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ON THE CONVERGENCE OF NONLINEAR SEMI-GROUPS

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1. Introduction. Let X be a Banach space and let $\{T(\xi);\ \xi \geqq 0\}$ be a family of (nonlinear) operators from X into itself satisfying the following conditions:

(i) $T(0) = I$(the identity) and $T(\xi + \eta) = T(\xi)T(\eta)$ for $\xi, \eta \geqq 0$.

(ii) For each $x \in X$, $T(\xi)x$ is strongly continuous in $\xi \geqq 0$.

We call such a family $\{T(\xi);\ \xi \geqq 0\}$ simply a *nonlinear semi-group.* If there is a non-negative constant c such that

(iii) $\|T(\xi)x - T(\xi)y\| \leqq e^{c\xi}\|x-y\|$ for $x, y \in X$ and $\xi \geqq 0$,

then a nonlinear semi-group $\{T(\xi);\ \xi \geqq 0\}$ is said to be *of local type.* (In particular, if $c = 0$, it is called a *nonlinear contraction semi-group.*) We define the *infinitesimal generator* A_0 of a nonlinear semi-group $\{T(\xi);\ \xi \geqq 0\}$ by

$$A_0x = \lim_{\delta\to 0+} \delta^{-1}(T(\delta) - I)x \tag{1.1}$$

and the *weak infinitesimal generator* A' by

$$A'x = \text{w-}\lim_{\delta\to 0+} \delta^{-1}(T(\delta) - I)x, \tag{1.2}$$

if the right sides exist. (The notation "lim" ("w-lim") means the strong limit (the weak limit) in X.)

REMARK. In case of *linear* semi-groups, it is well known that the weak infinitesimal generator coincides with the infinitesimal generator.

H. F. Trotter [9] proved the following convergence theorem of *linear* semi-groups.

THEOREM. *Let $\{T_n(\xi);\ \xi \geqq 0\}_{n=1,2,3,\cdots}$ be a sequence of semi-groups (of linear operators) of class (C_0) satisfying the stability condition*

$$\|T_n(\xi)\| \leqq Me^{\omega\xi} \text{ for } \xi \geqq 0,\ n = 1, 2, 3, \cdots,$$

where M and ω are independent of n and ξ. Let A_n be the infinitesimal generator of $\{T_n(\xi);\ \xi \geqq 0\}$ and define $Ax = \lim_n A_nx$.

Suppose that

(a) $D(A)$ *(the domain of* A*) is dense in* X,

(b) *for some* $\lambda > \omega$, $R(\lambda - A) = X$ *(or* $\overline{R(\lambda - A)} = X$).

Then A *(or the closure of* A*) generates a semi-group* $\{T(\xi);\ \xi \geqq 0\}$ *of class* (C_0); *and for each* $x \in X$

$$\lim_n T_n(\xi)\, x = T(\xi)\, x$$

for $\xi \geqq 0$ *and the convergence is uniform with respect to* ξ *in every finite interval.*

In this paper we shall study the convergence of nonlinear semi-groups $\{T_n(\xi);\ \xi \geqq 0\}$ $(n = 1, 2, 3, \cdots)$ of local type with the stability condition

$$\|T_n(\xi)\, x - T_n(\xi)\, y\| \leqq e^{\omega\xi}\, \|x-y\|\ ; \tag{1.3}$$

and we can prove the following (see Theorem 2.1):

"*Let* A_n *be the infinitesimal generator of* $\{T_n(\xi):\ \xi \geqq 0\}$, *and let* A' *be the weak infinitesimal generator of a semi-group* $\{T(\xi);\ \xi \geqq 0\}$ *of local type. If there exists a dense set* D_0 *such that for each* $x \in D_0$, $\lim\limits_n A_n x = A'x$ *and* $\lim\limits_n A_n T(\xi)\, x = A'T(\xi)\, x$ *for a.a.*ξ *(with additional conditions* $T_n(\xi)\, x \in D(A_n)$ *for a.a.* ξ*), then for each* $x \in X$,

$$T(\xi)x = \lim_n T_n(\xi)\, x$$

uniformly on every finite interval."

