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ON C-HARMONIC FORMS IN A COMPACT SASAKIAN SPACE

YÔSUKE OGAWA

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Introduction. It is well known that in a $2m$-dimensional compact Kählerian space, any harmonic p-form $(p \leqq m)$ can be written uniquely in terms of effective harmonic forms and the fundamental 2-form of the space. When we consider the analogy in a compact Sasakian space, it is insignificant as far as we are concerned about harmonic forms, because any harmonic form is effective. S. Tachibana [1] has introduced the notion of C-harmonic forms in a compact Sasakian space, which is wider than that of harmonic forms, and succeeded to prove the analogy of the decomposition theorem for C-harmonic forms. In this paper we try to make the definition of C-harmonic forms a little looser than that of Tachibana's original one. On the other hand, S. Tanno has drawn the relation of Betti numbers between the base space and the bundle space in the fibering of a regular K-contact Riemannian space. It is shown that a p-form $(p \leqq m)$ on the bundle space is C-harmonic if and only if it is induced from a harmonic p-form on the base space. Thus we can obtain the theorem of Tanno again. Lastly we investigate the C^*-harmonic forms which are dual to the C-harmonic forms, and in connection with them, we observe Killing forms and give one of its example on a Sasakian space.

Manifolds are assumed to be connected and the differentiable structures on them are assumed to be of class C^∞.

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Contentes are as follows:

1. Preliminaries. An n-dimensional Riemannian space M^n is called a Sasakian space if it admits a unit Killing vector field η^λ such that

$$\nabla_\lambda \nabla_\mu \eta_\nu = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}, \tag{1.1}$$

where $g_{\lambda\mu}$ is the metric tensor of M^n. Then n is necessarily odd $(=2m+1)$ and M^n is orientable. With respect to a local coordinates system $\{x^\lambda\}$, $\lambda=1, \cdots, n$, if we define a 2-form $\varphi=(1/2)\varphi_{\lambda\mu}dx^\lambda\wedge dx^\mu$ by

$$\varphi_{\lambda\mu}=\nabla_\lambda\eta_\mu,$$

then we have $d\eta=2\varphi$ and it holds

$$\nabla_\lambda\varphi_{\mu\nu}=\eta_\mu g_{\lambda\nu}-\eta_\nu g_{\lambda\mu}. \tag{1.2}$$

On a Sasakian space, the following identities are well known (cf. [2]):

$$\nabla^\lambda\varphi_{\lambda\mu}=-(n-1)\eta_\mu, \tag{1.3}$$

$$R_{\lambda\mu\nu\omega}\eta^\omega=\eta_\lambda g_{\mu\nu}-\eta_\mu g_{\lambda\nu}, \tag{1.4}$$

$$\varphi_\lambda{}^\varepsilon R_{\varepsilon\mu\nu\omega}=\varphi_\mu{}^\varepsilon R_{\varepsilon\lambda\nu\omega}+\varphi_{\nu\lambda}g_{\omega\mu}-\varphi_{\nu\mu}g_{\omega\lambda}+\varphi_{\omega\mu}g_{\nu\lambda}-\varphi_{\omega\lambda}g_{\nu\mu}, \tag{1.5}$$

$$R_{\lambda\mu}{}^{\rho\sigma}\varphi_{\rho\alpha}\varphi_{\sigma\beta}=R_{\lambda\mu\alpha\beta}+\varphi_{\lambda\beta}\varphi_{\mu\alpha}-\varphi_{\lambda\alpha}\varphi_{\mu\beta}+g_{\lambda\alpha}g_{\mu\beta}-g_{\lambda\beta}g_{\mu\alpha}, \tag{1.6}$$

$$(1/2)\varphi^{\alpha\beta}R_{\alpha\beta\lambda\mu}=R_{\lambda\varepsilon}\varphi_\mu{}^\varepsilon+(n-2)\varphi_{\lambda\mu}, \tag{1.7}$$

$$R_{\mu\varepsilon}\varphi_\lambda{}^\varepsilon=-R_{\lambda\varepsilon}\varphi_\mu{}^\varepsilon,\quad R_\mu{}^\varepsilon\varphi_\varepsilon{}^\lambda=R_\varepsilon{}^\lambda\varphi_\mu{}^\varepsilon. \tag{1.8}$$

In the following, we consider always an $n(=2m+1)$-dimensional Sasakian space M^n. If M^n is compact, then we denote the global inner product of any p-forms u and v by

$$(u, v)=\int_M(u\wedge *v),$$

where $*v$ is a dual form of v. The dual operator $*$ satisfies the relation

$$**u=u \tag{1.9}$$

for any p-form u. The adjoint operator δ of d is given by

$$\delta u=(-1)^p *d*u \tag{1.10}$$

for a p-form u.

Next we define the operators $e(\eta)$, $i(\eta)$ and L, Λ for any p-form u as follows:

$$e(\eta)u = \eta \wedge u, \quad i(\eta)u = (-1)^{p-1} *e(\eta)*u, \tag{1.11}$$

$$Lu = d\eta \wedge u, \quad \Lambda u = i(d\eta)u = *L*u, \tag{1.12}$$

and for 0-form a and 1-form b we define

$$i(\eta)a = 0, \quad \Lambda a = \Lambda b = 0.$$

Then we have for any forms u and v

$$d\eta \wedge u_{p-2} \wedge *v_p = u_{p-2} \wedge *(*L*v_p),$$

$$\eta \wedge u_{p-1} \wedge *v_p = (-1)^{p-1} u_{p-1} \wedge *(*e(\eta)*v_p),$$

where the subscripts of u and v denote the degree of them. Therefore if M^n is compact, then $e(\eta)$ (resp. L) and $i(\eta)$ (resp. Λ) are adjoint operators with respect to the global inner product. We shall call a form u to be effective if it satisfies $\Lambda u = 0$.

LEMMA 1.1. *In a Sasakian space, the following relations hold* (*cf.* [9]):

$$L = e(\eta)d + de(\eta), \tag{1.13}$$

$$\Lambda = i(\eta)\delta + \delta i(\eta). \tag{1.14}$$

PROOF. For any p-form u, we have

$$Lu = d\eta \wedge u = d(\eta \wedge u) - (-\eta \wedge du)$$
$$= de(\eta)u + e(\eta)du.$$

(1.14) is obtained easily from (1.10, 11, 12).

LEMMA 1.2. *In a Sasakian space, the operator* L (resp. Λ) *commutes with the operators* $i(\eta)$, $e(\eta)$ *and* d (*resp.* $i(\eta)$, $e(\eta)$ *and* δ). (*cf.* [9]).

PROOF. As $i(\eta)d\eta = 0$, we have for any form u

$$i(\eta)(d\eta \wedge u) = (i(\eta)d\eta) \wedge u + d\eta \wedge (i(\eta)u)$$
$$= Li(\eta)u.$$

From (1.11) and (1.12), we obtain $\Lambda e(\eta) = e(\eta)\Lambda$. The other relations are obtained by virtue of Lemma 1.1 and

$$i(\eta)^2 = e(\eta)^2 = d^2 = \delta^2 = 0.$$

We denote the Lie derivative with respect to η^λ by $\theta(\eta)$. It is well known that it holds

$$\theta(\eta) = di(\eta) + i(\eta)\,d\,.$$

LEMMA 1.3. *In a Sasakian space, we have* (*cf.* [9])

$$*\theta(\eta) = \theta(\eta)*\,. \tag{1.15}$$

or equivalently

$$\theta(\eta) = -\delta e(\eta) - e(\eta)\delta\,. \tag{1.16}$$

PROOF. Since η is a Killing form, it satisfies $\theta(\eta)g=0$, $\delta\eta=0$, where g is the metric tensor of M^n. Then it follows that for a p-form u of coefficients $u_{\lambda_1\cdots\lambda_p}$

$$\begin{aligned}((\theta(\eta) - *\theta(\eta)*)u)_{\lambda_1\cdots\lambda_p} &= \delta\eta\, u_{\lambda_1\cdots\lambda_p} + (p\,!)\sum_{i=1}^{p} g^{\rho\sigma}(\theta(\eta)g)_{\sigma\lambda_i}\, u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p} \\ &= 0\,.\end{aligned}$$

Therefore we have $\theta(\eta) = *\theta(\eta)*$. While from (1.10) and (1.11) it is shown that $\delta e(\eta)+e(\eta)\delta = -*\theta(\eta)*$, hence we have (1.16).

Next we introduce some operators on the graded algebra of differentiable forms on a Sasakian space. Let u be a p-form and its coefficients $u_{\lambda_1\cdots\lambda_p}$. Then the p-forms Φu, Ψu, $\nabla_\eta u$, $(p-1)$-form Du, and $(p+1)$-form Γu are defined by the following forms with coefficients respectively:

$$(\Phi u)_{\lambda_1\cdots\lambda_p} = \sum_{i=1}^{p} \varphi_{\lambda_i}{}^{\sigma}\, u_{\lambda_1\cdots\overset{i}{\sigma}\cdots\lambda_p} \qquad (p\geqq 1)$$

$$(\Psi u)_{\lambda_1\cdots\lambda_p} = \varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\, u_{\sigma_1\cdots\sigma_p} \qquad (p\geqq 1)$$

$$(\nabla_\eta u)_{\lambda_1\cdots\lambda_p} = \eta^\sigma\,\nabla_\sigma\, u_{\lambda_1\cdots\lambda_p} \qquad (p\geqq 0)$$

$$(Du)_{\lambda_2\cdots\lambda_p} = \varphi^{\rho\sigma}\,\nabla_\rho\, u_{\sigma\lambda_2\cdots\lambda_p} \qquad (p\geqq 1)$$

$$(\Gamma u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha=0}^{p} (-1)^\alpha \varphi_{\lambda_\alpha}{}^{\sigma} \nabla_\sigma u_{\lambda_0\cdots\overset{\alpha}{\hat{\sigma}}\cdots\lambda_p} \quad (p \geqq 1)$$

where $u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p}$ means that the subscript σ appears at the i-th position and $u_{\lambda_0\cdots\hat{\lambda}_\alpha\cdots\lambda_p}$ means that the α-th subscript λ_α is omitted.

LEMMA 1.4. *For any p-form u in a $(2m+1)$-dimensional Sasakian space, we haue*

$$(\Lambda L^k - L^k\Lambda)\, u = 4k[(m-p-k+1)\, L^{k-1} u + e(\eta)\, i(\eta)\, L^{k-1} u] \tag{1.17}$$

where k is a non-negative integer and $L^{-1}u=0$.

