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ON THE INTEGRAL REPRESENTATION OF SOME FUNCTIONAL ON A VON NEUMANN ALGEBRA

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1. Introduction. The purpose of this paper is to study the integral representation of a normal positive linear functional on a von Neumann algebra. This is a part of reduction theory in von Neumann algebra and it has been studied by many authors. In this paper, we shall show that a normal positive linear functional on a von Neumann algebra has an integral representation by factor states. Before going into the discussions, the author wishes to express his hearty thanks to Prof. M. Fukamiya and Prof. M. Takesaki in the presentation of this paper.

2. Notations and Definitions. Let M be a von Neumann algebra on a Hilbert space H with the predual M_* and the center Z; a positive linear functional ψ on M is included in a positive linear functional φ on M [notation: $\psi \ll \varphi$] if there exists a positive scalar α such that $\varphi - \alpha\psi$ is a positive linear functional on M. Then $\psi \ll \varphi$ if and only if, for each sequence $\{a_n\}$ in M, $\varphi(a_n^*a_n) \to 0$ as $n \to \infty$, implies $\psi(a_n^*a_n) \to 0$ as $n \to \infty$.

By using the above notation, we set the following definition.

DEFINITION 1. Let M be a von Neumann algebra and φ be a positive linear functional. If the normalized form of φ is a pure state, then φ is said to be pure. If, whenever ψ is a positive linear functional such that $\psi \ll \varphi$, there exists an element a_0 in M^+ (that is, the set of all positive elements of M) such that $\psi(a) = \varphi(aa_0)$ for all $a \in M$, then φ is said to be reducible.

Let A be a C^*-algebra with the identity and φ a positive linear functional on A. Putting

$$I_\varphi = \{a \in A\,;\ \varphi(a^*a) = 0\}$$

which is called the left kernel of φ, the quotient space A/I_φ becomes the pre-Hilbert space with the inner product canonically induced by φ. We denote the element of A/I_φ corresponding to $a \in A$ by $\eta_\varphi(a)$. Then we get a Hilbert space H_φ, the completion of A/I_φ, and a cyclic representation π_φ of A, as the left multiplication operators on H_φ.

Let A be a C^*-algebra acting on a Hilbert space H, let K be a subspace of H. If K is invariant under A, then we use a symbol $K\eta A'$ (that is, the set of all commuting operators for A). In particular, if K is a closed subspace, then the projection e from H onto K is an element of A'.

DEFINITION 2. Let A be a C^*-algebra with the identity 1 and φ a state on A. Then, if the weak closure $\widetilde{\pi_\varphi(A)}$ of $\pi_\varphi(A)$ is a factor, φ is called a factor state.

3. **Main theorems**. The purpose of this paper is to show that the normal reducible functional φ is a faithful normal trace on eMe (this notation is due to [1]) where e is the support of φ, and the normal reducible functional on a type I von Neumann algebra M has the integral representation by factor states. However, these factor states do not necessarily induce von Neumann representation, which we shall show.

Now we shall state the explained results in the following form:

THEOREM A. *Let M be a von Neumann algebra of type I with the center Z on a Hilbert space H and let φ be a reducible normal positive linear functional on M. Then φ admits the integral representation on the spectrum X of Z:*

$$(\alpha) \qquad \varphi(a) = \int_X \varphi_\zeta(a)\, d\nu(\zeta) \quad \text{for each } a \in M,$$

which satisfies the following conditions:

(1) ν is the spectral measure ν_ξ on X where the vector ξ is an element of H which arises from restricting φ on Z,

(2) for each $z \in Z$ and $a \in M$, $\varphi_\zeta(za) = z^\wedge(\zeta)\varphi_\zeta(a)$ where $z^\wedge$ is the Gelfand's representation of z,

(3) the mapping $\zeta \to \varphi_\zeta$ is weakly continuous on $\mathrm{supp}(\nu)$,

(4) there exists a non-dense set N in $\mathrm{supp}(\nu) = Y$ such that, for $\zeta \in Y - N$, φ_ζ is a factor state.