(We note here that we may take $\bigcup\limits_{x \in D_0} \{T(\xi)\, x;\ \lim\limits_n A_n T(\xi)\, x = A'T(\xi)\, x\}$ as a set D in Theorem 2.1.) In particular if X^* (the adjoint space of X) is uniformly convex, then the Trotter theorem holds good for our nonlinear case (see Theorem 2.3).

For *linear* semi-group $\{T(\xi);\ \xi \geqq 0\}$ of class (C_0), it is well known that

$$T(\xi)\, x = \lim_{\delta \to 0+} T_\delta(\xi)\, x \quad \text{for } x \in X,\ \xi \geqq 0,$$

where $A_\delta = \delta^{-1}(T(\delta) - I)$ and $\{T_\delta(\xi);\ \xi \geqq 0\}$ is the semi-group generated by A_δ. And, in this case, $T_\delta(\xi)(= \exp(\xi A_\delta))$ is continuous in $\xi \geqq 0$ with respect to the uniform operator topology (see [3]). In §4 we shall give similar results for nonlinear semi-groups of local type.

2. Theorems. The main theorems are as follows.

THEOREM 2.1. *Let* $\{T_n(\xi);\ \xi \geqq 0\}_{n=1,2,3,\ldots}$ *be a sequence of nonlinear semi-groups of local type satisfying the stability condition*

$$\|T_n(\xi)x - T_n(\xi)y\| \leqq e^{\omega\xi}\|x-y\| \tag{2.1}$$

for $\xi \geqq 0$, $n = 1, 2, 3, \cdots$ *and* $x, y \in X$, *where* ω *is a non-negative constant independent of* n, x, y, *and* ξ. *Let* A_n *be the infinitesimal generator of* $\{T_n(\xi);\ \xi \geqq 0\}$ *and let* $\lim_n A_n x = Ax$ *on a set* $D \subset \bigcap_{n=1}^{\infty} D(A_n)$.

Suppose that

(a) *A(defined on D) is a restriction of the weak infinitesimal generator of some nonlinear semi-group* $\{T(\xi);\ \xi \geqq 0\}$ *such that for any* $\beta > 0$, $\{T(\xi);\ 0 \leqq \xi \leqq \beta\}$ *is equi-Lipschitz continuous on every bounded set,*

(b) *there exists a set* $D_0 \subset D$ *such that for each* $x \in D_0$

(b_1) *for each* n, $T_n(\xi)x \in D(A_n)$ *for a.a.* $\xi \geqq 0$,

(b_2) $T(\xi)x \in D$ *for a.a.* $\xi \geqq 0$.

Then for each $x \in \overline{D}_0$ *(the strong closure of* D_0*) we have*

$$T(\xi)x = \lim_n T_n(\xi)x \quad \text{for each } \xi \geqq 0, \tag{2.2}$$

and the convergence is uniform with respect to ξ *in every finite interval.*

REMARKS 1. If for any bounded set B there is a positive constant M_B such that $\|T(\xi)x - T(\xi)y\| \leqq M_B\|x-y\|$ for $\xi \in [0, \beta]$ and $x, y \in B$, then the family $\{T(\xi);\ 0 \leqq \xi \leqq \beta\}$ is said to be *equi-Lipschitz continuous* on every bounded set.

2. The above theorem remains true even if the conditions "$D \subset \bigcap_{n=1}^{\infty} D(A_n)$" and ($b_1$) are replaced by "$D \subset \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} D(A_n)$" and the following ($b_1'$), respectively.

(b_1') For sufficiently large n, $T_n(\xi)x \in D(A_n)$ for a. a. $\xi \geqq 0$.

The proof is given in §3.

