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ON THE RETRACTIONS OF STUNTED PROJECTIVE SPACES

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Introduction. Let F denote the field of real numbers (i.e.R), or the field of complex numbers (i.e.C) and FP^n the real projective n-space RP^n or the complex projective n-space CP^n, according as $F=R$ or C. Now we consider stunted projective spaces FP^n/FP^m, FP^k/FP^m $(0<m<n<k)$, and the natural inclusion map $i: FP^n/FP^m \to FP^k/FP^m$. In this note, we give necessary and sufficient conditions for m, n and k under which FP^n/FP^m is a retract of FP^k/FP^m, i.e. i has a left inverse, or equivalently i has a left homotopy inverse $r: FP^k/FP^m \to FP^n/FP^m$.

In **2**, we treat the real case. For this case, the result of J.F. Adams [1] is fundamental, moreover, the problem is almost proved in [1], for the case where $m=n-1$.

Next, in **3**, we treat the complex case. For this case, the result of M.F. Atiyah [5] is known. We show that the problem is completely reduced to the known results in [2] and [3].

1. Preliminaries. All spaces considered in this paper are connected finite CW-complexes with base points $*$, and all maps are base-point-preserving.

Moreover, $[X, Y]$ denotes the set of base-point-preserving homotopy classes of maps from X to Y.

And X^n denotes the n-skeleton of X.

Following proposition follows from the "Suspension Theorem" of [9].

PROPOSITION 1.1. *Let X be an n-dimensional CW-complex and Y be an m-connected CW-complex* $(m \geqq 1)$.

If $2m \geqq n$, then the suspension map

$$S: [X, Y] \longrightarrow [SX, SY]$$

is bijective.

Now we give two results on the reducibility and co-reducibility. For the proof of main objects in **2** and **3**, we need not Proposition 1.3.

PROPOSITION 1.2. *Let X be an n-dimensional and simply connected CW-complex with one n-cell ($n>2$). Consider the maps*

$$X^{n-1} \xrightarrow{\ i\ } X \xrightarrow{\ \pi\ } X/X^{n-1} = S^n,$$

where i is the natural inclusion, and π is the natural projection. Then i has a left homotopy inverse r if and only if X is reducible, that is, π has a right homotopy inverse

$$s : S^n \longrightarrow X$$

PROOF. Suppose that π has a right homotopy inverse s. Then, since X^{n-1} is connected, we can assume that

$$i(*) = s(*).$$

We define a map

$$i \vee s : X^{n-1} \vee S^n \longrightarrow X$$

as follows:

$$(i \vee s)(x, *) = i(x) \qquad \text{for } x \in X^{n-1},$$
$$(i \vee s)(*, y) = s(y) \qquad \text{for } y \in X/X^{n-1} = S^n.$$

Now, we consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{H}_*(X^{n-1} \vee S^n) \arrow[rr, "\approx"] \arrow[dr, "(i\vee s)_*"'] & & \widetilde{H}_*(X^{n-1}) + \widetilde{H}_*(S^n) \arrow[dl, "i_* + s_*"] \\ & \widetilde{H}_*(X) & \end{tikzcd}$$

where $\approx$ is the canonical isomorphism.

As $\pi \cdot s$ is homotopic to the identity on S^n, we have

$$H_*(X) = \text{Ker } \pi_* + \text{Im } s_*,$$

and π_* is onto, and s_* is one-to-one. Furthermore, from the exactness of the sequence

$$H_{r+1}(X) \xrightarrow{\pi_*} H_{r+1}(X/X^{n-1}) \longrightarrow H_r(X^{n-1})$$
$$\xrightarrow{i_*} H_r(X) \xrightarrow{\pi_*} H_r(X/X^{n-1}),$$

we can see that $\mathrm{Ker}\pi_* = \mathrm{Im} i_*$, and that i_* is one-to-one. Thus $i_* + s_* : \tilde{H}_*(X^{n-1}) + \tilde{H}_*(S^n) \longrightarrow \tilde{H}_*(X)$ is an isomorphism.

Hence, $(i \vee s)_* : H_*(X^{n-1} \vee S^n) \longrightarrow H_*(X)$ is an isomorphism, from the above diagram.