Let π_ζ be the canonical representation induced by φ_ζ, then we have

THEOREM B. *Let M be a properly infinite von Neumann algebra of* *type* I *with the separable predual* M_* *and the center* Z *which is non-atomic. Then there exists an element* ζ *of the spectrum* X *of* Z *for which* $\pi_\zeta(M)$ *is not a von Neumann algebra.*

4. **Some lemmas.** To prove our theorems, we shall provide some considerations. A projection e in M is said to be abelian if a von Neumann algebra eMe is abelian. If $z(e)$ is the central support of e, then $Z_{z(e)}$ and eMe is *-isomorphic ([1] p.19, Proposition 2). We define the *-isomorphism Φ from $Z_{z(e)}$ onto eMe as $\Phi(a) = ae$ for $a \in Z_{z(e)}$, and the linear mapping τ_e from M onto $Z_{z(e)}$ as $\tau_e(a) = \Phi^{-1}(eae)$ for each $a \in M$.

For each $\zeta \in X$, the closed two-sided ideal in M generated by ζ will be denote by $[\zeta]$, for which the quotient algebra $M/[\zeta]$ is a C^*-algebra. For any $z \in Z$, $z^\wedge$ denotes the image of z by the Gelfand's representation of z.

LEMMA 1. *For each* ζ *in the set* $\{\zeta \in X;\ z(e)^\wedge(\zeta) = 1\}$, *the functional* $\varphi(a) = \tau_e(a)^\wedge(\zeta)$ *is pure.*

PROOF. It is clear that φ is a non-zero positive linear functional. Let ψ be a positive linear functional on M such that $\psi \ll \varphi$. Since $\varphi([\zeta]) = 0$, $\psi([\zeta]) = 0$. Therefore, there exists a functional ψ_1 on $M/[\zeta]$ such that $\psi_1(a(\zeta)) = \psi(a)$ for all $a \in M$ where $a(\zeta)$ is the element of $M/[\zeta]$ corresponding to a. Furthermore, $\psi(1-e) = 0$; this means that $\psi(a) = \psi(eae)$ for all $a \in M$ by the Schwartz's inequality. Therefore, we have:

$$\begin{aligned}\psi(a) &= \psi(eae) = \psi(\tau_e(a)\, e) = \psi_1(\tau_e(a)^\wedge(\zeta)\, e(\zeta)) \\ &= \varphi(a)\, \psi_1(e(\zeta)) = \varphi(a)\, \psi(e) \quad \text{for all } a \in M.\end{aligned}$$

This shows that ψ is a scalar multiple of φ, hence φ is pure.

Let φ be a normal positive linear functional on M and ψ be any positive linear functional on M such that $\psi \ll \varphi$, then, by the Radon-Nikodym theorem due to S. Sakai [6], there exists a positive element a_0 of M satisfying $\psi(a) = \varphi(a_0 a a_0)$ for all $a \in M$. But, a_0 is not necessarily $\|a_0\| \leqq 1$.

LEMMA 2. *Let* φ *be a faithful normal positive linear functional on* M, (π_φ, H_φ) *the canonical representation of* M *induced by* φ *and* ξ_φ *a cyclic vector for this representation. Then* φ *is reducible on* M *if and only if* ω_{ξ_φ} *is reducible on* $\pi_\varphi(M)$ *where* ω_{ξ_φ} *denote a normal state defined by* $\omega_{\xi_\varphi}(a) = (a\xi_\varphi \mid \xi_\varphi)$ *for all* $a \in \pi_\varphi(M)$.

PROOF. Suppose that φ is reducible. Since φ is faithful, π_φ is a

*-isomorphism and σ-weakly continuous. If ψ is a positive linear functional on the von Neumann algebra $\pi_\varphi(M)$ such that $\psi \ll \omega_{\xi_\varphi}$ and ψ_1 is defined by $\psi_1(a)=\psi(\pi_\varphi(a))$, then ψ_1 is an element of $(M_*)^+$ such that $\psi_1 \ll \varphi$. Obviously, if $\varphi(a_n^* a_n) \to 0$ as $n \to \infty$, then $\psi_1(a_n^* a_n) = \psi(\pi(a_n)^* \pi(a_n)) \to 0$ as $n \to \infty$.

Since φ is reducible, there exists a positive element a_0 of M such that $\psi_1(a) = \varphi(aa_0)$ for all $a \in M$, hence we get

$$\begin{aligned}\psi(\pi_\varphi(a)) &= (\pi_\varphi(aa_0)\,\xi_\varphi \,|\, \xi_\varphi) = (\pi_\varphi(a)\,\pi_\varphi(a_0)\,\xi_\varphi \,|\, \xi_\varphi) \\ &= \omega_{\xi_\varphi}(\pi_\varphi(a)\,\pi_\varphi(a_0)),\end{aligned}$$

which shows that ω_{ξ_φ} is reducible. The above argument can also be applied to prove the converse part. This completes the proof of Lemma 2.