In the above theorem, if X is a reflexive Banach space and $D \supset D(A_0)$, where A_0 is the infinitesimal generator of $\{T(\xi);\ \xi \geqq 0\}$ in the assumption (a), then the assumption (b) is automatically satisfied by taking $D_0 = D$. In fact, if $x \in D$, then $x \in D(A')$ and $x \in D(A_n)$, where A' is the weak infinitesimal generator of $\{T(\xi);\ \xi \geqq 0\}$; and hence $T(\xi)x$ and $T_n(\xi)x$ are strongly absolutely continuous on every finite interval (see the proof of Lemma 3.3). Thus the reflexivility of X shows that $T(\xi)x$ and $T_n(\xi)x$ are strongly differentiable at a.a. $\xi \geqq 0$ (for example, see Y. Kōmura [5]), so that the semi-group property (i) (in §1) implies

$$T_n(\xi)\, x \in D(A_n) \qquad \text{for a. a. } \xi \geqq 0$$

and

$$T(\xi)\, x \in D(A_0) \subset D \text{ for a. a. } \xi \geqq 0.$$

Thus we have the following

THEOREM 2.2. *Let* $\{T_n(\xi)\,;\ \xi \geqq 0\}_{n=1,2,3,\dots}$ *be a sequence of nonlinear semi-groups in Theorem* 2.1 *defined on a reflexive Banach space* X, *and let* A_n *be the infinitesimal generator of* $\{T_n(\xi)\,; \xi \geqq 0\}$ *and assume* $\lim_n A_n x = Ax$ *on a set* D.

If the condition (a) *in Theorem* 2.1 *is satisfied and* $D \supset D(A_0)$ (*i.e.*, $A' \supset A \supset A_0$), *then for each* $x \in \overline{D}$ [1] *we have*

$$T(\xi)\, x = \lim_n T_n(\xi)\, x \quad \textit{for all } \xi \geqq 0,$$

and the convergence is uniform with respect to ξ *in every finite interval.*

T.Kato proved a generation theorem of nonlinear contraction semi-groups defined on a Banach space such that the adjoint space is uniformly convex (see T.Kato [4] and F. E. Browder [1]), and his result has been extended to some class of nonlinear semi-groups (which contains semi-groups of local type) by S.Oharu [8]. Using Oharu's result, we can prove the following

THEOREM 2.3. *Let the adjoint space* X^* *of* X *be a uniformly convex Banach space. Let* $\{T_n(\xi)\,; \xi \geqq 0\}_{n=1,2,3,\dots}$ *be a sequence of nonlinear semi-groups in Theorem* 2.1, *and let* A_n *be the infinitesimal generator of* $\{T_n(\xi)\,;\ \xi \geqq 0\}$ *and define* $Ax = \lim_n A_n x$.

Suppose that

(a′) $D(A)$ (*the domain of* A) *is dense in* X,

(b′) *for some* $h_0 \in (0, 1/\omega)$, $R(1 - h_0 A) = X$.

Then A *is the weak infinitesimal generator of a nonlinear semi-group* $\{T(\xi)\,; \xi \geqq 0\}$ *of local type and for each* $x \in X$

$$(*) \qquad T(\xi)\, x = \lim_n T_n(\xi)\, x \quad \textit{for all } \xi \geqq 0,$$

and the convergence is uniform with respect to ξ *in every finite interval.*

1) It is easy to see that $\overline{D} = \overline{D(A')} = \overline{D(A_0)}$.

REMARK. If we omit the condition (a′), then A is the weak infinitesimal generator of a nonlinear semi-group $\{T(\xi);\ \xi \geqq 0\}$ of local type defined on $\overline{D(A)}$ and the convergence $(*)$ holds on $\overline{D(A)}$.

PROOF. If we can prove the following

$$(2.3)\qquad \begin{cases} \text{the limit operator } A \text{ is the weak infinitesimal generator of a} \\ \text{nonlinear semi-group } \{T(\xi);\ \xi \geqq 0\} \text{ such that for any } \beta > 0,\ \{T(\xi); \\ 0 \leqq \xi \leqq \beta\} \text{ is equi-Lipschitz continuous on every bounded set,} \end{cases}$$

then the convergence $(*)$ follows from Theorem 2.2 by taking $D = D(A)$ because X is reflexive with X^*, and the convergence implies

$$\|T(\xi)x - T(\xi)y\| \leqq e^{\omega\xi}\|x - y\|$$

for $\xi \geqq 0, x, y \in X$.

