$ if $1 \leqq p < \infty$ and weak*-closed if $p=\infty$.

(ii) $\mathfrak{M}$ is invariant under multiplication by functions in $A+\bar{A}_0$

(where the bar denotes complex conjugation). Say that a subspace $\mathfrak{M}$ of $L^p_{\mathfrak{h}}$ is simply invariant if it satisfies (i) above and

(ii′) $$[\mathfrak{M}A_0]_2 \subsetneqq \mathfrak{M}$$

where $[\quad]_2$ denotes the $L^2_{\mathfrak{h}}$-closure. The purpose of this paper is to prove the following theorem.

THEOREM 1. *The simply invariant subspaces $\mathfrak{M}$ of $L^p_{\mathfrak{h}}(1 \leqq p \leqq \infty)$ are precisely the subspaces of the form*

$$\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1} \oplus \widehat{\mathfrak{G}}L^p_{\mathfrak{h}}$$

where $\mathfrak{G}$ is a measurable range function, and $\boldsymbol{U}$ is a measurable operator function whose values are isometries of an auxiliary Hilbert space $\mathfrak{h}_1$ into $\mathfrak{h}$ with range perpendicular to $\mathfrak{G}$ a.e.

For the circle $|z|=1$, this theorem was proved in Helson [2] for $p=2$. The analogous theorem for doubly invariant subspaces was proved in Srinivasan [3] and Hasumi and Srinivasan [1]. Our discussion was suggested by that of Helson [2]. We first give a proof of the theorem for the case of $p=2$ and for general case apply the interpolation method of Srinivasan and Wang [4].

THEOREM 2. *Every doubly invariant subspace $\mathfrak{M}$ of $L^p_{\mathfrak{h}}(1 \leqq p \leqq \infty)$ is of the form $\widehat{\mathfrak{G}}L^p_{\mathfrak{h}}$ for some measurable range fuaction $\mathfrak{G}$; $\mathfrak{M}$ determines $\mathfrak{G}$ uniquely.*

SKETCH OF THE PROOF FOR THE CASE OF $p=2$. Let $\{e_k\}_{k=1}^{\infty}$ be some fixed c.n.o.s. for $\mathfrak{h}$ and q_k be the projection of the constant function e_k on $\mathfrak{M}$. Each q_k is defined a.e. on X and all q_k's together. Let $\mathfrak{G}(x)$ be the closed linear span of $\{q_k(x)\}_{k=1}^{\infty}$ in $\mathfrak{h}$. Then $\mathfrak{G}(x)$ is defined a.e. We conclude that

(i) $\mathfrak{G}$ is measurable

(ii) $\mathfrak{M}=\{f\in L^2_{\mathfrak{h}}; f(x)\in \mathfrak{G}(x) \text{ a.e.}\}$.

We shall refer to Srinivasan [3] for the details of the proof of Theorem 2.

Let $\mathfrak{M}$ be a closed subspace of $L^2_{\mathfrak{h}}$. The range function $\mathfrak{G}$ associated with the smallest doubly invariant subspace containing $\mathfrak{M}$, we shall call the range function of $\mathfrak{M}$.

PROPOSITION 3. *Let $\mathfrak{M}$ be a closed subspace of $L^2_{\mathfrak{h}}$, and let $\mathfrak{G}$ be the range function of $\mathfrak{M}$, then*

$$\mathfrak{G}(x)\subset [\{f(x); f\in\mathfrak{M}, \|f(x)\|_{\mathfrak{h}}<\infty\}]_{\mathfrak{h}} \quad \text{a.e.}$$

where $[\quad]_{\mathfrak{h}}$ denotes the closed linear span in $\mathfrak{h}$.