In Lemma 2, the assumsption that φ is faithful is not essential, but its proof will be left for readers.

The following lemma is due to H. Halpern [2].

LEMMA 3. *The vector state ω_ξ is reducible on M if and only if ξ is a trace element for eMe where e =* supp(ω_ξ).

Now we shall show our Prop. 1 which will give us an extended notion of normal reducible functional.

PROPOSITION 1. *Let M be a von Neumann algebra on a Hilbert space H, Z the center of M, φ an element of $(M_*)^+$ and e the support of φ. Then φ is reducible if and only if φ is a faithful normal trace on a von Neumann algebra eMe.*

PROOF. Suppose firstly that φ is a faithful normal trace on a von Neumann algebra *eMe*. If ψ is a positive linear functional such that $\psi \ll \varphi$, then $\psi(a)=\psi(eae)$ for all $a \in M$. Since $\psi \ll \varphi$ on *eMe*, by the Radon-Nikodym theorem due to S. Sakai [6], there eixsts a positive element a_0 of *eMe* such that $\psi(eae) = \varphi(a_0 eae a_0)$ for all a $\in M$. Since φ is a trace on *eMe*, we have

$$\begin{aligned}\psi(a) &= \psi(eae) = \varphi(a_0 eae a_0) = \varphi(eaea_0^2) \\ &= \varphi(eaa_0^2) = \varphi(aa_0^2) \quad \text{for all } a \in M.\end{aligned}$$

This shows that φ is reducible on M.

We shall show the converse. Suppose that φ is a faithful positive linear functional on eMe. Let (π_φ, H_φ) be the canonical representation of eMe induced by φ such that $\varphi(a) = \varphi(eae) = (\pi_\varphi(eae)\xi_0 | \xi_0)$ where ξ_0 is a cyclic vector for $\pi_\varphi(eMe)$. Then, by Lemma 2, ξ_0 is a trace element for $\pi_\varphi(eMe)$. Thus, φ is a normal trace on eMe. This completes the proof of Proposition 1.

LEMMA 4. *Let A be a C^*-algebra with the identity 1, $\{\varphi_i\}_{i=1,2}$ be two states on A and $\{\pi_i\}_{i=1,2}$ be the canonical representations of A induced by $\{\varphi_i\}_{i=1,2}$ respectively. If there exist two equivalent projections $\{e_i\}_{i=1,2}$ in A for which u is the partial isometry with the initial projection e_1 and the final projection e_2, and if $\{e_i\}_{i=1,2}$ satisfy the relations $\varphi_i(1-e_i)=0$, for $i=1,2$ and $\varphi_1(u^*au)=\varphi_2(a)$ for all $a \in A$, then π_1 and π_2 are unitarily equivalent.*

PROOF. For $i=1,2$, let I_i be the left kernel of φ_i, $K_i = A/I_i$ and H_i the completion of K_i with respect to the inner product induced by φ_i. Difine the mapping U of K_2 into K_1 by the following form: $U(\bar{a}^2) = \overline{au}^1$($\bar{a}^i$ is the class of K_i corresponding to a for $i=1,2$). Then, for $a, b \in \bar{a}^2$, we have

$$\begin{aligned}\varphi_1(((a-b)u)^*((a-b)u)) &= \varphi_1(u^*(a-b)^*(a-b)u)\\ &= \varphi_2((a-b)^*(a-b)) = 0\end{aligned}$$

This shows thet U is well-defined; it is clear that U maps K_2 onto K_1.

U has the unique extension to a unitary operator from H_2 onto H_1, because, for each $a,b \in A$, we have

$$\begin{aligned}(\bar{a}^2 | \bar{b}^2)_2 &= \varphi_2(b^*a) = \varphi_1(u^*(b^*a)u)\\ &= \varphi_1((bu)^*(au)) = (U(\bar{a}^2) | U(\bar{b}^2))_1,\end{aligned}$$

hence U is an isometry and has the unitary extension from H_2 onto H_1 (we denote it again U). Furthermore, since we have, for each $a,b \in A$,

$$\pi_1(a)U(\bar{b}^2) = \pi_1(a)(\overline{bu}^1) = \overline{abu}^1,$$

and

$$U(\pi_2(a)(\bar{b}^2)) = U(\overline{ab}^2) = \overline{abu}^1,$$

so we have, for all $a \in A$,

$$\pi_1(a)U = U\pi_2(a),$$

π_1 and π_2 are unitarily equivalent. This completes the proof of Lemma 4.