PROOF. We take the induction with respect to the integer k. For $k=0$, (1.17) is trivially valid. Now suppose that it is true for all $k=0, 1, \cdots, k$ and consider the $(k+1)$-case. Then we have

$$\begin{aligned}(\Lambda L^{k+1} - L^{k+1}\Lambda)\, u &= (\Lambda L^k - L^k\Lambda)\, Lu + L^k(\Lambda L - L\Lambda)\, u \\ &= 4k[(m-p-2-k+1)\, L^{k-1}\cdot Lu + e(\eta) i(\eta) L^{k-1}\cdot Lu] \\ &\qquad + L^k\cdot 4[(m-p)\, u + e(\eta)\, i(\eta)\, u] \\ &= 4(k+1)[(m-p-k)\, L^k u + e(\eta)\, i(\eta) L^k u]\end{aligned}$$

for any p-form u, which asserts that the lemma is true for all non-negative integer k.

2. C-harmonic forms. We consider an $n(=2m+1)$-dimensional Sasakian space. A p-form u on the space is called to be C-harmonic if it satisfies

$$du = 0, \tag{i}$$

$$\delta u = e(\eta)\, \Lambda u\,. \tag{ii}$$

As a form of degree 1 or 0 is effective, a C-harmonic 1- or 0-form is harmonic. As a harmonic p-form $(p \leqq m)$ is necessarily effective, in compact case, so it follows that a p-form $(p \leqq m)$ is harmonic if and only if it is effective C-harmonic.

In defining the C-harmonic forms S. Tachibana [1] imposed the condition $i(\eta)u=0$ in addition to (i) and (ii). In the following we prove that this relation follows necessarily from (i) and (ii).

LEMMA 2.1. *In a Sasakian space, we have (cf.* [9])

(2. 1) $$\Delta e(\eta) - e(\eta)\,\Delta = \delta L - L\delta\,,$$

(2. 2) $$\Delta i(\eta) - i(\eta)\,\Delta = d\Lambda - \Lambda d\,.$$

PROOF. By virtue of Lemma 1.1, we get

$$\begin{aligned}\Delta e(\eta) - e(\eta)\,\Delta &= \delta(de(\eta) + e(\eta)\,d) - (e(\eta)\,d + de(\eta))\,\delta \\ &\qquad - (\delta e(\eta) \oplus e(\eta)\,\delta)\,d + d(\delta e(\eta) + e(\eta)\,\delta) \\ &= \delta L - L\delta + \theta(\eta)\,d - d\theta(\eta) \\ &= \delta L - L\delta\,.\end{aligned}$$

(2. 2) is obtained in the same way.

THEOREM 2.1. *In a compact Sasakian space, we have for any C-harmonic form u*

(2. 3) $$\theta(\eta)\,u = 0\,.$$

PROOF. We put $u' = i(\eta)\,u$. Since

$$\Delta u = d\delta u = L\Lambda u - e(\eta)\,d\Lambda u\,,$$

we have taking account of (2. 2) and Lemma 1.2

$$\begin{aligned}\Delta u &= i(\eta)\,L\Lambda u - i(\eta)\,e(\eta)\,d\Lambda u + d\Lambda u \\ &= L\Lambda u + e(\eta)\,i(\eta)\,d\Lambda u\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}(u', \Delta u') &= (u', L\Lambda u') + (u', e(\eta)\,i(\eta)\,d\Lambda u) \\ &= (\Lambda u', \Lambda u')\,.\end{aligned}$$

On the other hand, it holds that

$$\delta u' = \Lambda u - i(\eta)\,\delta u = e(\eta)\,i(\eta)\,\Lambda u\,,$$

hence we obtain

$$(\delta u', \delta u') = (e(\eta)\,\Lambda u', e(\eta)\,\Lambda u') = (\Lambda u', \Lambda u').$$

Therefore we get $(du', du') = 0$, which means that $du' = 0$. Then

$$\theta(\eta)\,u = di(\eta)\,u + i(\eta)\,du = 0,$$

and the theorem is proved.

LEMMA 2.2. *In a compact Sasakian space, we have for any C-harmonic form u,*

$$\delta(e(\eta)\,u) = 0.$$

PROOF. From Lemma 1.3 and (2.3), we have

$$\delta(e(\eta)\,u) = -\theta(\eta)\,u - e(\eta)\,\delta u = -e(\eta)\,e(\eta)\,\Lambda u = 0.$$

LEMMA 2.3. *In a compact Sasakian space, $u' = i(\eta)u$ is C-harmonic for any C-harmonic form u.*

PROOF. From (2.3), it is evident $du' = 0$. Making use of (1.14) and Lemma 1.2 we have

$$\delta u' = \Lambda u - i(\eta)\,e(\eta)\,\Lambda u = e(\eta)\,i(\eta)\,\Lambda u = e(\eta)\,\Lambda u'.$$

LEMMA 2.4. *In a Sasakian space, we have for any p-form u(cf.* [9])

$$\begin{aligned} Du &= \delta\nabla_\eta u - \nabla_\eta \delta u + (n-p)\,i(\eta)\,u \\ &= (-1/2)(d\Lambda - \Lambda d)\,u + (p-1)\,i(\eta)\,u, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \Gamma u &= d\nabla_\eta u - \nabla_\eta du - pe(\eta)\,u \\ &= (1/2)(\delta L - L\delta)\,u - (n-p-1)\,e(\eta)\,u. \end{aligned} \tag{2.5}$$

PROOF. Let $u_{\lambda_1\cdots\lambda_p}$ be the coefficients of the p-form u. Then we have

$$\begin{aligned} (\delta\nabla_\eta u)_{\lambda_2\cdots\lambda_p} &= -\nabla^{\lambda_1}(\eta^\rho\,\nabla_\rho u_{\lambda_1\cdots\lambda_p}) \\ &= \varphi^{\rho\lambda_1}\,\nabla_\rho u_{\lambda_1\cdots\lambda_p} - \eta^\rho(\nabla_\rho\nabla_{\lambda_1} u^{\lambda_1}{}_{\lambda_2\cdots\lambda_p} + R_{\lambda_1\rho\sigma}{}^{\lambda_1} u^\sigma{}_{\lambda_2\cdots\lambda_p} - \sum_{i=2}^{p} R_{\lambda_1\rho\lambda_i}{}^\sigma u^{\lambda_1}{}_{\lambda_2\cdots\overset{i}{\sigma}\cdots\lambda_p}) \\ &= (Du)_{\lambda_2\cdots\lambda_p} + (\nabla_\eta\delta u)_{\lambda_2\cdots\lambda_p} + (1-n+p-1)\,\eta^\rho u_{\rho\lambda_2\cdots\lambda_p}. \end{aligned}$$

Next we have

$$(d\Lambda u)_{\lambda_2\cdots\lambda_p} = \nabla_{\lambda_2}(\varphi^{\alpha\beta}u_{\alpha\beta\lambda_3\cdots\lambda_p}) - \sum_{i=3}^{p}\nabla_{\lambda_i}(\varphi^{\alpha\beta}u_{\alpha\beta\lambda_3\cdots\overset{i}{\hat{\lambda}_2}\cdots\lambda_p})$$

$$= 2\eta^{\alpha}u_{\alpha\lambda_2\cdots\lambda_p} - 2\sum_{i=3}^{p}\eta^{\alpha}u_{\alpha\lambda_i\lambda_3\cdots\overset{i}{\hat{\lambda}_2}\cdots\lambda_p} + \varphi^{\alpha\beta}(\nabla_{\lambda_2}u_{\alpha\beta\lambda_3\cdots\lambda_p} - \sum_{i=3}^{p}\nabla_{\lambda_i}u_{\alpha\beta\lambda_3\cdots\overset{i}{\hat{\lambda}_2}\cdots\lambda_p})$$

$$= 2(p-1)\eta^{\alpha}u_{\alpha\lambda_2\cdots\lambda_p} + \varphi^{\alpha\beta}(\nabla_{\lambda_2}u_{\alpha\beta\lambda_3\cdots\lambda_p} - \sum_{i=3}^{p}\nabla_{\lambda_i}u_{\alpha\beta\lambda_3\cdots\overset{i}{\hat{\lambda}_2}\cdots\lambda_p}),$$

$$(\Lambda du)_{\lambda_2\cdots\lambda_p} = \varphi^{\alpha\beta}(\nabla_{\alpha}u_{\beta\lambda_2\cdots\lambda_p} - \nabla_{\beta}u_{\alpha\lambda_2\cdots\lambda_p} - \sum_{i=2}^{p}\nabla_{\lambda_i}u_{\beta\lambda_2\cdots\overset{i}{\hat{\alpha}}\cdots\lambda_p})$$

$$= 2\varphi^{\alpha\beta}\nabla_{\alpha}u_{\beta\lambda_2\cdots\lambda_p} - \varphi^{\alpha\beta}(\nabla_{\lambda_2}u_{\beta\alpha\lambda_3\cdots\lambda_p} - \sum_{i=3}^{p}\nabla_{\lambda_i}u_{\beta\alpha\lambda_3\cdots\overset{i}{\hat{\lambda}_2}\cdots\lambda_p}).$$

Hence $(2.4)_2$ is obtained. For $(2.5)_1$, we have

$$(d\nabla_{\eta}u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha=0}^{p}(-1)^{\alpha}\nabla_{\lambda_\alpha}(\eta^{\rho}\nabla_{\rho}u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p})$$

$$= \sum_{\alpha=0}^{p}(-1)^{\alpha}\varphi_{\lambda_\alpha}{}^{\rho}\nabla_{\rho}u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p} + \sum_{\alpha=0}^{p}(-1)^{\alpha}\eta^{\rho}[\nabla_{\rho}\nabla_{\lambda_\alpha}u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p} - \sum_{\beta\neq\alpha}R_{\lambda_\alpha\rho\lambda_\beta}{}^{\sigma}u_{\lambda_0\cdots\overset{\beta}{\hat{\sigma}}\cdots\hat{\alpha}\cdots\lambda_p}]$$

$$= (\Gamma u)_{\lambda_0\cdots\lambda_p} + (\nabla_{\eta}du)_{\lambda_0\cdots\lambda_p} + \sum_{\alpha\neq\beta}(-1)^{\alpha}\eta^{\rho}g_{\lambda_\alpha\lambda_\beta}u_{\lambda_0\cdots\overset{\beta}{\hat{\rho}}\cdots\hat{\alpha}\cdots\lambda_p} - \sum_{\alpha\neq\beta}(-1)^{\alpha}\eta_{\lambda_\beta}u_{\lambda_0\cdots\overset{\beta}{\hat{\lambda}_\alpha}\cdots\hat{\alpha}\cdots\lambda_p}$$

$$= (\Gamma u)_{\lambda_0\cdots\lambda_p} + (\nabla_{\eta}du)_{\lambda_0\cdots\lambda_p} + p(e(\eta)u)_{\lambda_0\cdots\lambda_p},$$