PROOF. Let $\mathfrak{M}_{-\infty}$ be the smallest doubly invariant subspace containing $\mathfrak{M}$. Then

$$\mathfrak{M}_{-\infty}=\{f\in L^2_{\mathfrak{h}};\ f(x)\in\mathfrak{G}(x) \quad \text{a.e.}\}$$

by Theorem 2. Now we define $\mathfrak{S}(x)=[\{f(x);\ f\in\mathfrak{M}, \|f(x)\|_{\mathfrak{h}}<\infty\}]_{\mathfrak{h}}$. Clearly $\mathfrak{S}(x)\supset\mathfrak{G}(x)$ a.e. Indeed, there exist $q_k\in\mathfrak{M}_{-\infty}$ such that $\mathfrak{G}(x)=[\{q_k(x)\}_{k=1}^{\infty}]_{\mathfrak{h}}$ a.e. by the construction of $\mathfrak{G}$ (See Srinivasan [3]). Hence

$$\mathfrak{G}(x)\subset[\{f(x); f\in\mathfrak{M}_{-\infty}; \|f(x)\|_{\mathfrak{h}}<\infty\}]_{\mathfrak{h}} \quad \text{a.e.}$$

Since $[(A+\overline{A}_0)\mathfrak{M}]_2=\mathfrak{M}_{-\infty}$, we have

$$[\{f(x); f\in\mathfrak{M}_{-\infty};\ \|f(x)\|_{\mathfrak{h}}<\infty\}]_{\mathfrak{h}}=\mathfrak{S}(x) \quad \text{a.e.}$$

we conclude $\mathfrak{G}(x)\subset\mathfrak{S}(x)$ a.e.

LEMMA 4. *We put $Z(f)=\{x\in X; f(x)=0\}$ and $K=\bigcap_{f\in A_0} Z(f)$, then $m(K)=0$.*

PROOF. Suppose $m(K)>0$. We take a measurable set E such that E contains K and put $\mathfrak{M}=C_E\cdot L^2(dm)$ (where C_E denotes the characteristic function of E), then $\mathfrak{M}$ is a doubly invariant subspace in $L^2(dm)$. Hence $[A_0\mathfrak{M}]_2=\mathfrak{M}$. Thus any $f\in\mathfrak{M}$ vanishes on $E^c\cup K$. We conclude that

$$\mathfrak{M}\subset C_{E\cap K^c}L^2=C_{E-K}L^2\subsetneqq C_EL^2=\mathfrak{M}$$

which is a contradiction.

PROPOSITION 5. *Let* $\mathfrak{M}$ *be a closed subspace of* L^2, *then* $\mathfrak{S}$ *associated with* $\mathfrak{M}$ *in the proof of Proposition* 3 *coincides with that of* $[A_0\mathfrak{M}]_2$ a.e.

PROOF. The assertion follows from Lemma 4.

PROOF OF THEOREM 1 (the case of $p=2$). Let $\mathfrak{M}_\infty$ be the largest doubly invariant subspace which is contained in $\mathfrak{M}$ and let $\mathfrak{M}_{-\infty}$ be the smallest doubly invariant subspace containing $\mathfrak{M}$. Clearly $L^2_{\mathfrak{h}} \supset \mathfrak{M}_{-\infty} \supsetneqq \mathfrak{M} \supsetneqq \mathfrak{M}_\infty \supset \{0\}$. We put $\mathfrak{N}=\mathfrak{M}\ominus\mathfrak{M}_\infty$.

(i) Since $\mathfrak{M}$ is simply invariant, it is easy to see that $\mathfrak{N}$ is simply invariant.

(ii) From the maximality $\mathfrak{M}_\infty$, it follows $\mathfrak{N}_\infty=\{0\}$.

(iii) By Theorem 2, $\mathfrak{M}_\infty=\widehat{\mathfrak{G}}L^2_{\mathfrak{h}}$ for some measurable range function $\mathfrak{G}$.

(vi) If $f\in\mathfrak{N}$, $g\in\mathfrak{M}_\infty$, then $f\perp\xi g$ for all $\xi\in A+\overline{A}_0$.
Hence

$$\int (f(x), g(x))_{\mathfrak{h}}\overline{\xi}(x)dm(x) = 0 \qquad (\forall\xi\in A+\overline{A}_0)$$

and so $(f(x), g(x))=0$ a.e. on X. We have $f(x)\perp\mathfrak{G}(x)$ a.e. and the range of $\mathfrak{N}$ is perpendicular to $\mathfrak{G}$ a.e.