LEMMA 5. *Let A be a C^*-algebra with the identity* 1, φ *a state on* A *and* $\{\varphi_i\}_{i=1}^n$ *a family of positive linear functionals on* A *with* $\varphi = \sum_{i=1}^n \varphi_i$. *Let* (π_φ, H_φ) *and* (π_i, H_i) *be the canonical representations of* A *induced by* φ *and* $\{\varphi_i\}_{i=1}^n$, *respectively. If there exists a family* $\{e_i\}_{i=1}^n$ *of orthogonal projections in* A *such that* $\varphi_i(1-e_i)=0$ *for* $i = 1, 2, \cdots, n$, *then there exists a family* $\{K_i\}_{i=1}^n$ *of closed subspaces of* H_φ *satisfying the following conditions: for each* $i, j = 1, 2, \cdots, n$,

(i) *if* $i \neq j$, K_i *and* K_j *are mutually orthogonal,*
(ii) K_i *are invariant subspaces under* $\pi_\varphi(A)$,
(iii) $H_\varphi = \sum_{i=1}^n \oplus K_i$ *and*
(iv) $\pi_\varphi | K_i$ *are unitarily equivalent to* π_i.

PROOF. For each $i = 1, 2, \cdots, n$, let $(\ |\ )_\varphi$ and $(\ |\ )_i$ be the inner product for H_φ and H_i, respectively. Let ξ_i be cyclic vectors for $\pi_\varphi(A)$ and $\pi_i(A)$ respectively. If we define a bilinear form $[\pi_\varphi(a)\xi_\varphi | \pi_\varphi(b)\xi_\varphi]$ on the dense subset $\{\pi_\varphi(A)\xi_\varphi\}$ of H_φ such that $[\pi_\varphi(a)\xi | \pi_\varphi(b)\xi_\varphi] = \varphi_i(b^*a)$, then this bilinear form is bounded and so may be extended on H_φ. Therefore, by Riesz' representation theorem, there exists the unique bounded operator t_i on H with $0 \leqq t_i \leqq 1$ such that $\varphi_i(b^*a) = [\pi_\varphi(a)t_i\xi_\varphi | \pi_\varphi(b)t_i\xi_\varphi]$ for every $a, b \in A$, and t_i is an element of $\pi_\varphi(A)'$.

For each $i = 1, 2, \cdots, n$, I_φ and I_i be the left kernel of φ and φ_i respectively and η_φ and η_i the canonical mappings from A onto A/I_φ and A/I_i, respectively. Then we have, for each $a,b \in A$,

$$(\eta_i(a) |\ \eta_i(b))_i = \varphi_i(b^*a) = (t_i\pi_\varphi(a)\xi_\varphi |\ t_i\pi_\varphi(b)\xi_\varphi)_\varphi.$$

Therefore, if we define the operator U_i from $\{\pi_i(A)\xi_i\}$ into $\{\pi_\varphi(A)\xi_\varphi\}$ by $U_i(\pi_i(a)\xi_i) = \pi_\varphi(a)t_i\xi_\varphi$ for each i and $a \in A$, it is a linear isometrical mapping. If we define the closed subspace K_i of H_φ by $K_i = [\pi_\varphi(A)t_i\xi_\varphi] = [t_i\pi_\varphi(A)\xi_\varphi] = \overline{t_i(H_\varphi)} = \mathrm{supp}(t_i)$, then U_i has the unique extension to a unitary operator H_i onto K_i, which we denote again it by U_i. Since t_i is an element of $\pi_\varphi(A)'$, K_i is an invariant subspace under $\pi_\varphi(A)$. Furthermore, for each $a, b \in A$, we have

$$U_i(\pi_i(\mathrm{a})\eta_i(b)) = U_i(\pi_i(ab)\xi_i) = \pi_\varphi(ab)t_i\xi_\varphi$$

and

$$\pi_\varphi(a)(U_i\eta_i(b)) = \pi_\varphi(a)U_i(\pi_i(b)\xi_i) = \pi_\varphi(a)\pi_\varphi(b)t_i\xi_\varphi$$
$$= \pi_\varphi(ab)t_i\xi_\varphi.$$

Therefore, π_i and $\pi_\varphi | K_i$ are unitarily equivalent.