and the last formula $(2.5)_2$ is the result of the following calculation;

$$(\delta Lu)_{\rho\lambda_1\cdots\lambda_p} = -2\nabla^{\sigma}[\varphi_{\sigma\rho}u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\sigma\lambda_i}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\lambda_i\rho}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j}\varphi_{\lambda_i\lambda_j}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}]$$

$$= 2(n-p-1)(\eta_{\rho}u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\eta_{\lambda_i}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}) - 2[\varphi_{\sigma\rho}\nabla^{\sigma}u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\sigma\lambda_i}\nabla^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}$$

$$- \sum_{i=1}^{p}\varphi_{\lambda_i\rho}\nabla^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j}\varphi_{\lambda_i\lambda_j}\nabla^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}],$$

$$(L\delta u)_{\rho\lambda_1\cdots\lambda_p} = -2[\varphi_{\rho\lambda_1}\nabla^{\sigma}u_{\sigma\lambda_2\cdots\lambda_p} - \sum_{j=2}^{p}\varphi_{\rho\lambda_j}\nabla^{\sigma}u_{\sigma\lambda_2\cdots\overset{j}{\hat{\lambda}_1}\cdots\lambda_p}$$

$$- \sum_{j=2}^{p}\varphi_{\lambda_j\lambda_1}\nabla^{\sigma}u_{\sigma\lambda_2\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p} + \sum_{2\leq i<j}\varphi_{\lambda_i\lambda_j}\nabla^{\sigma}u_{\sigma\lambda_2\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\lambda}_1}\cdots\lambda_p}]$$

$$= -2[-\sum_{i=1}^{p}\varphi_{\lambda_i\rho}\nabla^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j}\varphi_{\lambda_i\lambda_j}\nabla^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}].$$

Thus the lemma 2.4 is proved.

LEMMA 2.5. *In a Sasakian space, the operator* ∇_η *commutes with the operators* $i(\eta)$, $e(\eta)$, L *and* Λ.

PROOF. For the forms η_λ and $\varphi_{\lambda\mu}$, we have

$$(\nabla_\eta \eta)_\lambda = 0, \qquad (\nabla_\eta \varphi)_{\lambda\mu} = 0,$$

and hence the lemma follows easily.

THEOREM 2.2. *In a compact* $(2m+1)$*-dimensional Sasakian space, any C-harmonic p-form* $(p \leqq m)$ *is orthogonal to* η, *that is*

$$i(\eta)\, u = 0\,.$$

PROOF. If we set $u' = i(\eta)u$, $\alpha = e(\eta)u'$ and $\beta = u - \alpha$, then it holds $i(\eta)\beta = 0$. According to Lemmas 2.2 and 2.3, we have $\delta\alpha = 0$, hence

$$\delta\beta = \delta u$$

holds good. As u and u' are C-harmonic, we have

$$\begin{aligned} d\beta &= -\, de(\eta)u' = -\,(L - e(\eta)\,d)u' \\ &= -\, Lu'. \end{aligned}$$

Furthermore we have from (2. 1) and (2. 5)

$$\Delta\alpha = e(\eta)(\Delta u' + 4(m-p+1)u') + 2d\nabla_\eta u'\,,$$

therefore using Lemma 2.5 we get

$$\begin{aligned} (\beta, \Delta\alpha) &= (\beta, 2d\nabla_\eta u') = 2(\delta\beta, \nabla_\eta u') \\ &= 2(\delta u, \nabla_\eta u') = 2(e(\eta)\,\Lambda u', \nabla_\eta u') = 0\,. \end{aligned}$$

Thus we have

$$(\beta, \Delta\beta) = (\beta, \Delta u) = (\delta\beta, \delta\beta)\,,$$

and conseqently we get $(d\beta, d\beta) = 0$, which shows $d\beta = 0$. Therefore we have $Lu' = 0$. Applying Lemma 1.4 for the case $k=1$, we have

$$- L\Lambda u' = 4(m-p+1)\, u'$$

because $i(\eta)u'=0$, and hence

$$(u', -L\Lambda u') = -(\Lambda u', \Lambda u') = 4(m-p+1)(u', u').$$

Therefore if u is of degree $p\leqq m$, both sides of this equation must be zero, and we have $u'=0$. This proves the theorem.

COROLLARY 2.2.1. *In a compact $(2m+1)$-dimensional Sasakian space, we have*

$$i(\eta)\, Lu = 0, \qquad i(\eta)\, \Lambda u = 0$$

for any C-harmonic $(m+1)$-form u.

Next we study some properties of C-harmonic forms.

LEMMA 2.6. *In a $(2m+1)$-dimensional Sasakian space, for any p-form u we have* (*cf.* [9])

$$(\Delta L - L\Delta)\, u = 4(m-p-1)\, Lu + 4de(\eta)\, u, \tag{2.6}$$

$$(\Delta\Lambda - \Lambda\Delta)\, u = -4(m-p+2)\, \Lambda u + 4\delta i(\eta)\, u. \tag{2.7}$$

PROOF. First we verify the formula for any p-form u $(p\geqq 2)$

$$\delta Du + D\delta u = (n-p+1)\, \Lambda u - \delta i(\eta)\, u. \tag{2.8}$$

In fact, making use of $(2.4)_1$, we have

$$\delta Du = -\delta \nabla_\eta \delta u + (n-p)\, \delta\, i(\eta)\, u,$$

$$D\delta u = \delta \nabla_\eta \delta u + (n-p+1)\, i(\eta)\, \delta u,$$

and hence we get

$$\delta Du + D\delta u = (n-p+1)(i(\eta)\delta + \delta i(\eta))\, u - \delta i(\eta)\, u.$$

On the other hand, by virtue of $(2.4)_2$ we see that

$$\begin{aligned}\delta Du + D\delta u &= (-1/2)(\Delta\Lambda - \Lambda\Delta)\, u + (p-1)\, \delta i(\eta)\, u + (p-2)\, i(\eta)\, \delta u \\ &= (-1/2)\, (\Delta\Lambda - \Lambda\Delta)\, u + (p-2)\, \Lambda u + \delta i(\eta)\, u.\end{aligned}$$

Comparing the above two relations, we have

$$(-1/2)(\Delta\Lambda-\Lambda\Delta)\,u = (n-2p+3)\,\Lambda u - 2\delta i(\eta)\,u\,,$$

which shows (2.7). The formula (2.6) is only dual to (2.7).

THEOREM 2.3. ([1]) *In a compact $(2m+1)$-dimensional Sasakian space, if a p-form u is C-harmonic and $p\leqq m$, then Λu is C-harmonic, too.*

PROOF. As the operator Λ commutes with δ and $e(\eta)$, we have

$$\delta\Lambda u = \Lambda e(\eta)\,\Lambda u = e(\eta)\,\Lambda\Lambda u\,,$$

hence we have only to show that $d\Lambda u=0$. Since C-harmonic form u satisfies

$$\Delta u = L\Lambda u - e(\eta)\,d\Lambda u\,,$$

with the aid of Lemma 2.6, 1.4, and Theorem 2.2, we have

$$\begin{aligned}\Delta\Lambda u &= \Lambda\Delta u - 4(m-p+2)\,\Lambda u\\ &= L\Lambda^2 u - e(\eta)\,\Lambda d\Lambda u\,,\end{aligned}$$

if $p\leqq m$. Hence we obatain

$$(\Lambda u,\,\Delta\Lambda u) = (\Lambda u,\,L\Lambda^2 u - e(\eta)\,\Lambda d\Lambda u) = (\Lambda^2 u,\,\Lambda^2 u)\,.$$

From $\delta\Lambda u=e(\eta)\,\Lambda^2 u$, we again have

$$(\delta\Lambda u,\,\delta\Lambda u) = (\Lambda^2 u,\,\Lambda^2 u)\,.$$

These equations show that $(d\Lambda u, d\Lambda u)=0$, and hence we have $d\Lambda u=0$.

From this proof of the theorem, we see that $d\Lambda u = 0$ for a C-harmonic p $(\leqq m)$-form u. Thus we have

$$\Delta u = L\Lambda u\,, \tag{2.9}$$

$$Du = 0 \tag{2.10}$$

by taking account of $(2.4)_2$. Regarding to (2.9), we can give the following necessary and sufficient condition for a form to be C-harmonic.

THEOREM 2.4. *In a compact $(2m+1)$-dimensional Sasakian space, a p-form u $(p\leqq m)$ is C-harmonic if and only if it satisfies $i(\eta)u=0$ and $\Delta u = L\Lambda u$.*

PROOF. Let u be a C-harmonic $p(\leqq m)$-form. Then Theorem 2.2 and (2.9) imply the necessary condition of the theorem. Conversely let u be a p-form satisfying $i(\eta)u=0$. Then we have

$$(u, \Delta u - L\Lambda u) = (du, du) + (\delta u, \delta u) - (\Lambda u, \Lambda u)\,,$$

$$\begin{aligned}(\delta u - e(\eta)\Lambda u, \delta u - e(\eta)\Lambda u) &= (\delta u, \delta u) - 2(\delta u, e(\eta)\Lambda u) + (e(\eta)\Lambda u, e(\eta)\Lambda u)\\ &= (\delta u, \delta u) - 2(\Lambda u - \delta i(\eta)u, \Lambda u) + (\Lambda u, \Lambda u)\\ &= (\delta u, \delta u) - (\Lambda u, \Lambda u)\,.\end{aligned}$$

Hence we have the following integral formula for a p-form u orthogonal to η:

$$(2.11)\qquad (u, \Delta u - L\Lambda u) = (du, du) + (\delta u - e(\eta)\Lambda u, \delta u - e(\eta)\Lambda u)\,.$$

Therefore if $\Delta u - L\Lambda u = 0$, then we have $du=0$, $\delta u = e(\eta)\Lambda u$, which proves our theorem.

THEOREM 2.5. *In a compact $(2m+1)$-dimensional Sasakian space, if a p-form u is C-harmonic and $p\leqq m$, then Lu is also C-harmonic.*

PROOF. As L commutes with $i(\eta)$, we know that Lu is orthogonal to η. As we have from (2.6)

$$\begin{aligned}\Delta Lu &= L\Delta u + 4(m-p)\,Lu\\ &= LL\Lambda u + 4(m-p)\,Lu = L\Lambda Lu\,,\end{aligned}$$

Lu is also C-harmonic, because of Theorem 2.4.

COROLLARY 2.5.1. *In a compact $(2m+1)$-dimensional Sasakian space, if a p-form u is C-harmonic and $p\leqq m$, then Δu is also C-harmonic.*

This is a consequence of (2.9). Next we consider the operators Φ, Ψ and ∇_η for a C-harmonic form. For this purpose we give some lemmas.