(v) Let $\mathfrak{N}\ominus[A\mathfrak{N}]_2=R_0$ By the invariance of $\mathfrak{N}$ and the closedness of $\mathfrak{N}$, $[AR_0]_2\subset\mathfrak{N}$. Let $g\in\mathfrak{N}\ominus[AR_0]_2$. Then

$$0=\int (g, \xi q)dm=\int\overline{\xi}(g,q)dm \qquad (\forall\xi\in A,\ q\in R_0).$$

Also since $A_0g\subset[A_0\mathfrak{N}]_2\perp R_0$, we have

$$0=\int (\eta g, q)dm=\int \eta(g, q)dm \qquad (\forall\eta\in A_0,\ q\in R_0).$$

So

$$0=\int \xi(g, q)dm \qquad (\forall\xi\in A_0+\overline{A},\ q\in R_0),$$

and $(g(x), q(x))=0$ a.e. on X for any $q\in R_0$. We conclude that $g(x)$ is orthogonal to the range function of R_0 a.e. Now the range function of $R_0=\mathfrak{N}\ominus[A_0\mathfrak{N}]_2$ coincides with that of $\mathfrak{N}$. Indeed $(R_0)_{-\infty}=\mathfrak{N}_{-\infty}\ominus([A_0\mathfrak{N}]_2)_\infty$ and $\mathfrak{N}_\infty=\{0\}$ by (ii). Hence $g(x)$ is orthogonal a.e. to the range function of $\mathfrak{N}$. But $g\in\mathfrak{N}$, we have $g=0$ a.e. It follows that $\mathfrak{N}=[AR_0]_2$.

(vi) If $u,v\in R_0$ and $\int (u, v)dm=c$, then $(u(x), v(x))=c$ a.e. Indeed since $R_0=\mathfrak{N}\ominus[A_0\mathfrak{N}]_2$,

$$\int \xi(u, v)dm = 0 \qquad (\forall\xi\in A_0)$$

Let $f\in A$, then $f-\int fdm\in A_0$, and by the above formula,

$$\int f\cdot(u, v)dm=c\cdot\int fdm.$$

Hence $\int f\{(u, v)-c\}dm=0$ for all $f\in A$. Similarly we have $\int \overline{\eta}\{(u,v)-c\}dm=0$ for all $\eta\in A_0$. Thus

$$\int f\cdot\{(u, v)-c\}dm=0 \qquad (f\in A+\overline{A}_0).$$

We conclude that $(u(v),v(x))=c$ a.e.

(vii) Now we regard R_0 as a Hilbert space and denote it by $\mathfrak{h}_1$, abstractly. Let U the operator which maps u of $\mathfrak{h}_1$ to u of R_0 by considering u as an element of R_0. (Essentially U is the identity operator.) Extend U to an operator of $L^2(dm)\otimes\mathfrak{h}_1$ by setting

$$U\left(\sum_{j=1}^{N} f_j\otimes u_j\right)(x)=\sum_{j=1}^{N} f_j(x)u_j(x).$$

The extended operator U is an isometry of $L^2\otimes\mathfrak{h}_1$ into $L^2_{\mathfrak{h}}$. Indeed in the expression of $\sum_{j=1}^{N} f_j\otimes u_j$ we may consider that $(u_i, u_j)=\delta_{ij}$ by the definition of tensor products. Thus by (vi) we have

$$\left\|\sum_{j=1}^{N} f_j\otimes u_j\right\|^2_{L\mathfrak{h}_1}=\sum_{j=1}^{N}\int |f_j|^2(u_j, u_j)_{\mathfrak{h}_1}dm=\sum_{j=1}^{N}\int |f_j|^2dm$$

$$=\sum_{i,j=1}^{N}\int f_j(x)\bar{f}_i(x)(u_j(x), u_i(x))_{\mathfrak{h}}dm=\int\left\|\sum_{j=1}^{N} f_j(x)u_j(x)\right\|_{\mathfrak{h}} dm$$

$$=\left\|\sum_{j=1}^{N} f_j(x)u_j(x)\right\|_{L^2_{\mathfrak{h}}}.$$

Hence U has a unique extension to an isometry of $L^2{}_{\mathfrak{h}_1}$ into $L^2{}_{\mathfrak{h}}$. We also denote this extended isometry by U.