Next, we shall show that $\{K_i\}_{i=1}^n$ are mutually orthogonal. By the assumption, we have: $\varphi_k(1-e_k)=0$ for all k. Since $\{e_k\}_{k=1}^n$ are mutually orthogonal, we have, for each $a,b \in A$ and for $i, k=1, 2, \cdots, n$, if $i \neq k$,

$$|\varphi_k((1-e_i)\,b^*ae_i)|^2 \leqq \varphi_k(a^*b(1-e_i)\,b^*a)\,\varphi_k(e_i)$$

$$\leqq \varphi_k(a^*b(1-e_i)\,b^*a)\,\varphi_k(1-e_k)=0,$$

and

$$|\varphi_i((1-e_i)\,b^*ae_i)|^2 \leqq \varphi_i(1-e_i)\,\varphi_i(b^*ae_ia^*b)=0.$$

Therefore, for each i, we have

$$H_\varphi = \overline{\eta_\varphi(Ae_i)} \oplus \overline{\eta_\varphi(A(1-e_i))}$$

and

$$K_i = \mathrm{supp}(t_i) \subset \overline{\eta_\varphi(Ae_i)}.$$

Therefore, if $i \neq j$, K_i and K_j are mutually orthogonal.

Now, we shall show that $H_\varphi = \sum_{i=1}^n \oplus K_i$. For every $a,b \in A$, we have

$$\begin{aligned}(\eta_\varphi(a)\,|\,\eta_\varphi(b))_\varphi &= \varphi(b^*a) = \sum_{i=1}^n \varphi_i(b^*a)\\ &= \sum_{i=1}^n (t_i\pi_\varphi(a)\,\xi_\varphi\,|\,t_i\pi_\varphi(b)\,\xi_\varphi)_\varphi\\ &= \left(\pi_\varphi(a)\xi_\varphi\,\middle|\,\left(\sum_{i=1}^n t_i^2\right)\pi_\varphi(b)\,\xi_\varphi\right)_\varphi\\ &= \left(\eta_\varphi(a)\,\middle|\,\left(\sum_{i=1}^n t_i^2\right)\eta_\varphi(b)\right)_\varphi\end{aligned}$$

Since the subspace $\{\eta_\varphi(a)\,;\, a \in A\}$ is dense in H_φ, we have $\sum_{i=1}^n t_i^2=1$. Therefore, we have: $H_\varphi = \sum_{i=1}^n \oplus K_i$. This completes the proof of Lemma 5.

By the mentioned lemmas, we have the following theorem.

THEOREM 1. *Let A be a C^*-algebra with the identity* 1. *Let φ be a state on A and $\{\varphi_i\}_{i=1}^n$ be a family of pure, positive linear functionals on*

A such that $\varphi = \sum_{i=1}^{n} \varphi_i$. If there exists a family $\{e_i\}_{i=1}^{n}$ of orthogonal equivalent projections in A such that $\varphi_i(1-e_i) = 0$ for $i = 1, 2, \cdots, n$, and if, for the partial isometry u_i with the initial projection e_1 and the final projection e_i, $\varphi_1(u_i^* a u_i) = \varphi_i(a)$ for each $a \in A$, then φ is a factor state.