LEMMA 2.7. *In a Sasakian space, the operator Φ commutes with the operators $i(\eta)$, $e(\eta)$, L and Λ, and the operator Ψ commutes with the operators L and Λ. (cf.* [9])

PROOF. Since it is easily shown that the Lie derivative $\theta(\eta)$ commutes with $i(\eta)$, $e(\eta)$, L and Λ, the first part of the lemma comes from Lemma 2.5 and the formula

$$\Phi u = \theta(\eta)\, u - \nabla_\eta u\,. \tag{2.12}$$

For the second part, we calculate directly, for any p-form u

$$(\Lambda\Psi u)_{\lambda_3\cdots\lambda_p} = \varphi^{\alpha\beta}(\varphi_\alpha{}^{\sigma_1}\varphi_\beta{}^{\sigma_2}\varphi_{\lambda_3}{}^{\sigma_2}\varphi\cdots{}_{\lambda_p}^{\sigma_p})\, u_{\sigma_1\cdots\sigma_p} = \varphi^{\sigma_1\sigma_2}\varphi_{\lambda_3}{}^{\sigma_3}\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\sigma_p}$$

$$(\Psi\Lambda u)_{\lambda_3\cdots\lambda_p} = (\varphi_{\lambda_3}{}^{\sigma_3}\cdots\varphi_{\lambda_p}{}^{\sigma_p})(\varphi^{\sigma_1\sigma}u_{\sigma_1\sigma_2\cdots\sigma_p})\,.$$

Hence we obtain $\Lambda\Psi=\Psi\Lambda$. Next

$$\begin{aligned}
(1/2)(\Psi L u)_{\alpha\beta\lambda_1\cdots\lambda_p} &= \varphi_\alpha{}^\rho\varphi_\beta{}^\sigma\varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\Big[\varphi_{\rho\sigma}u_{\sigma_1\cdots\sigma_p} - \sum_{i=1}^{p}\varphi_{\rho\sigma_i}u_{\sigma_1\cdots\overset{i}{\hat{\sigma}}\cdots\sigma_p} \\
&\qquad - \sum_{i=1}^{p}\varphi_{\sigma_i\sigma}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\sigma_p} + \sum_{i<j}\varphi_{\sigma_i\sigma_j}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\sigma_p}\Big] \\
&= \varphi_{\alpha\beta}(\varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\sigma_p}) + \sum_{i=1}^{p}\varphi_{\lambda_i\alpha}(\varphi_{\lambda_1}{}^{\sigma_1}\cdots\overset{i}{\hat{\varphi}}{}_\beta{}^\sigma\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\overset{i}{\hat{\sigma}}\cdots\sigma_p}) \\
&\qquad - \sum_{i=1}^{p}\varphi_{\lambda_i\beta}(\varphi_{\lambda_1}{}^{\sigma_1}\cdots\overset{i}{\hat{\varphi}}{}_\alpha{}^\rho\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\sigma_p}) \\
&\qquad + \sum_{i<j}\varphi_{\lambda_i\lambda_j}(\varphi_{\lambda_1}{}^{\sigma_1}\cdots\overset{i}{\hat{\varphi}}{}_\alpha{}^\rho\cdots\overset{j}{\hat{\varphi}}{}_\beta{}^\sigma\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}) \\
&= (1/2)(L\Psi u)_{\alpha\beta\lambda_1\cdots\lambda_p}\,,
\end{aligned}$$

which proves our lemma.

LEMMA 2.8. *In a Sasakian space we have for any p-form u (cf.* [9])

$$(\Delta\Phi-\Phi\Delta)\, u = 2(e(\eta)\,\delta u + di(\eta)\, u)\,. \tag{2.13}$$

PROOF. By virtue of Lemma 2.4 and (2.12), we have the following four relations operating to any p-form u

$$dDu = d\Phi\delta u - d\delta\Phi u + (n-p)\, di(\eta)\, u = (-1/2)(-d\Lambda du) + (p-1)\, di(\eta)\, u\,,$$

$$Ddu = \Phi\delta du - \delta\Phi du + (n-p-1)\, i(\eta)\, du = (-1/2)(d\Lambda du) + pi(\eta)\, du\,,$$

$$\delta\Gamma u = \delta\Phi du - \delta d\Phi u - p\delta e(\eta)\, u = (1/2)(-\delta L\delta u) - (n-p-1)\,\delta e(\eta)\, u\,,$$

$$\Gamma\delta u = \Phi d\delta u - d\Phi\delta u - (p-1)\, e(\eta)\,\delta u = (1/2)(\delta L\delta u) - (n-p)\, e(\eta)\,\delta u\,.$$

Adding sides by sides of these relations it follows that

$$(\Phi\Delta-\Delta\Phi)\,u = 2(\theta(\eta)+\delta e(\eta)-di(\eta))\,u\,,$$

from which our lemma is easily deduced.

COROLLARY 2.8.1. ([2]) *In a compact $(2m+1)$-dimensional Sasakian space, Φu is harmonic for any harmonic p-form u ($p\leqq m$).*

LEMMA 2.9. *In a Sasakian space we have for any p-form u*

$$\Psi^2 u = (-1)^p\,(i(\eta)\,e(\eta)\,u)\,, \tag{2.14}$$

$$(\Delta\Psi-\Psi\Delta)\,u = 2(d\Psi i(\eta)\,u + e(\eta)\,\Psi\delta u)\,. \tag{2.15}$$

PROOF. Let $u_{\lambda_1\cdots\lambda_p}$ be the coefficients of the p-form u. Then

$$\begin{aligned}(\Psi^2 u)_{\lambda_1\cdots\lambda_p} &= \varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\varphi_{\sigma_1}{}^{\rho_1}\cdots\varphi_{\sigma_p}{}^{\rho_p}u_{\rho_1\cdots\rho_p}\\ &= (-1)^p\,u_{\lambda_1\cdots\lambda_p}+\sum_{i=1}^{p}(-1)^{p-1}\,\eta_{\lambda_i}\,\eta^\rho u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}\\ &= (-1)^p\,(u_{\lambda_1\cdots\lambda_p}-(e(\eta)\,i(\eta)\,u)_{\lambda_1\cdots\lambda_p})\,,\end{aligned}$$

which shows (2.14). The second formula is obtained by a little complicated and straightforward computation, so we only point out the outline. At first we have

$$(\Delta\Psi u)_{\lambda_1\cdots\lambda_p} = -\nabla^\rho\nabla_\rho(\Psi u)_{\lambda_1\cdots\lambda_p}+\sum_{i=1}^{p}R_{\lambda_i}{}^{\rho}(\Psi u)_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}+\sum_{i<j}R_{\lambda_i\lambda_j}{}^{\rho\sigma}(\Psi u)_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}$$

and we put A_1, A_2, A_3 the three terms of the right hand side respectively. Then

$$\begin{aligned}A_1 &= -\varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\nabla^\rho\nabla_\rho u_{\sigma_1\cdots\sigma_p}\\ &\quad+2[p\varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}u_{\sigma_1\cdots\sigma_p}+\sum_{i=1}^{p}\varphi_{\lambda_1}{}^{\sigma_1}\cdots\widehat{i}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\eta^{\sigma_i}\nabla_{\lambda_i}u_{\sigma_1\cdots\sigma_i\cdots\sigma_p}]\\ &\quad-2\sum_{i=1}^{p}\varphi_{\lambda_1}{}^{\sigma_1}\cdots\widehat{i}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\eta_{\lambda_i}\nabla^{\sigma_i}u_{\sigma_1\cdots\sigma_p}+2\sum_{i\neq j}\varphi_{\lambda_1}{}^{\sigma_1}\cdots\widehat{j}\cdots\widehat{i}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\eta_{\lambda_j}\eta^{\sigma_i}u_{\sigma_1\cdots\overset{j}{\hat{\lambda}_i}\cdots\sigma_p}\\ &= -\varphi_{\lambda_1}{}^{\sigma_1}\cdots\varphi_{\lambda_p}{}^{\sigma_p}\nabla^\rho\nabla_\rho u_{\sigma_1\cdots\sigma_p}-2\sum_{i=1}^{p}(-1)^i\,\eta_{\lambda_i}(\Psi\delta u)_{\lambda_1\cdots\hat{i}\cdots\lambda_p}\end{aligned}$$

$$+2\sum_{i=1}^{p}(-1)^{i-1}\varphi_{\lambda_1}^{\sigma_1}\cdots\widehat{i}\cdots\varphi_{\lambda_p}^{\sigma_p}\nabla_{\lambda_i}(i(\eta)\,u)_{\sigma_1\cdots\hat{i}\cdots\sigma_p}$$

$$+2\sum_{i\neq j}(-1)^{j}\varphi_{\lambda_1}^{\sigma_1}\cdots\widehat{j}\cdots\widehat{i}\cdots\varphi_{\lambda_p}^{\sigma_p}\eta_{\lambda_j}(i(\eta)\,u)_{\sigma_1\cdots\overset{i}{\hat{\lambda_i}}\cdots\hat{j}\cdots\sigma_p}$$

$$=-\varphi_{\lambda_1}^{\sigma_1}\cdots\varphi_{\lambda_p}^{\sigma_p}\nabla^{\rho}\nabla_{\rho}u_{\sigma_1\cdots\sigma_p}+2[(e(\eta)\,\Psi\delta u)_{\lambda_1\cdots\lambda_p}+(d\Psi i(\eta)\,u)_{\lambda_1\cdots\lambda_p}]\,.$$

Next we calculate

$$(\Psi\Delta u)_{\lambda_1\cdots\lambda_p}=-\varphi_{\lambda_1}^{\sigma_1}\cdots\varphi_{\lambda_p}^{\sigma_p}\nabla^{\rho}\nabla_{\rho}u_{\sigma_1\cdots\sigma_p}+\sum_{i=1}^{p}\varphi_{\lambda_1}^{\sigma_1}\cdots\varphi_{\lambda_p}^{\sigma_p}R_{\sigma_j}{}^{\rho}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\sigma_p}$$

$$+\sum_{i<j}\varphi_{\lambda_1}^{\sigma_1}\cdots\varphi_{\lambda_p}^{\sigma_p}R_{\sigma_i\sigma_j}{}^{\rho\sigma}u_{\sigma_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\sigma_p}\,,$$

and we see by virtue of (1. 7) and (1. 8) the latter two terms of the right hand side are equal to A_2, A_3 respectively. Therefore we have (2. 15).