(viii) $UH^2_{\mathfrak{h}_1}=[AR_0]_2=\mathfrak{N}$. Because if $A\otimes\mathfrak{h}_1\ni f=\sum_{j=1}^{N} f_j\otimes u_j$, then by the definition of U

$$U(f)(x)=\sum_{j=1}^{N} f_j(x)u_j(x)\in[AR_0]_2.$$

Therefore $UH^2_{\mathfrak{h}_1}\subset[AR_0]_2$. On the other hand, for $h=Fg\in AR_0(F\in A, g\in R_0)$, we put $f=F\otimes g$, then $f\in H^2_{\mathfrak{h}_1}$ and $U(f)=h$. Hence $[AR_0]_2\subset UH^2_{\mathfrak{h}_1}$.

(ix) For $x\in X$, we define an operator $\boldsymbol{U}(x)$ of $\mathfrak{h}_1$ into $\mathfrak{h}$ by $\boldsymbol{U}(x)u=u(x)$ for $u\in\mathfrak{h}_1=R_0\subset L^2{}_{\mathfrak{h}}$. It is easy to see that for almost all $x\in X$, this operator $\boldsymbol{U}(x)$ is measurable and isometric. Now we have that for all $F\in L^2{}_{\mathfrak{h}_1}$,

$$(UF)(x)=\boldsymbol{U}(x)\ \ F(x).$$

Indeed this holds for constant functions by definition, and for $F\in(A+\overline{A_0})\otimes\mathfrak{h}_1$ because the construction of U. Finally the formula holds on all of $L^2{}_{\mathfrak{h}_1}$ by continuity. This clompletes the proof for the case of $p=2$.

LEMMA 6. *Let* $1\leqq p<2$ *and* $1/r+1/2=1/p$. *If* $f\in L^p_{\mathfrak{h}}$ and $f\notin[A_0f]_p$, *then* $f=Fh$ *where* $h\in H^2$ *is outer*(*) *and* $F\in[fA]_p\cap L^r_{\mathfrak{h}}$.

PROOF. We put that

$$f_1(x)=\|f(x)\|^{p/2}_{\mathfrak{h}}$$

$$f_2(x)=\begin{cases}0 & \text{if } f_1(x)=0\\ \dfrac{f(x)}{f_1(x)} & \text{if } f_1(x)\neq0\end{cases}$$

Then $f_1\in L^2, f_2\in L^r_{\mathfrak{h}}, f=f_1f_2$ and $f_1\notin[f_1A_0]_2$. Hence by the factorization Lemma of the scalar case, we have $f_1=qh$ where $q\in[f_1A]_2$ is unitary and $h\in H^2$ is outer. Define $F=qf_2$, then $F\in L^r_{\mathfrak{h}}$ and $F\in[fA]_p$. (See[4]).

Let $\{e_n\}_{n=1}^{\infty}$ be some fixed c.n.o.s. for $\mathfrak{h}$. We define $f=\sum_{n=1}^{\infty} f_n\otimes e_n$ by $f(x)$

(*) A function $h\in H^2$ is said to be outer if $[hA]_2=H^2$. For the details of the scalar case, see Srinivasan and Wang [4].

$= \sum_{n=1}^{\infty} f_n(x)e_n$ in the algebraic sense.

LEMMA 7. *Let* $1 \leqq p \leqq \infty$.