We shall now prove (2.3). Let x and y be elements in $D(A)$. By Lemma 3.1, for each n, we have

$$\mathrm{Re}(A_n x - A_n y, f) \leqq \omega\|x - y\|^2$$

for $f = F(x-y)$, where F denotes the duality map from X into X^*. Letting $n \to \infty$

$$(2.4)\qquad \mathrm{Re}(Ax - Ay, f) \leqq \omega\|x - y\|^2$$

This means that $B = A - \omega$ is a dissipative (i.e., $\mathrm{Re}(Bx - By, f) \leqq 0$). And the assumption (b') implies

$$R(1 - h_0(1 - h_0\omega)^{-1}B) = X,$$

so that $R(1 - \varepsilon B) = X$ for all $\varepsilon > 0$ (see S. Oharu [7], Y. Kōmura [5], T. Kato [4]). This leads

$$(2.5)\qquad R(1 - hA) = X \quad \text{for all } h \in (0, 1/\omega).$$

Let $h \in (0, 1/\omega)$. Since $\|x - y - h(Ax - Ay)\|\,\|x - y\| \geqq \mathrm{Re}(x - y - h(Ax - Ay), f) = \|x - y\|^2 - h\ \mathrm{Re}(Ax - Ay, f) \geqq (1 - h\omega)\|x - y\|^2 (x, y \in D(A), f = F(x - y))$ by (2.4), we obtain

$$\|x - y - h(Ax - Ay)\| \geqq (1 - h\omega)\|x - y\|$$

for each $x, y \in D(A)$. Consequently

(2.6) $\qquad$ for each $h \in (0, 1/\omega)$, $(1 - hA)^{-1}$ exists on X.

Now (2.3) follows from Oharu's results ([8; Theorems 4.1 and 4.2]).[2] Q.E.D.

3. Proof of Theorem 2.1.

We start from the following

LEMMA 3.1 *If* $\{T(\xi); \xi \geqq 0\}$ *is a nonlinear semi-group of local type with* $\|T(\xi)x - T(\xi)y\| \leqq e^{\omega\xi}\|x - y\| (\xi \geqq 0, x, y \in X)$, *and if* A' *is its weak infinitesimal generator, then for each* $x, y \in D(A')$ *we have*

$$\mathrm{Re}(A'x - A'y, f) \leqq \omega\|x - y\|^2$$

for any $f \in F(x - y)$, *where* F *is the duality map from* X *into* X^*.

PROOF. Let $x, y \in D(A')$, and let $f \in F(x - y)$.

$$\begin{aligned}
&\mathrm{Re}\ (\xi^{-1}[T(\xi)x - x] - \xi^{-1}[T(\xi)y - y], f) \\
&= \xi^{-1}\mathrm{Re}(T(\xi)x - T(\xi)y, f) - \xi^{-1}\mathrm{Re}(x - y, f) \\
&\leqq \xi^{-1}\|T(\xi)x - T(\xi)y\|\,\|x - y\| - \xi^{-1}\|x - y\|^2 \\
&\leqq \xi^{-1}(e^{\omega\xi} - 1)\|x - y\|^2.
\end{aligned}$$

Letting $\xi \to 0+$, we get

$$\mathrm{Re}(A'x - A'y, f) \leqq \omega\|x - y\|^2.$$

Q.E.D.

LEMMA 3.2 (T.Kato [4]). *Let* $x(\xi)$ *be an* X*-valued function on an interval of real numbers. Suppose* $x(\xi)$ *has a weak derivative* $x'(\eta) \in X$ *at* $\xi = \eta$ *and* $\|x(\xi)\|$ *is differentiable at* $\xi = \eta$. *Then*

$$\|x(\eta)\| \left[\frac{d}{d\xi}\|x(\xi)\|\right]_{\xi=\eta} = \mathrm{Re}(x'(\eta), f)$$

for any $f \in F(x(\eta))$.