PROOF. Let (π_φ, H_φ) and $\{(\pi_i, H_i)\}_{i=1}^{n}$ be the canonical representations induced by φ and $\{\varphi_i\}_{i=1}^{n}$ respectively. Since, by Lemma 4, π_i and π_j are unitarily equivalent, there exists an n-dimensional Hilbert space $H(n)$ such that $\left(\sum_{i=1}^{n} \oplus \pi_i\right)(A)$ and $\pi_1(A) \otimes \boldsymbol{C}_{H(n)}$ are unitarily equivalent where $\boldsymbol{C}_{H(n)}$ is the algebra of all scalar multiples of the identity on $H(n)$. Since φ_1 is pure, the weak closure $\widetilde{\pi_1(A)}$ of $\pi_1(A)$ is the algebra $B(H_1)$ of all bounded operators on H_1. Therefore, the weak closures of $\left(\sum_{i=1}^{n} \oplus \pi_i\right)(A)$ and $B(H_1) \otimes \boldsymbol{C}_{H(n)}$ are unitarily equivalent. Furthermore, by Lemma 5, $\pi_\varphi(A)$ and $\left(\sum_{i=1}^{n} \oplus \pi_i\right)(A)$ are unitarily equivalent. Therefore the weak closure $\widetilde{\pi_\varphi(A)}$ of $\pi_\varphi(A)$ and the weak closure $\widetilde{\left(\sum_{i=1}^{n} \oplus \pi_i\right)(A)}$ of $\left(\sum_{i=1}^{n} \oplus \pi_i\right)(A)$ are unitarily equivalent, and so $\widetilde{\pi_\varphi(A)}$ and $B(H_1) \otimes \boldsymbol{C}_{H(n)}$ are unitarily equivalent. Thus, φ is a factor state.

In Theorem 1, it is obvious that the weak closure $\widetilde{\pi_\varphi(A)}$ of $\pi_\varphi(A)$ is a factor of type I.

5. The proof of Theorem A and B. At first, we shall show Theorem A.

PROOF OF THEOREM A. Let e be the support of φ, then, by Proposition 1, φ is a faithful normal trace on eMe.

Since M is type I and eMe is finite, eMe is a finite von Neumann algebra of type I, and there exists a family $\{e_{n(i)}\}_{i=1}^{\infty}$ of orthogonal projections in Ze such that $e_{n(i)}$ is an $n(i)$-homogeneous projection and $e = \sum_{i=1}^{\infty} e_{n(i)}$. Thus, $eMe = \sum_{i=1}^{\infty} \oplus (eMe)e_{n(i)}$. In the following, we shall pass the argument by considering e_n for $e_{n(i)}$.

First, we suppose $e = 1$, and let Y_1 be the spectrum of Z and X_n the closed and open set corresponding to e_n for each n, then the set $Y_1 - \bigcup_{n=1}^{\infty} X_n = N'$ is a non-dense set in Y_1 (we distinguish Y_1 from X, because we shall use Y_1 for the spectrum of Ze).

Since Me_n is n-homogeneous, there exists a family $\{p_i^{(n)}\}_{i=1}^{n}$ of projections in M such that they are equivalent, orthogonal and abelian projections and their sum is e_n. Thus, there exists an abelian von Neumann algebra $\mathfrak{A}_n$ which is the center of Me_n, and we have $Me_n = \mathfrak{A}_n \otimes B(H_n)$ where H_n an n-dimensional Hilbert space.

Now, since $p_1^{(n)} M p_1^{(n)}$ is abelian, $p_1^{(n)} M p_1^{(n)} = Z p_1^{(n)} \overset{(*)}{\cong} Ze_n$. The above $*$-isomorphism Φ_n is defined by $\Phi_n^{-1}(p_1^{(n)} a p_1^{(n)}) = be_n$, for $p_1^{(n)} a p_1^{(n)} = bp_1^{(n)}$ in $p_1^{(n)} M p_1^{(n)} = Z p_1^{(n)}$ where a is an element of M and b is an element of Z. Let $u_i^{(n)}$ be the partial isometry with the initial projection $p_1^{(n)}$ and the final projection $p_i^{(n)}$ for $i = 1, 2, \cdots, n$, then, for $a \in Me_n$, we have

$$a = \sum_{i,j=1}^{n} a_{ij} u_i^{(n)} u_j^{(n)*}$$

where

$$a_{ij} = \Phi_n^{-1}(p_1^{(n)} u_i^{(n)*} a u_j^{(n)} p_1^{(n)}) \in \mathfrak{A}_n.$$

Furthermore, we have

$$p_i^{(n)} a p_i^{(n)} = a_{ii} p_i^{(n)}.$$

This shows that $a_{ii} = \tau_{p_i^{(n)}}(a)$ where τ denotes the mapping defined in the beginning of §4. Thus we denote a by $\{a_{ij}\}$ with $a_{ij} \in \mathfrak{A}_n$ and call it the matrix representation of a. [See also [7], p.2.11]

Since M is a finite von Neumann algebra, there exists the center-valued trace $\natural$. Moreover we have the following integral representation of φ: for each $a \in M$,

$$\varphi(a) = \varphi(a^\natural) = \int_{Y_1} a^{\natural\wedge}(t)\, d\mu(t) \tag{1}$$

$$= \sum_{n=1}^{\infty} \int_{X_n} a^{\natural\wedge}(t)\, d\mu(t) + \int_{N'} a^{\natural\wedge}(t) d\mu(t)$$

where μ is the spectral measure on Y_1 and the support of μ is Y_1.