As these two spaces are simply connected, it follows that $X^{n-1} \vee S^n$ and X have the same homotopy type. Therefore i has a left homotopy inverse.

Conversely, assume that i has a left homotopy inverse. By the homotopy extension property, it follows that X^{n-1} is a retract of X. By primary obstruction theory we can see that the attaching map of the n-dimensional cell of X is null homotopic. Hence π has a right homotopy inverse.

PROPOSITION 1.3. *Let dim. X be less than $2n$ $(1 < n)$ and $X^n = S^n$. Consider the maps,*

$$X^n = S^n \xrightarrow{i} X \xrightarrow{\pi} X/X^n,$$

where i is the natural inclusion and π is the natural projection.

Then X is co-reducible, that is, i has a left homotopy inverse if and only if π has a right homotopy inverse.

PROOF. Suppose that i has a left homotopy inverse r. Then, $r_* \cdot i_*$ is the identity on $H_*(S^n)$, and, by use of exactness of the homology sequence of the pair (X, X^n), it follows, $r_* : H_k(X^n) \to H_k(S^n)$ is an isomorphism for $k \leqq n$, and $\pi_* : H_k(X) \to H_k(X/X^n)$ is an isomorphism for $k > n$.

Next, let $p_1 : S^n \times (X/X^n) \to S^n$, $p_2 : S^n \times (X/X^n) \to X/X^n$, be the natural projections, then $p_{1*} : H_k(S^n \times (X/X^n)) \to H_k(S^n)$ is an isomorphism for $k \leqq n$ and $p_{2*} : H_k(S^n \times (X/X^n)) \to H_k(X/X^n)$ is an isomorphism for $n < k < 2n$.

Now consider the maps

$$X \xrightarrow{d} X \times X \xrightarrow{r \times \pi} S^n \times (X/X^n),$$

where d is the diagonal map, then $r = p_1 \cdot (r \times \pi) \cdot d$, $\pi = p_2 \cdot (r \times \pi) \cdot d$.

Thus $(\pi \times r)_* \cdot d_* : H_k(X) \to H_k(S^n \times (X/X^n))$ is an isomorphism for $k < 2n$.

On the other hand, since dim X is less than $2n$, $(S^n \times (X/X^n))^{2n} = S^n \vee (X/X^n)$.

Hence there exists a cellular map $h : X \to S^n \vee (X/X^n)$ such that $i' \cdot h \simeq (r \times \pi) \cdot d$, where, $i' : S^n \vee (X/X^n) \to S^n \times (X/X^n)$ is the natural inclusion, which induces the isomorphism of the homology groups in dimensions less than 2n.

Thus $h_* : H_*(X) \to H_*(S^n \vee (X/X^n))$ is an isomorphism, so it follows that $S^n \vee (X/X^n)$ and X have the same homotopy type. Therefore, π has a right homotopy inverse.

Conversely, assume that π has a right homotopy inverse. Then it follows that $S^n \vee (X/X^n)$ and X have the same homotopy type, from the analogous arguments to the proof of Proposition 1.2. Hence, i has a left homotopy inverse.

2. The real case. From the cohomology structure of the real projective spaces (see[10]), and the cohomology exact sequence for the pair (RP^n, RP^m), we have

PROPOSITION 2.1

$$\begin{aligned} \tilde{H}^s(RP^n/RP^m : Z_2) &= Z_2, \textit{ for } m < s \leqq n, \\ &= 0, \textit{ otherwise.} \end{aligned}$$

For the generator $w_s \in \tilde{H}^s(RP^n/RP^m : Z_2)$, *we have*

$$\begin{aligned} Sq^i w_s &= \binom{s}{i} w_{s+i} && \textit{for } s+i \leqq n, \\ &= 0 && \textit{for } s+i > n\,. \end{aligned}$$

PROPOSITION 2.2. *If the map*

$$i : RP^n/RP^m \longrightarrow RP^k/RP^m$$

has a left homotopy inverse

$$r : RP^k/RP^m \longrightarrow RP^n/RP^m,$$

then n is even and, $m=n-1$ or $k=n+1$.