(i) *If* $f \in L^p_{\mathfrak{h}}$, *then* $f = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in L^p$

(ii) *If* $f \in A_{\mathfrak{h}}$, *then* $f = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in A$

(iii) *If* $f \in H^p_{\mathfrak{h}}$, *then* $f = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in H^p$, *in particular, if* $f = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in H^2$ *and* $\sum_{n=1}^{\infty} \int |f_n|^2 dm < \infty$, *then* $f \in H^2_{\mathfrak{h}}$.

PROOF. (i) is trivial. We shall prove (ii). If $g \in A \otimes \mathfrak{h}$, then $g = \sum_{j=1}^{N} f'_j \otimes u_j$ ($f'_j \in A$, $u_j \in \mathfrak{h}$ ($j=1, 2, \cdots, N$)). If we express u_j as $u_j = \sum_{n=1}^{\infty} \alpha_n^{(j)} e_n$, then

$$g(x) = \sum_{j=1}^{N} f'_j(x) \sum_{n=1}^{\infty} \alpha_n^{(i)} e_n = \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{N} \alpha_n^{(j)} f'_j(x) \right\} e_n.$$

Since $f_n = \sum_{j=1}^{N} \alpha_n^{(j)} f'_j \in A$, g has the expression $g = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in A$. Now for $f \in A_{\mathfrak{h}}$, there exist $g_i = \sum_{n=1}^{\infty} g_n^{(j)} \otimes e_n \in A \otimes \mathfrak{h}$ such that $g_i \to f$(unif.). If we put $f = \sum_{n=1}^{\infty} f_n \otimes e_n$, $f_n \in L^2$ then

$$\|f(x) - g_i(x)\|^2_{\mathfrak{h}} = \sum_{n=1}^{\infty} |f_n(x) - g_n^{(i)}(x)|^2 \geqq |f_n(x) - g_n^{(i)}(x)|^2 \quad (n=1, 2, \cdots)$$

It follows that $f_n \in A$. The proof of (iii) is similar and the last assertion follows from Lemma 8.

LEMMA 8. *Let* $1 \leqq p < \infty$. $H^p_{\mathfrak{h}} = [H^p \otimes \mathfrak{h}]_p$.

PROOF. $H^p_{\mathfrak{h}} \subset [H^p \otimes \mathfrak{h}]_p$ is clear. Conversely, if $f \in H^p \otimes \mathfrak{h}$, $f = \sum_{j=1}^{N} f_j \otimes u_j$ then for any $\varepsilon > 0$, there exists $g_j \in A$ such that $\|f_j - g_j\|_p < \varepsilon$. We have that

$g_j\otimes u_j \in A^p_{\mathfrak{h}}$ and $\|g_j\otimes u_j - f_j\otimes u_j\|_p < \varepsilon\|u_j\|$ $(j=1, 2, \cdots, N)$. Therefore $f_j\otimes u_j \in [A_{\mathfrak{h}}]_p (j=1, 2, \cdots, N)$. Hence $\sum_{j=1}^{N} f_j\otimes u_j \subset [A\ ]_p$ and $H^p\otimes\mathfrak{h}\subset H^p_{\mathfrak{h}}$. Thus $[H^p\otimes\mathfrak{h}]_p \subset H^p_{\mathfrak{h}}$.

LEMMA 9. *Let* $1\leqq p\leqq\infty$. *Then*

$$H^p_{\mathfrak{h}} = \{f\in L^p_{\mathfrak{h}};\ \int (f, \bar{g})dm=0\ (\forall g\in A_{\mathfrak{h},0})\},$$

where $A_{\mathfrak{h},0}$ *is defined by* $A_0\otimes_\lambda\mathfrak{h}$.