Let $\natural_{e_n}$ be the center-valued trace on Me_n, then it satisfies that $a^{\natural_{e_n}} = a^\natural$ for all $a \in Me_n$. Furthermore we can show that $a^{\natural_{e_n}} = \dfrac{1}{n} \sum_{i=1}^{n} a_{ii}$ for $a \in Me_n$ where $\{a_{ij}\}$ is the matrix representation of a, we have

$$\varphi(a) = \varphi(a^{\natural}) = \int_{Y_1} \left(\frac{1}{n} \sum_{i=1}^{n} a_{ii}{}^{\wedge}(t) \right) d\mu(t)$$

$$= \int_{X_n} \left(\frac{1}{n} \sum_{i=1}^{n} \Phi_n^{-1}(p_1^{(n)} u_i^{(n)*} a u_i^{(n)} p_1^{(n)})^{\wedge}(t) d\mu(t) \right)$$

for each $a \in Me_n$.

We return the argument from eMe to M, then

$$\varphi(a) = \varphi(eae) = \int_{Y_1} (eae)^{\natural_e \wedge}(t) d\mu(t).$$

Let Y be the closed and open set to which the central support $z(e)$ of e corresponds and Y_1 the spectrum of Ze. Then there exists a $*$-isomorphism π from $Z_{z(e)}$ onto Ze given by $a \to ae$ for $a \in Z_{z(e)}$. Considering the transpose ${}^t\pi$ of π, it is the linear isomorphism of $(Ze)^*$ onto $(Z_{z(e)})^*$, and ${}^t\pi = \delta$ induces a homeomorphism from Y_1 onto Y. Furthermore, $\eta = \delta^{-1}$ is a homeomorphism from Y onto Y_1.

Put $\varphi_\zeta(a) = (eae)^{\natural_e \wedge}(\eta(\zeta))$ for each $a \in M$. Then we get, for each $a \in Z$ and $\zeta \in Y$, $(ae)^{\wedge}(\eta(\zeta)) = a^{\wedge}(\zeta)$ and so, for each $a \in Z$ and $b \in M$, we have

$$\varphi_\zeta(ab) = (eabe)^{\natural_e \wedge}(\eta(\zeta)) = a^{\wedge}(\zeta)\varphi_\zeta(b).$$

Let us define ν in the following form:

$$C(Y_1)^* \ni \mu \longrightarrow {}^t\pi(\mu) = \nu \in C(Y)^*,$$

then we have: for each $a \in M$,

$$\varphi(a) = \int_{Y_1} (eae)^{\natural_e \wedge}(t)\, d\mu(t) = \int_Y \varphi_\zeta(a)\, d\nu(\zeta).$$

It is obvious that $\mathrm{supp}(\nu) = Y$ and ν is a spectral measure on Y. The set N' is non-dense in Y_1 and so $N = \delta(N')$ is non-dense in Y. Hence, for each $\zeta \in Y - N$, there exists a positive integer n such that $\eta(\zeta) \in X_n$. For such n, we define a linear functional $\varphi_{i\zeta}$ in the following form:

$$\varphi_{i\zeta}(a) = \frac{1}{n} \Phi_n^{-1}(p_1^{(n)} u_i^{(n)*}(e_n a e_n) u_i^{(n)} p_1^{(n)})^{\wedge}(\eta(\zeta))$$

$$= \frac{1}{n} \tau_{p_i^{(n)}}(e_n a e_n)^{\wedge}(\eta(\zeta)).$$

Then $\varphi_{i\zeta}(1-e_n)=0$, $\varphi_{i\zeta}(a)=\varphi_{i\zeta}(e_nae_n)$ and φ_i is pure on e_nMe_n by Lemma 1. Therefore $\varphi_{i\zeta}$ is pure on M.