PROOF. If i has a left homotopy inverse r, then $i^* \cdot r^*$ must be the identity on $\tilde{H}^*(RP^n/RP^m : Z_2)$. By Proposition 2.1 and the fact that each $S_q{}^i$ commutes with $i^* \cdot r^*$, it follows that n is even and

$$\begin{aligned} n &\equiv 0 \ (4), && \text{if } k > n+1, \\ n &\equiv 2 \ (4), && \text{if } m < n-1\,. \end{aligned}$$

Therefore, the proposition follows immediately.

By this proposition, we have only to consider the cases where $m=n-1$ and $k=n+1$.

(*A*) *The case* $m=n-1$. In this case, $RP^n/RP^m=S^n$, and we have

THEOREM 2.3. *The map*

$$i: S^n \longrightarrow RP^k/RP^{n-1}$$

has a left homotopy inverse, if and only if $k<n+\rho(n)$, *where* $\rho(n)$ *is an integer defined in* [1].

PROOF. The "only if" part is shown by J.F. Adams [1].

Suppose that $k<n+\rho(n)$. From [6], S^{n-1} admits a tangent $(k-n)$-frame. Then, for any positive integer p, S^{pn-1} admits a tangent $(k-n)$-frame. Thus, the fiber bundle $V_{pn,k-n+1}\to S^{pn-1}$ admits a cross-section. On the other hand, if $pn\geqq 2(k-n+1)$, the pn-skeleton of $V_{pn,k-n+1}$ is homeomorphic to $RP^{pn-1}/RP^{pn-k+n-2}$ (see[10]), hence $RP^{pn-1}/RP^{pn-k+n-2}$ is reducible. For any positive integer q such that $qr>pn$, $RP^{qr-pn+k-n}/RP^{qr-pn-1}$ is an S-dual of $RP^{pn-1}/RP^{pn-k+n-2}$, where r is order of $J(\xi)$ in $J(RP^{k-n})$, and ξ is the canonical line bundle (see [4]). Hence, $RP^{qr-pn+k-n}/RP^{qr-pn-1}$ is S-co-reducible.

For a sufficiently large integer h, let $p=hr-1$, $q=hn$. Then RP^k/RP^{n-1} is S-co-reducible. Hence, by Proposition 1.1, for $k\leqq 2(n-1)$, if $k<n+\rho(n)$, then i has a left homotopy inverse. Therefore the assertion is proved, for the cases $n\neq 2,4,8$.

Next, for the cases $n=2,4$, and 8, RP^2/RP^1, RP^3/RP^3, and RP^{14}/RP^7 are co-reducible in the above argument. On the other hand, $(V_{4,2})^4=RP^3/RP^1$, $(V_{8,4})^8=RP^7/RP^3$, and $(V_{16,8})^{16}=RP^{15}/RP^7$. Hence, RP^3/RP^1, RP^7/RP^3 and RP^{15}/RP^7 are reducible. Therefore, by Proposition 1.2 the proof is completed.

(*B*) *The case* $k=n+1$.

THEOREM 2.4. *The map*

$$i: RP^n/RP^m \longrightarrow RP^{n+1}/RP^m$$

has a left homotopy inverse if and only if

$$m\geqq \frac{1}{2}n \text{ and } m>n-\rho(n+2).$$

PROOF. Suppose that i has a left homotopy inverse r.

By Proposition 1.2, the natural projection $\pi: RP^{n+1}/RP^m\to RP^{n+1}/RP^n=S^{n+1}$ has a right homotopy inverse. Then, $n+2\geqq 2(n-m+1)$(see [8]), i.e.

$m \geqq \frac{1}{2}n$. And the $(n+2)-$skeleton of the Stiefel manifold $V_{n+2,n-m+1}$ is homeomorphic to RP^{n+1}/RP^m. Thus, the fiber bundle

$$V_{n+2,n-m+1} \longrightarrow S^{n+1}$$

admits a cross-section, hence, $n-m < \rho(n+2)$(see[1]), i.e. $m > n-\rho(n+2)$.

Conversely, suppose that $m \geqq \frac{1}{2} n$ and $m > n-\rho(n+2)$.

Then, the fiber bundle

$$V_{n+2,n-m+1} \longrightarrow S^{n+1}$$

admits a cross-section. Since $n+2 \geqq 2(n-m+1)$, π has a right homotopy inverse. Thus, by Proposition 1.2, the proof is completed.