PROOF. Let $f\in A_{\mathfrak{h}}, f=\sum_{n=1}^{\infty} f_n\otimes e_n (f_n\in A, n=1, 2, \cdots)$ and let $g\in A_{\mathfrak{h},0}$ $g=\sum_{n=1}^{\infty} g_n\otimes e_n, (g_n\in A_0; n=1, 2, \cdots)$. Then we have

$$\int (f, \bar{g})dm = \sum_{n=1}^{\infty}\int f_n g_n dm = \sum_{n=1}^{\infty}\int f_n dm \int g_n dm = 0\,.$$

From this, it is easy to see that $\int (f, \bar{g})dm=0$ for $f\in H^p_{\mathfrak{h}}$. Let $p=2$. We take $f\in L^2_{\mathfrak{h}}$ such that $\int (f, \bar{g})dm=0$ for all $g\in A_{\mathfrak{h},0}$. We put $f=\sum_{n=1}^{\infty} f_n\otimes e_n$, $f_n\in L^2$, then we have $\sum_{n=1}^{\infty}\int |f_n|^2 dm = \int \|f\|^2_{\mathfrak{h}} dm < \infty$. Since $\xi\otimes e_n\in A_{\mathfrak{h},0}$ for all $\xi\in A_0$,

$$0=\int (f, \bar{\xi}\otimes e_n)dm = \int f_n\xi dm (n=1, 2, \cdots)\,.$$

Hence $f_n\in H^2$ and by Lemma 7 (iii), $f\in H^2_{\mathfrak{h}}$. Next let $p=1$. Take $f\in L^1_{\mathfrak{h}}$ such that $\int (f, \bar{g})dm=0$ for all $g\in A_{\mathfrak{h},0}$. We may assume that $f\notin [fA_0]_1$. From Lemma 6, it follows that $f=Fh$ where $F\in [fA]_1\cap L^2_{\mathfrak{h}}$ and $h\in H^2$ is outer. There exist $\xi_\alpha\in A$ such that $\xi_\alpha f\to F$ in $L^1_{\mathfrak{h}}$. Therefore for all $g\in A_{\mathfrak{h},0}$, we have

$$\int (\xi_\alpha f, \bar{g})dm = \int (f, g\bar{\xi}_\alpha)dm=0.$$

Hence $\int (F, \bar{g})dm=0$ $(\forall g\in A_{\mathfrak{h},0})$. By the case of $p=2$, it follows that $F\in H^2_{\mathfrak{h}}$.

Now,

$$f=Fh\in H^2_{\mathfrak{h}}\cdot H^2\subset H^1_{\mathfrak{h}}.$$

The case of $p=\infty$ follows immediately from the definition of $H^\infty_{\mathfrak{h}}$ and the above case. For the other case we shall show $H^p_{\mathfrak{h}}=H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$, then the proof will be complete. Let $1<p<2$. For $f\in H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$, we may assume $f\notin[fA_0]_p$ and by Lemma 6, one have $f=Fh$ where $F\in[fA]_p\cap L^r_{\mathfrak{h}}$ and $h\in H^2$ is outer. Since $r>2$, $F\in L^2_{\mathfrak{h}}$ and since $f\in H^1_{\mathfrak{h}}$, $F\in[fA]_p\subset H^1_{\mathfrak{h}}$. Therefore $F\in H^1_{\mathfrak{h}}\cap L^2_{\mathfrak{h}}=H^2_{\mathfrak{h}}\subset H^p_{\mathfrak{h}}$ ($p<2$!). Hence $f=Fh\in FH^2=F[A]_2\subset[FA]_p\subset H^p_{\mathfrak{h}}$. Thus $H^p_{\mathfrak{h}}\supset H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$. The converse is trivial. Let $2<p<\infty$. We put $1/p+1/q=1$. In this case again $H^p_{\mathfrak{h}}\subset H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$ is clear, and suffices to show that if $H^p_{\mathfrak{h}}\perp g\in L^q_{\mathfrak{h}}$, then $g\perp H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$. By the case of $p=1$, it follows that $\bar{g}\in H^1_{\mathfrak{h},0}$ where $H^1_{\mathfrak{h},0}$ is the $L^1_{\mathfrak{h}}$-closure of $A_{\mathfrak{h},0}$. As $1<q<2$, by the above case, $\bar{g}\in H^1_{\mathfrak{h},0}\cap L^q_{\mathfrak{h}}=H^q_{\mathfrak{h},0}$. So there exist $g_n\in A_{\mathfrak{h}0}$, such that $g_n\to\bar{g}$ in $L^q_{\mathfrak{h}}$. Hence

$$0=\int(h,\bar{g}_n)dm\to\int(h,g)dm$$

for all $h\in H^1_{\mathfrak{h}}\cap L^p_{\mathfrak{h}}$. So the proof is completed.