LEMMA 3.3. *Let* $\{T(\xi); \xi \geqq 0\}$ *be a nonlinear semi-group with the*

2) We note that (2.4) implies the condition (S) in his theorem.

weak infinitesimal generator A', and let for any $\beta > 0$ the family $\{T(\xi);\ 0 \leqq \xi \leqq \beta\}$ be equi-Lipschitz continuous on every bounded set. If $x \in D(A')$ and $T(\xi)x \in D(A')$ for a.a. $\xi \geqq 0$, then $A'T(\xi)x$ is strongly measurable and essentially bounded (and hence, Bochner integrable) on every finite interval, and

$$T(\xi)x - x = \int_0^\xi A'T(\eta)x d\eta \quad \text{for all } \xi \geqq 0.$$

Consequently $T(\xi)x$ is strongly differentiable at a.a. ξ and

$$(d/d\xi)T(\xi)x = A'T(\xi)x \quad \text{for a.a. } \xi \geqq 0.$$

PROOF. Let $\beta > 0$ be an arbitrary given. If we put

$$B = \{T(\xi)x;\ 0 \leqq \xi \leqq 1\} \quad \text{and} \quad K = \sup_{0<\delta\leqq 1} \delta^{-1}\|T(\delta)x - x\|,$$

then B is a bounded set and K is finite. Since the family $\{T(\xi);\ 0 \leqq \xi \leqq \beta\}$ is equi-Lipschitz continuous on B, there exists a constant M_B such that

$$\|T(\xi)y - T(\xi)z\| \leqq M_B\|y - z\|$$

for all $y, z \in B$ and $\xi \in [0, \beta]$. Therefore, for $0 \leqq \xi \leqq \beta$ and $0 \leqq \delta \leqq 1$, we have

$$\|T(\xi + \delta)x - T(\xi)x\| \leqq M_B\|T(\delta)x - x\| \leqq M_B K\delta. \tag{3.1}$$

This shows that $T(\xi)x$ is strongly absolutely continuous on $[0, \beta]$. Since $T(\xi)x \in D(A')$ for a.a. $\xi \geqq 0$,

$$\begin{cases} A'T(\xi)x = \text{w-}\lim\limits_{\delta\to 0+} \delta^{-1}(T(\delta) - I)T(\xi)x \\ \qquad\qquad = \text{w-}\lim\limits_{\delta\to 0+} \delta^{-1}(T(\xi + \delta)x - T(\xi)x) \end{cases} \tag{3.2}$$

for a.a. $\xi \geqq 0$; hence $A'T(\xi)x$ is strongly measurable (for example, see [3, Theorem 3.5.4]). By (3.1) and (3.2)

$$\|A'T(\xi)x\| \leqq M_B K \quad \text{for a.a. } \xi \in [0, \beta],$$

so that $A'T(\xi)x$ is essentially bounded on $[0, \beta]$. Consequently $A'T(\xi)x$ is Bochner integrable on $[0, \beta]$.

Let $f \in X^*$. Since $(T(\xi)x, f)$ $(= f(T(\xi)x))$ is absolutely continuous on $[0, \beta]$, $(T(\xi)x, f)$ is differentiable at a.a. $\xi \in [0, \beta]$ and

$$(T(\xi)x,f)-(x,f)=\int_0^\xi \frac{d}{d\eta}(T(\eta)x,f)\,d\eta$$

for any $\xi\in[0,\beta]$. Moreover it follows from (3.2) that

$$\frac{d}{d\xi}(T(\xi)x,f)=(A'T(\xi)x,f)$$

for a. a. $\xi\in[0,\beta]$. Thus the above equalities and the Bochner integrability of $A'T(\xi)x$ on $[0,\beta]$ show that

$$\begin{aligned}(T(\xi)x,f)-(x,f)&=\int_0^\xi (A'T(\eta)x,f)d\eta\\&=\left(\int_0^\xi A'T(\eta)x\,d\eta,f\right)\end{aligned}$$

for all $\xi\in[0,\beta]$. Hence we get

$$T(\xi)x-x=\int_0^\xi A'T(\eta)x\,d\eta \qquad \text{for all } \xi\in[0,\beta]$$

and $(d/d\xi)T(\xi)x=A'T(\xi)x$ for a.a. $\xi\in[0,\beta]$.

Q. E. D.

LEMMA 3.4. *Under the assumptions of Theorem* 2.1, *for each* $x\in D_0$ *we have the following*:

(3.3) $\left\{\begin{array}{l}AT(\xi)x \textit{ is strongly measurable and essentially bounded on every finite interval.}\end{array}\right.$

$$T(\xi)x-x=\int_0^\xi AT(\eta)x\,d\eta \qquad \textit{for all } \xi\geqq 0 \tag{3.4}$$

and $(d/d\xi)T(\xi)x=AT(\xi)x$ *for* a.a. $\xi\geqq 0$.

$$T_n(\xi)x-x=\int_0^\xi A_nT_n(\eta)x d\eta \qquad \textit{for all } \xi\geqq 0 \tag{3.5}$$

and $(d/d\xi)T_n(\xi)x=A_nT_n(\xi)x$ *for a.a.* $\xi\geqq 0$.