COROLLARY 2.9.1. *In a compact $(2m+1)$-dimensional Sasakian space, Ψu is harmonic for any harmonic p-form u $(p\leqq m)$.*

THEOREM 2.6. *In a compact $(2m+1)$-dimensional Sasakian space, if a p-form u is C-harmonic and $p\leqq m$, then Φu, Ψu and $\nabla_\eta u$ are C-harmonic, too.*

PROOF. Since $p\leqq m$, it holds that $i(\eta)\,\Phi u=0$, using Theorem 2.2 and Lemma 2.7. Then we have

$$\Delta\Phi u=\Phi\Delta u+2(e(\eta)\,\delta u+di(\eta)\,u)$$

$$=\Phi L\Lambda u=L\Lambda\Phi u\,,$$

hence we see that Φu is also C-harmonic by virtue of Theorem 2.4. Next for Ψu, it is evident from the definition of the operator Ψ that $i(\eta)\,\Psi u$ and $\Psi e(\eta)\,u$ are zero. From (2. 15) we have also

$$\Delta\Psi u=\Psi\Delta u+2(d\Psi i(\eta)\,u+e(\eta)\,\Psi e(\eta)\,\Lambda u)$$

$$=\Psi L\Lambda u=L\Lambda\Psi u\,,$$

and Ψu is again C-harmonic. As $\theta(\eta)\,u=0$ for a C-harmonic form u, we know that $\Phi u=-\nabla_\eta u$. Therefore if u is a C-harmonic form, then so is $\nabla_\eta u$.

Owing to the relation (2. 14), it follows that if $p(\leqq m)$ is odd, then Ψ is a complex structure of the vector space of all C-harmonic (or harmonic) p-forms, hence we have

THEOREM 2.7. *In a compact $(2m+1)$-dimensional Sasakian space, if $p(\leqq m)$ is odd, then the dimension of the vector space of all C-harmonic (or harmonic) p-forms is even.*

3. The decomposition theorem. S. Tachibana has showed the following theorem for a C-harmonic p-form analogous to the Kählerian space.

THEOREM 3.1. ([1]) *In a compact $(2m+1)$-dimensional Sasakian space, any C-harmonic p-form u_p $(p\leqq m)$ can be written uniquely in the form:*

$$u_p = \sum_{k=0}^{r} L^k \phi_{p-2k}$$

where ϕ_{p-2k} is a harmonic $(p-2k)$-form and r is the integral part of $p/2$. Conversely any p-form written as in the right hand side is C-harmonic.

The assumption of p in Tachibana's original theorem is $p\leqq m+1$. This difference is due to the definition of C-harmonic forms. Our theorem 2.3 and 2.5 require the assumption $p\leqq m$, and the theorem can be proved with the aid of these theorems. If p satisfies $p\leqq m$, then our definition of C-harmonic p-forms coincides with that of [1], therefore the proof of Theorem 3.1 is completely the same as [1], and we omit it.

Let C_p and H_p be the vector space of C-hamonic p-forms and harmonic p-forms, and put $c_p=\dim C_p$, $b_p=\dim B_p$ $(=p$-th Betti number). As any 0- or 1-form is effective, and a C-harmonic form is harmonic if and only if it is effective, we have $b_0=c_0$ $(=1)$, $b_1=c_1$. Next we show that the forms $(d\eta)^k=L^k\cdot 1$ $(0\leqq k\leqq m)$ are C-harmonic. Since d commutes with L, we see easily that $dL^k\cdot 1=0$. We want to calculate $\delta(L^k\cdot 1)$. Making use of Lemma 2.5 we have $\nabla_\eta(L^k\cdot 1)=0$. Therefore by virtue of (2. 5) it holds

$$\delta L^k\cdot 1 - L\delta L^{k-1}\cdot 1 = 4(m-2k+2)\, e(\eta)\, L^{k-1}\cdot 1\,,$$

and hence we can obtain

$$\delta L^k\cdot 1 = 4k(m-k+1)\, e(\eta)\, L^{k-1}\cdot 1\,.$$

On the other hand we have by virtue of (1. 17)

$$e(\eta)\,\Lambda(L^k\cdot 1) = e(\eta)(L^k\Lambda\cdot 1 + 4k(m-k+1)\,L^{k-1}\cdot 1 + e(\eta)\,i(\eta)\,L^{k-1}\cdot 1)$$
$$= 4k(m-k+1)\,e(\eta)\,L^{k-1}\cdot 1\,,$$

hence it holds

$$\delta L^k\cdot 1 = e(\eta)\,\Lambda(L^k\cdot 1)\,.$$

This shows that $L^k\cdot 1$ is C-harmonic for $k=0,\cdots,m$. Hence we have $c_{2k}\geqq 1$ for all $k=0,\cdots,m$. Thus

THFOREM 3.2. *In a $(2m+1)$-dimensional Sasakian space, we have*

$$c_{2k}\geqq 1\,,\qquad k=0,\cdots,m\,,$$
$$c_0=b_0=1\,,\quad c_1=b_1\,.$$

As a corollary of Theorem 3.1, we have the following

THEOREM 3.3. ([1]) *In a compact $(2m+1)$-dimensional Sasakian space, we have*

$$b_p = c_p - c_{p-2}\,,$$
$$c_p = b_p + b_{p-2} + \cdots + b_{p-2r},$$

where r denotes the integral part of $p/2$, and $p\leqq m$.

PROOF. From Theorem 3.1, the vector space C_p and H_p satisfy the relation

$$C_p = H_p \oplus LH_{p-2} \oplus \cdots \oplus L^r H_{p-2r}\,,$$

where $\oplus$ denotes the direct sum and $p\leqq m$. We assume $p\leqq m-2$. Then for $p+2(\leqq m)$ we have

$$C_{p+2} = H_{p+2} \oplus LH_p \oplus \cdots \oplus L^{r+1} H_{p+2-2(r+1)}\,.$$

Since $L: C_p\rightarrow C_{p+2}$ is into isomorphic, we have

$$LC_p = LH_p \oplus L^2H_{p-2} \oplus \cdots \oplus L^{r+1}H_{p-2r}\,,$$

and comparing these two relations we have

$$C_{p+2} = H_{p+2} \oplus LC_p ,$$

this proves the theorem.

4. **Regular Sasakian structure.** Suppose that a compact n-dimensional Sasakian space M^n has regular structure. Then we have a principal circle bundle (M^n, p, B^{n-1}) over the Kählerian space $B^{n-1}=M^n/\eta$, and $p: M^n \to B^{n-1}$ is the projection. S. Tanno has showed that (in the case of regular K-contact space M^n) the Betti numbers of M^n and B^{n-1} have the relation

$$b_p(M) = b_p(B) - b_{p-2}(B), \qquad (p \leqq m)$$

and if $p=1$, then the vector space $H_1(B)$ of harmonic 1-forms on B^{n-1} is isomorphic to the vector space $H_1(M)$. We shall show that in a Sasakian space the vector space $H_p(B)$ is isomorphic to the vector space $C_p(M)$ for $p \leqq m$.

As the 1-form η on M^n is an infinitesimal connection of (M, p, B), there exists a lift $L: T(B) \to T(M)$ with respect to this connection. ($T(B)$ and $T(M)$ denote the tangent bundles of the spaces B^{n-1} and M^n.) Let $g = (g_{\lambda\mu})$ be the metric tensor of M^n, then the metric g' of B^{n-1} is defined by

$$g' = L^* g . \tag{4.1}$$

We investigate the relation between Riemannian connections of these metrics g and g'. We fix a point x_0 in M^n and $u_0 = p(x_0)$ in B^{n-1}, and take local coordinate systems (x^λ) at x_0 and (u^a) at u_0. We denote the right translation $M^n \to M^n$ of the structural group by

$$(x^1, \cdots, x^n, t) \to \varphi^\lambda(x^1, \cdots, x^n, t)$$

for sufficiently small t with respect to the local coordinates system. Since each fibre of M^n is a trajectory of the vector field η^λ, we get

$$\eta^\lambda(x) = \left(\frac{\partial \varphi^\lambda}{\partial t}\right)_{t=0} \tag{4.2}$$

Next we construct some local cross-section over the neighbourhood of u_0 as follows: let X be a vector at u_0 and $u(s)$ be the geodesic starting at u_0 and having the tangent vector X. Take a lift LX at x_0, and the geodesic $\bar{x}(\bar{s})$ starting at x_0 and having the tangent vector LX. The curve $\bar{x}(\bar{s})$ is projected to the curve $u(s)$. Thus there exists a local cross-section over a sufficiently small neighbourhood of u_0 as every point can be united to the original point u_0 by a unique geodesic. In the local coordinate systems, we represent it by

$$x^\lambda = l^\lambda(u^1, \cdots, u^{n-1}), \qquad x_0{}^\lambda = l^\lambda(u_0),$$

and we call this local cross-section an adapted one at x_0. Then the equation $x_t^\lambda = l_t^\lambda(u^1, \cdots, u^{n-1})$ of the adapted local cross-section at $x^\lambda = \varphi^\lambda(x_0, t)$ can be written by

$$l_t^\lambda(u) = \varphi^\lambda(l(u), t),$$

because the right translation of the group on M^n is an isometry. Hence we have

$$\frac{\partial l_t(u)}{\partial u^a} = \frac{\partial \varphi^\lambda(x, t)}{\partial x^\mu}\frac{\partial l^\mu(u)}{\partial u^a}, \qquad x = l(u). \tag{4.3}$$

In particular, we have

$$\left(\frac{\partial \varphi^\lambda}{\partial x^\mu}\right)_{t=0} = \delta_\mu^\lambda. \tag{4.4}$$

We express the projection p: $M^n \to B^{n-1}$ by

$$u^a = p^a(x^1, \cdots, x^n).$$

Then for sufficiently small t, we have

$$u^a = p^a(\varphi^1(l(u), t), \cdots, \varphi^n(l(u), t)), \tag{4.5}$$

and therefore we get differentiating it

$$\frac{\partial p^a(x)}{\partial x^\lambda}\frac{\partial l^\lambda(u)}{\partial u^b} = \delta_b^a, \qquad x = l(u), \tag{4.6}$$

$$\frac{\partial p^a}{\partial x^\lambda}\eta^\lambda = 0, \tag{4.7}$$

where the latter equation is nothing but the projection of the vector η^λ to the base space and it holds good at every point in the neighbourhood of x_0. We denote $\partial l^\lambda(u)/\partial u^a$ (resp. $\partial p^a(x)/\partial x^\lambda$) by $l_a^\lambda(u)$ (resp. $p_\lambda^a(x)$).