PROOF OF THEOREM 1 (the case of $1\leqq p<2$). Put $\mathfrak{N}=L^2_{\mathfrak{h}}\cap\mathfrak{M}$. It is clear that $\mathfrak{N}$ is $L^2_{\mathfrak{h}}$-closed subspace and $[A_0\mathfrak{N}]_2\subset\mathfrak{N}$. We wish to show that $\mathfrak{N}$ is simply invariant. As $\mathfrak{M}$ is simply invariant, there exists an $f\neq 0$ shch that $f\in\mathfrak{M}-[A_0\mathfrak{M}]_p$. So $f\notin[fA_0]_p$, and by lemma 6, $f=Fh$ where $h\in H^2$ is outer and $F\in[fA]_p\cap L^r_{\mathfrak{h}}\subset\mathfrak{M}\cap L^2_{\mathfrak{h}}=\mathfrak{N}$. Also $F\notin[\mathfrak{N}A_0]_2$, since $f\notin[\mathfrak{M}A_0]_p$. Thus $\mathfrak{N}$ is simply invariant and by the case of $p=2$, we have

$$\mathfrak{N}=\boldsymbol{U}\cdot H^2_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{S}}L^2_{\mathfrak{h}}.$$

Now $\mathfrak{M}\supset\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{S}}L^p_{\mathfrak{h}}$ is trivial. To see the reverse inclusion, let $f\in\mathfrak{M}-[\mathfrak{M}A_0]_p$, $f\neq 0$. Then already we have $f=Fh$ where $h\in H^2$ is outer and $F\in[fA]_p\cap L^r_{\mathfrak{h}}$. It follows that

$$f=Fh\in F[A]_2\subset[FA]_p\subset[\mathfrak{H}A]_p\subset[\mathfrak{N}]_p=\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{S}}L^p_{\mathfrak{h}}.$$

Thus $\mathfrak{M}-[\mathfrak{M}A_0]_p\subset\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{S}}L^p_{\mathfrak{h}}$. The algebraic sum

$$\{\mathfrak{M}-\mathfrak{M}A_0]_p\}+[\mathfrak{M}A_0]_p\subset\mathfrak{M}-[\mathfrak{M}A_0]_p$$

shows that $[\mathfrak{M}A_0]_p\subset\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\mathfrak{H}L^p_{\mathfrak{h}}$. We get that

$$\mathfrak{M}=\{\mathfrak{M}-[\mathfrak{M}A_0]_p\}\cup[\mathfrak{M}A_0]_p\subset \boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\mathfrak{G}L^p_{\mathfrak{h}}.$$

(the case of $2<p\leqq\infty$) Put $1/p+1/q=1$. We define $\mathfrak{N}$ by $[\mathfrak{M}A_0]^{\perp}_p=\{f\in L^q_{\mathfrak{h}};\int(f,\bar{g})\,dm=0,(\forall g\in[\mathfrak{M}A_0]_p)\}$, then it is easy to check that $\mathfrak{N}$ is a simply invariant subspace of $L^q_{\mathfrak{h}}$. By the case of $1\leqq p<2$, we have

$$\mathfrak{N}=\boldsymbol{U}\cdot H^q_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{G}'}L^q_{\mathfrak{h}}.$$