Assume that $\eta(\zeta)\in X_n$, then $a_{11}^{(n)}=\Phi_n^{-1}(p_1^{(n)}e_nae_np_1^{(n)})$, $a_{ii}^{(n)}=\Phi_n^{-1}(p_1^{(n)}u_i^{(n)*}e_nae_n u_i^{(n)}p_1^{(n)})$ and $u_i^{(n)}e_n=e_nu_i^{(n)}$.

Hence we have

$$\begin{aligned}\varphi_{1\zeta}(u_i^{(n)*}au_i^{(n)}) &= \frac{1}{n}\Phi_n^{-1}(p_1^{(n)}e_n(u_i^{(n)*}au_i^{(n)})e_np_1^{(n)})\hat{\ }(\eta(\zeta))\\ &= \frac{1}{n}\Phi_n^{-1}(p_1^{(n)}u_i^{(n)*}(e_nae_n)u_i^{(n)}p_1^{(n)})\hat{\ }(\eta(\zeta))\\ &= \varphi_{i\zeta}(a) \quad \text{for all } a\in M.\end{aligned}$$

Since $\varphi_{i\zeta}(1-p_i^{(n)})=0$ and $\{p_i^{(n)}\}$ are mutually equivalent, by considering Theorem 1, $\varphi_\zeta=\sum_{i=1}^{n}\varphi_{i\zeta}$ is a factor state. This completes the proof of Theorem A.

Let π_ζ be the canonical representation induced by φ_ζ, then $\pi_\zeta(M)$ is not necessary a von Neumann algebra. Considering π_ζ, then, for $\zeta\in X$, $\pi_\zeta^{-1}(0)\cap Z=\{z\in Z\,;\, z\hat{\ }(\zeta)=0\}$. Therefore, π_ζ is σ-weakly continuous if and only if the one-point set $\{\zeta\}$ is a closed and open set in X, and π_ζ is σ-weakly continuous for all $\zeta\in X$ if and only if Z is an atomic abelian von Neumann algebra. By this consideration, we have the following proof of Theorem B.

PROOF OF THEOREM B. Since Z is non-atomic, there exists an element ζ of X such that π_ζ is not σ-weakly continuous. If $\pi_\zeta(M)$ is a von Neumann algebra for such an element ζ, then π_ζ is a *-homomorphism from M onto a von Neumann algebra $\pi_\zeta(M)$, and, by Theorem 1 in [8], π_ζ must be σ-weakly continuous, which is a contradiction. Therefore, $\pi_\zeta(M)$ can't be a von Neumann algebra, which completes the proof.

We may construct a von Neumann algebra M and a positive linear functional φ which satisfy the conditions in Theorem A and B. Let H be a countably infinite dimensional Hilbert space, $B(H)$ the von Neumann algebra of all bounded operators on H and $L^\infty(0,1)$ a von Neumann algebra of all essentially bounded functions under the Lebesgue measure on the open interval $(0,1)$. As $B(H)$ is properly infinite and $L^\infty(0,1)$ is a finite von Neumann algebra, the W^*-tensor product $M=L^\infty(0,1)\otimes B(H)$ is a properly infinite von Neumann algebra (p.3.40 in [7]) and the center Z of M is $L^\infty(0,1)\otimes \boldsymbol{C}_H$ where $\boldsymbol{C}_H$ is the algebra of all scalar multiples of the identity on H, and it is a non-atomic abelian von Neumann algebra. Since the Hilbert space $L^2(0,1)\otimes H$

is separable, M_* is separable. Therefore, M satisfies the assumption in Theorem B.

Next, we shall construct a positive linear functional φ which satisfies the assumption in Theorem A and B. Let e be a non-zero, finite dimensional projection on H, then the W^*-tensor product $N = L^\infty(0,1) \otimes eB(H)e$ is a finite von Neumann algebra (p. 3. 40 in [7]) and σ-finite, because the Hilbert space $L^2(0,1) \otimes e(H)$ is separable. Therefore there exists a faithful normal trace φ' on $N = (1 \otimes e) M (1 \otimes e)$. If we define a positive linear functional φ by $\varphi(a) = \varphi'(a(1 \otimes e))$ for all $a \in M$, is a normal positive linear functional on M. We see, by the definition of φ, $\mathrm{supp}(\varphi) = 1 \otimes e$. Therefore, by Proposition 1, φ is a reducible normal positive linear functional on M. By the above construction, we see that φ and M satisfy the assumptions in Theorem A and B.

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