The lift L: $T(B) \to T(M)$ is a differentiable distribution in M^n and a linear mapping at each point of B^{n-1}. We denote it by

$$L_a^\lambda : T_u(B) \to T_{l(u)}(M)$$

with respect to the local coordinates systems. Then from the construction of the adapted local cross-section, we have

$$L_a^\lambda(u_0) = l_a^\lambda(u_0)\,. \tag{4.8}$$

The lift at the point $\varphi^\lambda(l(u), t)$ is given by $(\partial\varphi^\lambda(l(u), t)/\partial x^\mu)\, L_a^\mu(u)$, hence it holds

$$\eta_\lambda(\varphi(l(u), t)) \frac{\partial\varphi^\lambda(l(u), t)}{\partial x^\mu} L_a^\mu(u) = 0\,. \tag{4.9}$$

Corresponding to (4.6), we have

$$p_\lambda^a(l(u))\, L_b^\lambda(u) = \delta^a{}_b\,. \tag{4.10}$$

Let X^λ be any vector at the point $\varphi^\lambda(l(u), t)$, then we see that the vector $X^\lambda - \eta_\mu X^\mu \eta^\lambda$ is horizontal and has the projection $p_\lambda^a X^\lambda$. Thus we have

$$\frac{\partial\varphi^\lambda(l(u), t)}{\partial x^\mu} L_a^\mu(u)\, p_\nu^a(\varphi(l(u), t))\, X^\nu = X^\lambda - \eta_\mu X^\mu \eta^\lambda\,,$$

and especially,

$$L_a^\lambda(u)\, p_\mu^a(l(u)) = \delta_\mu^\lambda - \eta^\lambda(l(u))\, \eta_\mu(l(u)) \tag{4.11}$$

holds good. Differentiating (4.9) at $t=0$, we have

$$\partial_\mu \eta_\lambda \frac{\partial l^\mu}{\partial u^b} L_a^\lambda + \eta_\lambda L_{a,b}^\lambda = 0$$

where $L_{a,b}^\lambda = \partial L_a^\lambda/\partial u^b$. Hence the following equation

$$\eta_\lambda(L_{a,b}^\lambda - L_{b,a}^\lambda) = \partial_\mu \eta_\lambda \left(\frac{\partial l^\mu}{\partial u^a} L_b^\lambda - \frac{\partial l^\mu}{\partial u^b} L_a^\lambda\right) \tag{4.12}$$

is valid at the point $x=l(u)$. Similarly we have from (4.11)

$$p_\lambda^a(L_{b,c}^\lambda - L_{c,b}^\lambda) = p_{\lambda,\mu}^a(l_b^\mu L_c^\lambda - l_c^\mu L_b^\lambda)$$

and therefore we get by virtue of (4.11) and (4.12)

(4.13) $$L^{\lambda}_{a,b} - L^{\lambda}_{b,a} = (\eta^{\lambda}\partial_{\mu}\eta_{\nu} + L^{\lambda}_{c}\, p^{c}_{\mu,\nu})(l^{\mu}_{a}L^{\nu}_{b} - l^{\mu}_{b}L^{\nu}_{a}).$$

In particular, at the points u_0 and x_0, it follows

(4.14) $$L^{\lambda}_{a,b}(u_0) = L^{\lambda}_{b,a}(u_0) + \eta^{\lambda}(d\eta)_{\mu\nu}(x_0)\, L^{\mu}_{a}L^{\nu}_{b}(u_0).$$

Now the metric tensor g'_{ab} is, by definition, given by

(4.15) $$g'_{ab}(u) = L^{\lambda}_{a}(u)\, L^{\mu}_{b}(u)\, g_{\lambda\mu}(l(u)).$$

As the metric g and the 1-form η are invariant on the trajectory of η, we have at an arbitrary point $x^{\lambda} = \varphi^{\lambda}(l(u), t)$

(4.16) $$g_{\lambda\mu}(x) = p^{a}_{\lambda}(x)\, p^{b}_{\mu}(x)\, g'_{ab}(p(x)) + \eta_{\lambda}(x)\,\eta_{\mu}(x),$$

(4.17) $$g^{\lambda\mu}(x) = L^{\lambda}_{a}(p(x))\, L^{\mu}_{b}(p(x))\, g'^{ab}(p(x)) + \eta^{\lambda}(x)\,\eta^{u}(x).$$

From them we can investigate the relation between the Christoffel symbol $\left\{ {\lambda \atop \mu\nu} \right\}_{x_0}$ at the point x_0 with respect to the metric $g_{\lambda\mu}$ and that $\left\{ {a \atop bc} \right\}'_{u_0}$ at the point $u_0 = p(x_0)$ with respect to the metric g'_{ab}. From (4.15), we have at the points x_0 and u_0

(4.18) $$\left\{ {a \atop bc} \right\}'_{u_0} = p^{a}_{\lambda}(x_0)\, L^{\mu}_{b}(u_0)\, L^{\nu}_{c}(u_0) \left\{ {\lambda \atop \mu\nu} \right\}_{x_0} + p^{a}_{\lambda}(x_0)\, L^{\lambda}_{b,c}(u_0).$$

While we have $\eta^{\lambda} p^{a}_{\lambda} = 0$ at every point on M^n, hence it follows that

$$p^{a}_{\rho}\left\{ {\rho \atop \lambda\sigma} \right\}\eta^{\sigma} = \nabla_{\lambda}\eta^{\rho}\, p^{a}_{\rho} + p^{a}_{\lambda,\rho}\,\eta^{\rho}.$$

If we differentiate $p^{a}_{\rho}(l(u))\, L^{\rho}_{b}(u) = \delta^{a}{}_{b}$ and consider it at $u = u_0$, then we get

$$p^{a}_{\rho}L^{\rho}_{b,c} = -p^{a}_{\rho,\sigma}L^{\rho}_{b}L^{\sigma}_{c}.$$

Thus we have at the points x_0 and $u_0 = p(x_0)$

(4.19) $$p^{a}_{\rho}\left\{ {\rho \atop \lambda\mu} \right\}_{x_0} = p^{b}_{\lambda}\, p^{c}_{\mu}\left\{ {a \atop bc} \right\}'_{u_0} + p^{a}_{\lambda,\mu} + (\nabla_{\lambda}\eta^{\rho}\,\eta_{\mu} + \nabla_{\mu}\eta^{\rho}\,\eta_{\lambda})\, p^{a}_{\rho}.$$

Let $u = (u_{a_1 \cdots a_p})$ be a p-form on the base space B^{n-1}, and put $\bar{u} = p^{*}u$. Then the p-form $\bar{u}$ on M^n has the coefficients

(4.20)
$$\overline{u}_{\lambda_1\cdots\lambda_p} = p_{\lambda_1}^{a_1}\cdots p_{\lambda_p}^{a_p} u_{a_1\cdots a_p}\,.$$

It is well known that $\overline{u}$ satisfies

(4.21)
$$\mathrm{i}(\eta)\,\overline{u} = 0, \qquad d\overline{u} = p^* du\,.$$

We calculate $\delta\overline{u}$ in the following. At the points x_0 and u_0, we see

$$\begin{aligned}
\nabla_\mu \overline{u}_{\lambda_1\cdots\mu_p}(x_0) &= \partial_\mu \overline{u}_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p} \left\{ {\sigma \atop \mu\lambda_i} \right\}_{x_0} \overline{u}_{\lambda_1\cdots\overset{i}{\sigma}\cdots\lambda_p} \\
&= \sum_{i=1}^{p} (p_{\lambda_1}^{a_1}\cdots p_{\lambda_i,\mu}^{a_i}\cdots p_{\lambda_p}^{a_p})\, u_{a_1\cdots a_p}(u_0) + (p_{\lambda_1}^{a_1}\cdots p_{\lambda_p}^{a_p})\, p_\mu^b\, \partial_b u_{a_1\cdots a_p}(u_0) \\
&\qquad - \sum_{i=1}^{p} \left\{ {\sigma \atop \mu\lambda_i} \right\}_{x_0} p_\sigma^{a_i} (p_{\lambda_1}^{a_1}\cdots \widehat{i} \cdots p_{\lambda_p}^{a_p})\, u_{a_1\cdots a_p} \\
&= p_{\lambda_1}^{a_1}\cdots p_{\lambda_p}^{a_p}\, p_\mu^a\, \nabla'_a u_{a_1\cdots a_p}(u_0) - \sum_{i=1}^{p} (\nabla_\mu \eta^\rho \eta_{\lambda_i} + \nabla_{\lambda_i}\eta^\rho \eta_\mu)\, \overline{u}_{\lambda_1\cdots\overset{i}{\rho}\cdots\lambda_p}(x_0).
\end{aligned}$$

Contracting this by $g^{\mu\lambda_1}$, we have

$$\begin{aligned}
(\delta\overline{u})_{\lambda_2\cdots\lambda_p} &= (p^*\delta u)_{\lambda_2\cdots\lambda_p} + \sum_{i=1}^{p} g^{\mu\lambda_1}(\varphi_\mu{}^\rho \eta_{\lambda_i} + \varphi_{\lambda_i}{}^\rho \eta_\mu)\, \overline{u}_{\lambda_1\cdots\overset{i}{\rho}\cdots\lambda_p} \\
&= (p^*\delta u)_{\lambda_2\cdots\lambda_p} + g^{\mu\lambda} \varphi_\mu{}^\rho \eta_\lambda \overline{u}_{\rho\lambda_2\cdots\lambda_p} + \sum_{i=2}^{p} g^{\mu\lambda}\varphi_\mu{}^\rho \eta_{\lambda_i} \overline{u}_{\lambda\lambda_2\cdots\overset{i}{\rho}\cdots\lambda_p} \\
&\qquad + g^{\mu\lambda}\varphi_\lambda{}^\rho \eta_\mu \overline{u}_{\rho\lambda_2\cdots\lambda_p} + \sum_{i=2}^{p} g^{\mu\lambda}\varphi_{\lambda_i}{}^\rho \eta_\mu \overline{u}_{\lambda\lambda_2\cdots\overset{i}{\rho}\cdots\lambda_p} \\
&= (p^*\delta u)_{\lambda_2\cdots\lambda_p} + (e(\eta)\Lambda\overline{u})_{\lambda_2\cdots\lambda_p}\,.
\end{aligned}$$

Thus we have

(4.22)
$$\delta\overline{u} = p^*\delta u + e(\eta)\,\Lambda u\,.$$

at every points x in M^n and $p(x)$ in B^{n-1}.