3. The complex case. From the cohomology structure of the complex projective spaces and the cohomology exact sequence for the pair (CP^n, CP^m), we have the result analogous to Proposition 2.1. So, it follows that Proposition 2.2. is valid for the complex case.

Thus we have only to consider the cases where $m=n-1$ and $k=n+1$.

(A) *The case* $m=n-1$. In this case, $CP^n/CP^m = S^{2n}$, and we have

THEOREM 3.1. *The map*

$$i: S^{2n} \longrightarrow CP^k/CP^{n-1}$$

has a left homotopy inverse, if and only if the coefficients of x^l *in* $\left(\frac{1}{x}\log(1+x)\right)^n$ *are all integers, for* $0 \leqq l \leqq k-n$.

PROOF. Let ξ be the canonical complex line bundle over CP^{k-n}, then the Thom complex $(CP^{k-n})^{n\xi}$ is homeomorphic to CP^k/CP^{n-1}(see [4]).

The following facts are known:

(a) $J(n\xi)=nJ(\xi)=0$ in $J(CP^{k-n})$ if and only if CP^k/CP^{n-1} is S-co-reducible (see [4]).

(b) Atiyah-Todd number M_{k-n+1} is equal to the order of $J(\xi)$ in $J(CP^{k-n})$ (see [2]).

(c) The coefficients of x^l in $\left(\frac{1}{x}\log(1+x)\right)^n$ are all integers, for $0 \leqq l \leqq k-n$ if and only if n is a multiple of M_{k-n+1} (see [3]).

By these facts, the proof of the necessity of the Theorem is trivial.

Conversely, assume that the coefficients of x^l in $\left(\frac{1}{x}\log(1+x)\right)^n$ are all integers, for $0 \leqq l \leqq k-n$. Then, by (b), (c) and the Theorem (6.5) of [4], the complex Stiefel fiber bundle

$$W_{n,k-n+1} \longrightarrow S^{2n-1}$$

admits a cross-section, and hence the fiber bundle $V_{2n,2(k-n+1)} \to S^{2n-1}$ admits a cross-section (see [7]). Thus, $2(k-n+1) \leqq \rho(2n)$ i.e. $2k \leqq \rho(2n)-1+2n-1 \leqq 2(2n-1)$.

On the other hand, by (a), (b), and (c), CP^k/CP^{n-1} is S-co-reducible, and because of $2k \leqq 2(2n-1)$, CP^k/CP^{n-1} is co-reducible, by Proposition 1.1.

(*B*) *The case* $k=n+1$.

THEOREM 3.2. *The map*

$$i: CP^n/CP^m \longrightarrow CP^{n+1}/CP^m$$

has a left homotopy inverse, if and only if m and n satisfy the following condition:

(*) *the coefficients of* x^l *in* $\left(\frac{1}{x}\log(1+x)\right)^{n+2}$ *are all integers, for* $0 \leqq l \leqq n-m$.

PROOF. Suppose that i has a left homotopy inverse, then by Proposition 1.2, CP^{n+1}/CP^m is reducible. Hence, by Theorem (1.6) of [3], (*) follows.

Conversely, assume that m and n satisfy the above condition (*). Then, CP^{n+1}/CP^m is S-reducible (see [2], and [3]).

Thus, by [8], the fiber bundle

$$W_{n+2,n-m+1} \longrightarrow S^{2n+3}$$

admits a cross-section, hence the fiber bundle

$$V_{2(n+2),2(n-m+1)} \longrightarrow S^{2n+3}$$

admits a cross-section. Thus, it follows

$$2(n-m+1) \leqq \rho(2n+4) \text{ i.e., } 2m \geqq 2n+4-\rho(2n+4)-2.$$

Next, for the cases $n \neq 2, 6$, it follows immediately that

$$2n+4-\rho(2n+4)-2 \geqq n, \text{ hence } 2m \geqq n.$$

And, for the cases $n=2$ and 6, by checking the condition (*), we have

$$n=m-1, \text{ hence } 2m \geqq n.$$

So, in every case, $2(n+1) \leqq 2(2m+1)$, thus by Proposition 1.1, CP^{n+1}/CP^m is reducible.
This completes the proof.

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