PROOF. If we denote the weak infinitesimal generator of $\{T(\xi);\ \xi\geqq 0\}$ by A', then the condition (a) is as follows;

(3.6) $$D \subset D(A') \text{ and } Ax = A'x \quad \text{for } x \in D.$$

Let $x \in D_0$. By (3.6) and (b_2)

$$x \in D(A'),\ T(\xi)x \in D(A') \text{ and } A'T(\xi)x = AT(\xi)x$$

for a.a. $\xi \geqq 0$. Therefore it follows from Lemma 3.3 that $AT(\xi)x$ $(=A'T(\xi)x$ a.a.) is strongly measurable and essentially bounded on every finite interval, and

$$T(\xi)x - x = \int_0^\xi AT(\eta)x\, d\eta \quad \text{for all } \xi \geqq 0,$$

$$(d/d\xi)T(\xi)x = AT(\xi)x \quad \text{for a.a. } \xi \geqq 0.$$

We remark that for any $\beta > 0$, $\{T_n(\xi);\ 0 \leqq \xi \leqq \beta\}$ is equi-Lipschitz continuous on X, because it is of local type. Since $x \in D(A_n)$ and $T_n(\xi)$x $\in D(A_n)$ for a.a. $\xi \geqq 0$ (see $M(b_1)$), (3.5) also follows from Lemma 3.3.

Q. E. D.

PROOF OF THEORE 2.1. Let $x \in D_0$ and put

(3.7) $$z_n(\xi) = T_n(\xi)x - T(\xi)x.$$

By Lemma 3.4

$$z_n(\xi) = \int_0^\xi (A_nT_n(\eta)x - AT(\eta)x)\, d\eta,$$

and each $z_n(\xi)$ has the strong derivative

$$z'_n(\xi) = A_nT_n(\xi)x - AT(\xi)x \quad \text{for a.a. } \xi \geqq 0;$$

moreover each $\|z_n(\xi)\|$ is differentiable at a.a. $\xi \geqq 0$ since $\|z_n(\xi)\|$ is absolutely continuous in $\xi \geqq 0$. Therefore it follows from Lemma 3.2 that for a.a. $\xi \geqq 0$

(3.8) $$\begin{cases} \|z_n(\xi)\|\,[(d/d\xi)\|z_n(\xi)\|] = \mathrm{Re}(z'_n(\xi), f_\xi) \\ \qquad = \mathrm{Re}(A_nT_n(\xi)x - AT(\xi)x, f_\xi) \end{cases}$$

for every $f_\xi \in F(z_n(\xi))$. And

$$(3.9)\qquad \|z_n(\xi)\|^2 = \int_0^\xi (d/d\eta)\|z_n(\eta)\|^2 d\eta = 2\int_0^\xi \|z_n(\eta)\|\,[(d/d\eta)\|z_n(\eta)\|]\,d\eta$$

for all $\xi \geqq 0$.

Let $\beta > 0$ be arbitrarily given. We shall show that the sequence $\{\|z_n(\xi)\|\,[(d/d\xi)\|z_n(\xi)\|]\}$ is uniformly (essentially) bounded on $[0, \beta]$. Put

$$K_1 = \operatorname*{ess\,sup}_{0\leqq\xi\leqq\beta} \|AT(\xi)\,x\| \ (<\infty)$$