For any harmonic p-form u on the base space, we see from (4.21) and (4.22) that the p-form $\overline{u}=p^*u$ satisfies

$$d\overline{u} = 0, \quad \delta\overline{u} = e(\eta)\,\Lambda\overline{u}\,,$$

and hence $\overline{u}$ is a C-harmonic p-form. Conversely, for any C-harmonic p-form w on M^n, $i(\eta)w = 0$ and $\theta(\eta)w = 0$ are valid if $p \leqq m$. Therefore there exists a p-form w' on B^{n-1} such that $w=p^*w'$. Then from (4.21) and (4.22) again we see that w must be harmonic. Consequently we have proved

THEOREM 4.1. *In a compact regular Sasakian space M^{2m+1}, let B^{2m} be the base space of the fibering of Boothby-Wang. Then the vector space of C-harmonic p-forms on M^{2m+1} is isomorphic to the vector space of harmonic p-forms on B^{2m} if $p\leqq m$.*

Thus we have $\dim H_p(B)(=b_p(B))=\dim C_p(M)$, if $p\leqq m$. Taking account of Theorem 3.3, we can obtain Tanno's theorems again.

COROLLARY 4.1.1. *In the same condition as Theorem 4.1, we have*

$$b_p(M) = b_p(B) - b_{p-2}(B), \quad 2\leqq p\leqq m.$$

COROLLARY 4.1.2. *In the same condition as Theorem 4.1, the vector space of harmonic 1-forms of M^{2m+1} and that of B^{2m} is isomorphic.*

5. C^*-harmonic forms. Let M^n be an $n(=2m+1)$-dimensional compact Sasakian space. As a dual form of a harmonic form in a Riemannian space is also harmonic, it is natural to ask for the properties of a dual form of a C-harmonic form in a Sasakian space.

We shall call a form u to be C^*-harmonic if it satisfies

$$du = i(\eta)\, Lu,$$

$$\delta u = 0.$$

From the definition, the following theorem is evident.

THEOREM 5.1. *In a Sasakian space, a p-form u is C-harmonic if and only if the $(n-p)$-form $*u$ is C^*-harmonic.*

Therefore the dual form $ce(\eta)L^{m-k}\cdot 1$ (where c is a constant) of $L^k\cdot 1$ is a C^*-harmonic form. By virtue of Theorem 2.2 we see that for any C^*-harmonic p-form u $(p\geqq m+1)$ it holds

$$e(\eta)\, u = 0.$$

Moreover we see from Theorem 2.1 and Lemma 1.3,

$$\theta(\eta)\, u = 0, \tag{5.1}$$

for any C^*-harmonic p-form u (p is arbitrary). In the proof of Theorem 2.2 we have $Li(\eta)u=0$ for any C-harmonic p-form u, and therefore we have $\Lambda e(\eta)u=0$ for any C^*-harmonic form u. We denote by $C^*{}_p$ the vector space of all C^*-harmonic p-forms.

LEMMA 5.1. *In a compact Sasakian space, we have*

$$H_p = C_p \cap C_p^*$$

for an arbitrary p.

PROOF. It is evident from the definition that $C_p \cap C_p^*$ is included in H_p. Conversely let u be a harmonic p-form. If $p \leqq m$, then we have $i(\eta)u = 0$, and $\Lambda u = 0$. Therefore

$$e(\eta)\Lambda u = 0, \quad i(\eta)Lu = 0 \tag{5.2}$$

hold good. Hence u is both C-harmonic and C^*-harmonic. If $p \geqq m+1$, then we have $e(\eta)u = 0$, from which (5.2) follows too, and $H_p \subset C_p \cap C_p^*$ is proved.

LEMMA 5.2. *Let u be a p-form in $C_p \cup C_p^*$. Then $e(\eta)u$ is a C^*-harmonic form, and $i(\eta)u$ is a C-harmonic form. The mapping $e(\eta)|C_p$ is an into isomorphism and $i(\eta)|C_{p+1}^*$ is a homomorphism onto C_p, if $p \leqq m$.*

PROOF. Let u be a C-harmonic p-form, then we have by virtue of Lemma 2.2 $\delta(e(\eta)u) = 0$. We have $Li(\eta)u=0$. As $du=0$ we get $d(e(\eta)u)=Lu$, and we have

$$Lu = Li(\eta)e(\eta)u = i(\eta)L(e(\eta)u).$$

Hence $e(\eta)u$ is C^*-harmonic. If v is C^*-harmonic, then we have

$$\delta(e(\eta)v) = -e(\eta)\delta v = 0,$$

$$d(e(\eta)v) = Lv - e(\eta)dv = Lv - e(\eta)(i(\eta)Lv)$$

$$= i(\eta)L(e(\eta)v),$$

which shows that $e(\eta)v$ is also C^*-harmonic. Moreover if $e(\eta)u = 0$ for a C-harmonic p-form u ($p \leqq m$), then we have

$$u = e(\eta)i(\eta)u + i(\eta)e(\eta)u = 0.$$

Therefore $e(\eta)$ is an isomorphism of C_p into C^*_{p+1} ($p \leqq m$). In the same way, we can prove the statement with respect to $i(\eta)$.

THEOREM 5.2. *In a compact* $(2m+1)$*-dimensional Sasakian space, it holds*

$$C_p^* = H_p \oplus e(\eta)\, C_{p-1}$$

if $p \leqq m$.

PROOF. The vector space H_p, and $e(\eta)C_{p-1}$ are the subspaces of C_p^* and $H_p \cap e(\eta)C_{p-1} = (0)$ if $p \leqq m$. For any C^*-harmonic form u we decompose it as

$$u = i(\eta)(e(\eta)u) + e(\eta)(i(\eta)u)\,.$$

Then $e(\eta)u$ is a C^*-harmonic $(p+1)$-form, and we see $i(\eta)e(\eta)u$ belongs to C_p. Similarly, as $i(\eta)u$ is a C-harmonic $(p-1)$-form, $e(\eta)\,i(\eta)u$ is C^*-harmonic. Therefore $i(\eta)e(\eta)u = u - e(\eta)\,i(\eta)u$ is at the same time C-harmonic and C^*-harmonic, hence belongs to H_p. Thus the theorem is proved.

COROLLARY 5.2.1. *In a compact* $(2m+1)$*-dimensional Sasakian space, the relation*

$$H_p = i(\eta)\, e(\eta)\, C_p^*$$

is valid for $p \leqq m$. *Hence* $b_p = 0$ *if and only if* $e(\eta)C_p^* = 0$ *for* $p \leqq m$.

COROLLARY 5.2.2. *In a compact* $(2m+1)$*-dimensional Sasakian space, if* u *is a C-harmonic form* ($p \leqq m$), *then* δu *is* C^**-harmonic. If* u *is a* C^**-harmonic p-form* ($p \leqq m$), *then* du *is C-harmonic.*

PROOF. The first half is an easy result from Theorem 2.3 and Lemma 4.2. Let u be a C^*-harmonic p-form ($p \leqq m$), then there exsits a harmonic p-form ψ and a C-harmonic $(p-1)$-form w such that $u = \psi + e(\eta)w$. Hence we have $du = de(\eta)w = Lw$, which is C-harmonic.

COROLLARY 5.2.3. *In a compact* $(2m+1)$*-dimensional Sasakian space, if a p-form* u ($p \leqq m+1$) *is* C^**-harmonic, then* Λu *is also* C^**-harmonic.*

COROLLARY 5.2.4. *In a compact* $(2m+1)$*-dimensional Sasakian space,* *we suppose that a p-form u* ($p \leqq m-2$) *is C^*-harmonic. Then Lu is also C^*-harmonic if and only if* $e(\eta)u=0$.

PROOF. Let u be a C^*-harmonic p-form ($p \leqq m-2$). If $e(\eta)u = 0$, then $Lu=e(\eta)du$ is also C^*-harmonic by virtue of Corollary 5.2.2 and Lemma 5.2. Conversely we assume that Lu is a C^*-harmonic form. There exist a harmonic p-form ψ and a C-harmonic $(p-1)$-form w such that $u=\psi+e(\eta)w$. Hence we have $L\psi=Lu-e(\eta)Lw$ is C^*-harmonic. Since a harmonic form ψ is C-harmonic, $L\psi$ is also C-harmonic. Therefore $L\psi$ is again a harmonic $(p+2)$-form. As $p+2 \leqq m$, we have $\Lambda L\psi = 0$. Now ΛL is an automorphism of the vector space which consists of the p-forms v such that $i(\eta)v=0$ if $p \leqq m-1$ (see [1]). Consequently we have $\psi=0$. This shows that $u=e(\eta)w$ and hence $e(\eta)u=0$.

By virtue of Theorem 3.1 and Theorem 4.2, we can set the following decomposition theorem for the C^*-harmonic form.

THEOREM 5.3. *In a compact* $(2m+1)$*-dimensional Sasakian space, any C^*-harmonic p-form u_p ($p \leqq m$) can be written uniquely in the form*:

$$u_p = \psi_p + \sum_{k=0}^{r} e(\eta)L^k\psi_{p-1-2k},$$

where ψ_p and ψ_{p-1-2k} are harmonic forms and r is the integral part of $(p-1)/2$. *Conversely any form written as in the right hand side is C^*-harmonic.*

Yano-Bochner [8] has defined the Killing p-form which can be considered as a natural extension of Killing 1-form. We show in the following an example of a Killing p-form where p is odd and ask for some relations between Killing forms and C^*-harmonic forms in a Sasakian space.

A p-form $v_{\lambda_1\cdots\lambda_p}$ in a Riemannian space is called to be Killing if its covariant derivative $\nabla_\mu v_{\lambda_1\cdots\lambda_p}$ is skew-symmetric in the indices $(\mu, \lambda_1, \cdots, \lambda_p)$. Therefore a p-form $v_{\lambda_1\cdots\lambda_p}$ is a Killing form if and only if it satisfies

$$(dv)_{\lambda_0\cdots\lambda_p} = (p+1)\nabla_{\lambda_0}v_{\lambda_1\cdots\lambda_p}. \tag{5. 3}$$

In the first place we show the following

THEOREM 5.4. *In a Sasakian space, the* $(2k+1)$*-form*

$$u^{(k)} = e(\eta)L^k\cdot 1$$

are Killing forms, where $k=0, 1, \cdots, m$.

To prove this we prepare some lemmas.

LEMMA 5.3. *In a Sasakian space, the* $(p+2)$*-tensor* $(p=2k, k\geqq 1)$

$$A^{(k)}_{\sigma\rho\lambda_1\cdots\lambda_p}=\sum_{i=1}^{p}\varphi_{\sigma\lambda_i}\varphi^{k}_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}$$

is skew-symmetric in the indices (σ, ρ), *where* $\varphi^{k}_{\lambda_1\cdots\lambda_p}$ *is the coefficients of the* $2k$*-form* $\varphi \overbrace{\wedge\cdots\wedge}^{k} \varphi$.