So $[A_0\mathfrak{M}]_p=\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1,0}\oplus\widehat{\mathfrak{G}}L^p_{\mathfrak{h}}$, and $\mathfrak{M}\supset\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}\oplus\widehat{\mathfrak{G}}L^p_{\mathfrak{h}}$.
Now for $f\in\mathfrak{M}$, put

$$F_1=\widehat{\mathfrak{G}}^{\perp}f,\qquad F_2=\widehat{\mathfrak{G}}f.$$

We shall show that $F_1\in\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}$. For $f=F_1+F_2$, we have $\xi f=\xi F_1+\xi F_2$ and $\xi f\in[\mathfrak{M}A_0]_p$ for all $\xi\in A_0$. But $\xi F_2\in\widehat{\mathfrak{G}}L^p_{\mathfrak{h}}$, so $\xi F_1\in\boldsymbol{U}\cdot H^p_{\mathfrak{h},0}$. Let $\Theta=\boldsymbol{U}^*F_1$. For fixed $g\in A_{\mathfrak{h}_1,0}$,

$$\int\xi(\Theta,\bar{g})dm=\int(\boldsymbol{U}^*\xi F_1,\bar{g})dm=0\qquad(\forall\xi\in A).$$

Because, for $g=\sum_{j=1}^{N}g_j\otimes u_j\in A_0\otimes\mathfrak{h}_1$, we get

$$\int(\boldsymbol{U}^*\xi F_1,\bar{g})dm=\sum_{j=1}^{N}\int(g_j\xi F_1,\boldsymbol{U}u_j)dm=0$$

by Lemma 9. We conclude that for each $g\in A_{\mathfrak{h}_1,0}$, $(\Theta,\bar{g})\in H^p_0(dm)$ as a scalar function. Thus

$$\int(\Theta,\bar{g})dm=0\qquad(\forall g\in A_{\mathfrak{h}_1,0}).$$

Hence $\Theta\in H^p_{\mathfrak{h}_1}$, so $\boldsymbol{U}\boldsymbol{U}^*F_1\in\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}$. Since $F_1(x)$ is contained in the range of $\boldsymbol{U}(x)$, $\boldsymbol{U}\boldsymbol{U}^*F_1=F_1$ and $F_1\in\boldsymbol{U}\cdot H^p_{\mathfrak{h}_1}$.

The following theorem is a generalization of Theorem 6 of Srinivasan [3] for a general Dirichlet algebra.

THEOREM 10. *A measurable range function $\mathfrak{G}$ is of constant dimension a.e. if and only if it is the range function of a simply invariant subspace $\mathfrak{M}$ such that $\mathfrak{M}_\infty=\{0\}$.*

PROOF. The sufficiency follows from Theorem 1. We shall show the necessity. Since $\mathfrak{G}$ is of constant dimension, there exist $q_k \in L^2_{\mathfrak{h}}(k=1, 2, \cdots)$ such that $\{q_k(x)\}$ is a c.n.o.s. of $\mathfrak{G}(x)$ a.e. (Srinivasan [3], Theorem 5). We put $\mathfrak{M}=[\{Aq_k; k=1, 2, \cdots\}]_2$ and let $f \in \mathfrak{M}$. Then f has the expression

$$f=\sum_{k=1}^{\infty} f_k q_k,\ f_k \in H^2,\ \sum_{k=1}^{\infty}\int |f_k|^2 dm < \infty$$

Now $f=\sum_{k=1}^{\infty} f_k C_{E_k} \otimes e_k$. For $n=1, 2, \cdots, e_n - q_n \perp [\{(A+\bar{A})q_k\}_{k=1}^{\infty}]_2 \supset \mathfrak{M}$ by the construction of q_k (see [3]). So for all $g \in A_0$,

$$0=\int (f, \bar{g}(e_n-q_n))dm=\int f_n C_{E_n} g dm - \int f_n g dm$$

$$=\int f_n C_{E_n} g dm - \int f_n dm \int g dm = \int f_n C_{E_n} g dm\,.$$

Thus $\int f_n C_{E_n} g dm = 0$ for all $g \in A_0$ and $n=1, 2, \cdots,$ and so $f_n C_{E_n} \in H^2$. Of course, $\sum_{n=1}^{\infty} \int |f_n C_{E_n}|^2 dm < \infty$, and $f \in H^2_{\mathfrak{h}}$ by Lemma 7. Therefore $\mathfrak{M} \subset H^2_{\mathfrak{h}}$ and $\mathfrak{M}_\infty = \{0\}$.

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