(see (3.3)). Since $\|A_nT_n(\xi)\,x\| = \lim_{\delta\to0+}\|\delta^{-1}(T_n(\xi+\delta)x - T_n(\xi)x)\| \leqq e^{\omega\xi}\lim_{\delta\to0+}\delta^{-1}\|T_n(\delta)x - x\| = e^{\omega\xi}\|A_nx\|$(a.a. ξ) and since $\lim_n A_nx = Ax$, there is a constant K_2 independent of n such that

$$\operatorname*{ess\,sup}_{0\leqq\xi\leqq\beta} \|A_nT_n(\xi)\,x\| \leqq K_2.$$

Consequently, for all n, we get

$$\operatorname*{ess\,sup}_{0\leqq\xi\leqq\beta} \|z'_n(\xi)\| = \operatorname*{ess\,sup}_{0\leqq\xi\leqq\beta} \|A_nT_n(\xi)x - AT(\xi)\,x\| \leqq K_1 + K_2$$

and

$$(3.10)\qquad \|z_n(\xi)\| \leqq \int_0^\xi \|A_nT_n(\eta)\,x - AT(\eta)x\|d\eta \leqq (K_1+K_2)\,\beta$$

for every $\xi \in [0, \beta]$. Hence by (3.8)

$$\begin{aligned}|\,\|z_n(\xi)\|\,[(d/d\xi)\|z_n(\xi)\|]\,| &\leqq \|z'_n(\xi)\|\,\|f_\xi\| = \|z'_n(\xi)\|\,\|z_n(\xi)\| \\ &\leqq (K_1+K_2)^2\beta\end{aligned}$$

for a.a. $\xi \in [0, \beta]$; so that $\{\|z_n(\xi)\|\,[(d/d\xi)\|z_n(\xi)\|]\}$ is uniformly (essentially) bounded on $[0, \beta]$. Thus by the Lebegue convergence theorem

$$(3.11)\qquad \begin{cases} \limsup_{n\to\infty} \|z_n(\xi)\|^2 = \limsup_{n\to\infty} 2\int_0^\xi \|z_n(\eta)\|\,[(d/d\eta)\|z_n(\eta)\|]\,d\eta \\ \qquad\qquad \leqq 2\int_0^\xi \limsup_{n\to\infty} \|z_n(\eta)\|\,[(d/d\eta)\|z_n(\eta)\|]\,d\eta \end{cases}$$

for all $\xi \in [0, \beta]$.

Since $T(\xi)\,x \in D \subset D(A_n)$ and $T_n(\xi)\,x \in D(A_n)$ for a.a.ξ, it follows from

Lemma 3.1 that for a.a. $\xi \geqq 0$

$$\text{Re}(A_n T_n(\xi)\, x - A_n T(\xi)\, x,\ f_\xi) \leqq \omega \|z_n(\xi)\|^2 \tag{3.12}$$

for every $f_\xi \in F(z_n(\xi))$. Combining this with (3. 8), for a.a. $\xi \in [0, \beta]$

$$\begin{aligned} \|z_n(\xi)\|\, [(d/d\,\xi)\|z_n(\xi)\|] &\leqq \text{Re}(A_n T(\xi)\, x - AT(\xi)\, x, f_\xi) + \omega\|z_n(\xi)\|^2 \\ &\leqq \|A_n T(\xi)\, x - AT(\xi)\, x\|\ \|z_n(\xi)\| + \omega\|z_n(\xi)\|^2 \\ &\leqq (K_1 + K_2)\beta\|A_n T(\xi)\, x - AT(\xi) x\| + \omega\|z_n(\xi)\|^2 \text{ (see (3.10))}; \end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|z_n(\xi)\|\, [(d/d\xi)\|z_n(\xi)\|] \leqq \omega \limsup_{n\to\infty} \|z_n(\xi)\|^2 \tag{3.13}$$

for a.a. $\xi \in [0, \beta]$. If we put

$$g(\xi) = \limsup_{n\to\infty} \|z_n(\xi)\|^2 \quad \text{for } \xi \in [0, \beta],$$

then $0 \leqq g(\xi) \leqq (K_1 + K_2)^2\beta^2$ on $[0, \beta]$ (see (3. 10)), and from (3. 11) and (3. 13) we obtain

$$0 \leqq g(\xi) \leqq 2\omega \int_0^\xi g(\eta)\, d\eta$$

for every $\xi \in [0, \beta]$. It is easy to see that the above inequality implies $g(\xi) = 0$ for $\xi \in [0, \beta]$. Thus we get