PROOF. For $k=1$, $\varphi_{\sigma\lambda_1}\varphi_{\rho\lambda_2}+\varphi_{\sigma\lambda_2}\varphi_{\lambda_1\rho}$ is clearly skew-symmetric in (σ, ρ). We assume that the lemma is true for $k=1, \cdots, k-1$, then

$$A^{(k-1)}_{\sigma\rho\lambda_3\cdots\lambda_p}=\sum_{j=3}^{p}\varphi_{\sigma\lambda_j}\varphi^{k-1}_{\lambda_3\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p},$$

$$B^{(h)}_{\sigma\rho\lambda\lambda_3\cdots\lambda_p}=\sum_{3\leqq j\neq h}\varphi_{\sigma\lambda_j}\varphi^{k-1}_{\lambda_3\cdots\overset{h}{\hat{\lambda}}\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}+\varphi_{\sigma\lambda}\varphi^{k-1}_{\lambda_3\cdots\overset{h}{\hat{\rho}}\cdots\lambda_p} \quad \text{for} \quad h\geqq 3,$$

$$C^{(hl)}_{\sigma\rho\lambda\mu\lambda_3\cdots\lambda_p}=\sum_{j\neq h,\, l\geqq 3}\varphi_{\sigma\lambda_j}\varphi^{k-1}_{\lambda_3\cdots\overset{h}{\hat{\lambda}}\cdots\overset{l}{\hat{\mu}}\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}+\varphi_{\sigma\lambda}\varphi^{k-1}_{\lambda_3\cdots\overset{h}{\hat{\rho}}\cdots\overset{l}{\hat{\mu}}\cdots\lambda_p}+\varphi_{\sigma\mu}\varphi^{k-1}_{\lambda_3\cdots\overset{h}{\hat{\lambda}}\cdots\overset{l}{\hat{\rho}}\cdots\lambda_p}$$

$$\text{for} \quad h, l \quad (3\leqq h<l)$$

are all skew-symmetric in (σ, ρ). Calculating $A^{(k)}$ directly we have

$$\begin{aligned}
A^{(k)}_{\sigma\rho\lambda_1\cdots\lambda_p}&=\varphi_{\sigma\lambda_1}\varphi^{k}_{\rho\lambda_2\cdots\lambda_p}-\varphi_{\sigma\lambda_2}\varphi^{k}_{\rho\lambda_1\cdots\lambda_p}+\sum_{j=3}^{p}\varphi_{\sigma\lambda_j}\varphi^{k}_{\lambda_1\cdots\overset{j}{\hat{\rho}}\cdots\lambda_p}\\
&=\varphi_{\lambda_1\lambda_2}A^{(k-1)}_{\sigma\rho\lambda_3\cdots\lambda_p}-\sum_{h=3}^{p}\varphi_{\lambda_1\lambda_h}B^{(h)}_{\sigma\rho\lambda_2\cdots\lambda_p}+\sum_{h=3}^{p}\varphi_{\lambda_2\lambda_h}B^{(h)}_{\sigma\rho\lambda_1\lambda_3\cdots\lambda_p}+\sum_{3\leqq h<l}\varphi_{\lambda_h\lambda_l}C^{(hl)}_{\sigma\rho\lambda_1\cdots\lambda_p}\\
&\quad+(\varphi_{\sigma\lambda_1}\varphi_{\rho\lambda_2}-\varphi_{\sigma\lambda_2}\varphi_{\rho\lambda_1})\varphi^{k-1}_{\lambda_3\cdots\lambda_p}-\sum_{j=3}^{p}(\varphi_{\sigma\lambda_1}\varphi_{\rho\lambda_j}-\varphi_{\rho\lambda_1}\varphi_{\sigma\lambda_j})\varphi^{k-1}_{\lambda_3\cdots\overset{j}{\hat{\lambda}}_2\cdots\lambda_p}\\
&\quad+\sum_{j=3}^{p}(\varphi_{\sigma\lambda_2}\varphi_{\rho\lambda_j}-\varphi_{\rho\lambda_2}\varphi_{\sigma\lambda_j})\varphi^{k-1}_{\lambda_3\cdots\overset{j}{\hat{\lambda}}_1\cdots\lambda_p}-\sum_{3\leqq j<h}(\varphi_{\sigma\lambda_j}\varphi_{\rho\lambda_h}-\varphi_{\sigma\lambda_h}\varphi_{\rho\lambda_j})\varphi^{k-1}_{\lambda_3\cdots\overset{j}{\hat{\lambda}}_1\cdots\overset{h}{\hat{\lambda}}_2\cdots\lambda_p}.
\end{aligned}$$

The latter four terms are clearly skew-symmetric in (σ, ρ), hence so is $A^{(k)}_{\sigma\rho\lambda_1\cdots\lambda_p}$.

LEMMA 5.4. *In a Sasakian space, we have*

$$\nabla_\rho(Lu^{(k)})=L(\nabla_\rho u^{(k)}),$$

where the form $\nabla_\rho u$ is defined by $(\nabla_\rho u)_{\lambda_1\cdots\lambda_p} = \nabla_\rho u_{\lambda_1\cdots\lambda_p}$ for a p-form u.

PROOF. Put $\varphi = (1/2)d\eta$. For $p(=2k+1)$-form $u^{(k)}$, we have $e(\eta)u^{(k)}=0$. Hence it follows that

$$\begin{aligned}\nabla_\rho(\varphi\wedge u^{(k)})_{\sigma\tau\lambda_1\cdots\lambda_p} &= (\varphi\wedge\nabla_\rho u^{(k)})_{\sigma\tau\lambda_1\cdots\lambda_p} + \nabla_\rho\varphi_{\sigma\tau}u^{(k)}_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\nabla_\rho\varphi_{\sigma\lambda_i}u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p}\\ &\quad - \sum_{i=1}^{p}\nabla_\rho\varphi_{\lambda_i\tau}u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j}\nabla_\rho\varphi_{\lambda_i\lambda_j}u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\tau}}\cdots\lambda_p}\\ &= g_{\rho\tau}(e(\eta)u^{(k)})_{\sigma\lambda_1\cdots\lambda_p} - g_{\rho\sigma}(e(\eta)u^{(k)})_{\tau\lambda_1\cdots\lambda_p}\\ &\quad - \sum_{i=1}^{p}g_{\rho\lambda_i}(e(\eta)u^{(k)})_{\sigma\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p} + (\varphi\wedge\nabla_\rho u^{(k)})_{\sigma\tau\lambda_1\cdots\lambda_p}\\ &= (\varphi\wedge\nabla_\rho u^{(k)})_{\sigma\tau\lambda_1\cdots\lambda_p}.\end{aligned}$$

PROOF OF THEOREM 5.4. We prove it by the induction again. In case $k=0$, $u^{(0)}=\eta$ is a Killing form. Assuming that the theorem is true for $k=0, 1, \cdots, k$, we set $p=2k+1$. Then the p-form $u^{(k)}$ satisfies

$$(du^{(k)})_{\lambda_0\cdots\lambda_p} = (p+1)\nabla_{\lambda_0}u^{(k)}_{\lambda_1\cdots\lambda_p} = 2\varphi^{k+1}_{\lambda_0\cdots\lambda_p}. \tag{5.4}$$

We have by virtue of Lemma 5.4

$$\begin{aligned}\nabla_\rho(\varphi\wedge u^{(k)})_{\sigma\tau\lambda_1\cdots\lambda_p} &= \varphi_{\sigma\tau}\nabla_\rho u^{(k)}_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\sigma\lambda_i}\nabla_\rho u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p}\\ &\quad - \sum_{i=1}^{p}\varphi_{\lambda_i\tau}\nabla_\rho u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j}\varphi_{\lambda_i\lambda_j}\nabla_\rho u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\tau}}\cdots\lambda_p},\end{aligned}$$

then the latter two terms in the right hand side is skew-symmetric in (σ, ρ) by the assumption. Considering (5.4), we have

$$\begin{aligned}&\varphi_{\sigma\tau}\nabla_\rho u^{(k)}_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\sigma\lambda_i}\nabla_\rho u^{(k)}_{\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p}\\ &= \frac{2}{p+1}\left[\varphi_{\sigma\tau}\varphi^{k+1}_{\rho\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p}\varphi_{\sigma\lambda_i}\varphi^{k+1}_{\rho\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p}\right]\\ &= \frac{2}{p+1}\sum_{\alpha=0}^{p}\varphi_{\sigma\lambda_\alpha}\varphi^{k+1}_{\lambda_0\cdots\overset{\alpha}{\hat{\rho}}\cdots\lambda_p},\end{aligned}$$

where we set $\lambda_0 = \tau$. This is skew-symmetric in (σ, ρ) from Lemma 5.3. Hence we see that $\nabla_\rho (u^{(k+1)})_{\sigma\lambda_0\cdots\lambda_p}$ is also skew-symmetric in the indices (ρ, σ), and the theorem is proved.

Now the Killing form $u^{(k)} = e(\eta) L^{(k)} \cdot 1$ satisfies $e(\eta) u^{(k)} = 0$. Though how many of the Killing forms satisfy this condition is not clear, we next only concern about such Killing forms in a Sasakian space. Then we can see that there exists a relation between Killing forms and C^*-harmonic forms.

Let u be a Killing form and assume that it satisfies $e(\eta)u=0$. From the definition of Killing form, we have easily

$$\delta u = 0,$$

$$i(\eta)\, du = (p+1)\, \nabla_\eta u\,.$$

We get

$$\theta(\eta)\, u = 0 \tag{5.5}$$

for such a Killing form from (1.18). Then we see

THEOREM 5.5. *In a Sasakian space, if a Killing p-form u satisfies $e(\eta)u=0$, then u is C^*-harmonic and $i(\eta)u$ is C-harmonic for all p.*

PROOF. We have from (5.5)

$$di(\eta)\, u = -i(\eta)\, du = -(p+1)\, \nabla_\eta u\,.$$

As $e(\eta)u = 0$, it holds $u = e(\eta)\, i(\eta) u$. Then

$$du = Li(\eta)\, u - e(\eta)(-(p+1)\, \nabla_\eta u) = Li(\eta)\, u\,.$$

This shows with $\delta u=0$ that u is *a* C^*-harmonic p-form. We have, therefore, $-i(\eta)\, du = di(\eta)\, u = 0$. Moreover we get $\delta(i(\eta)u) = \Lambda u$ and $e(\eta)\Lambda(i(\eta)u) = \Lambda u$, hence $i(\eta)\, u$ is a C-harmonic $(p-1)$-form.

COROLLARY 5.5.1. *In a Sasakian space, if a Killing form u satisfies $e(\eta)u=0$, then*

$$\nabla_\eta u = 0\,, \quad \Phi u = 0$$

are valid.

COROLLARY 5.5.2. *In a Sasakian space, we assume that a Killing form u satisfies $e(\eta)u=0$. Then u is effective if and only if $i(\eta)u$ is harmonic.*

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OCHANOMIZU UNIVERSITY
TOKYO, JAPAN.