$$\lim_n \|T_n(\xi)\, x - T(\xi)\, x\| (= \lim_n \|z_n(\xi)\|) = 0$$

for all $\xi \in [0, \beta]$. We shall show that the above convergence is uniform. Since

$$\|z_n(\xi)\|^2 \leqq 2\int_0^\xi \|z'_n(\eta)\|\ \|z_n(\eta)\|\, d\eta$$

(see (3. 8) and (3. 9)),

$$\sup_{0\leqq\xi\leqq\beta} \|z_n(\xi)\|^2 \leqq 2\int_0^\beta \|z'_n(\eta)\|\ \|z_n(\eta)\|\, d\eta \to 0$$

as $n \to \infty$, because the integrand converges boundedly to zero. Thus the theorem holds for $x \in D_0$.

Finally let $x \in \bar{D}_0$. There is a sequence $\{x_k\}$ $(x_k \in D_0)$ such that $\lim_k x_k = x$. Now

$$\begin{aligned}\|T_n(\xi)x - T(\xi)x\| &\leqq \|T_n(\xi)x - T_n(\xi)x_k\| \\ &\quad + \|T_n(\xi)x_k - T(\xi)x_k\| + \|T(\xi)x_k - T(\xi)x\| \\ &\leqq e^{\omega\xi}\|x - x_k\| + \|T_n(\xi)x_k - T(\xi)x_k\| + M_B\|x_k - x\|\end{aligned}$$

for $\xi \in [0, \beta]$. (Note there is a constant M_B such that $\|T(\xi)x_k - T(\xi)x\| \leqq M_B\|x_k - x\|$ for $\xi \in [0, \beta]$ and k, since the set $B = \{x, x_1, x_2, \cdots\}$ is bounded and the family $\{T(\xi);\ 0 \leqq \xi \leqq \beta\}$ is equi-Lipschitz continuous on bounded set.) Hence we get

$$\lim_n \|T_n(\xi)x - T(\xi)x\| = 0$$

uniformly on $[0, \beta]$.

Q. E. D.

4. **Approximation of semi-groups**. Let $\{T(\xi);\ \xi \geqq 0\}$ be a nonlinear semi-group of local type with $\|T(\xi)x - T(\xi)y\| \leqq e^{\omega\xi}\|x - y\|$, and let A_0 be its infinitesimal generator, and put

$$A_\delta = \delta^{-1}(T(\delta) - I) \qquad \text{for } \delta > 0.$$

THEOREM 4.1. I. *Each A_δ is the infinitesimal generator of a semi-group $\{T_\delta(\xi);\ \xi \geqq 0\}$ of local type satisfying the following conditions*:

(a) *For each $x \in X$, $T_\delta(\xi)x \in C^1([0, \infty); X)$*[3] *and*

$$(d/d\xi)T_\delta(\xi)x = A_\delta T_\delta(\xi)x \qquad \textit{for all } \xi \geqq 0.$$

(b) *For each $\xi \geqq 0$*

$$\sup_{x \neq y} \|T_\delta(\xi+h)x - T_\delta(\xi+h)y - (T_\delta(\xi)x - T_\delta(\xi)y)\| / \|x - y\| \to 0 \text{ as } h \to 0.$$

II. Suppose that

$$(4.1)\quad \begin{cases} \textit{there exists a set } D_0 \textit{ such that } D_0 \subset D(A_0) \textit{ and for any } x \in D_0, \\ T(\xi)x \in D(A_0) \textit{ for a.a. } \xi \geqq 0. \end{cases}$$

3) $C^1([0, \infty); X)$ denotes the set of all strongly continuously differentiable X-valued functions defined on $[0, \infty)$.

Then for each $x \in \overline{D_0}$ *we have*

$$T(\xi)x = \lim_{\delta \to 0+} T_\delta(\xi)x \quad \textit{for all } \xi \geqq 0, \tag{4.2}$$

and the convergence is uniform with respect to ξ *in every finite interval.*

REMARK. In case of nonlinear contraction semi-groups, the theorem has been proved by the author [6] (see also J. R. Dorroh [2]).

If X is a reflexive Banach space, then the assumption (4.1) is satisfied by taking $D_0 = D(A_0)$. (For, if $x \in D(A_0)$, then $T(\xi)x$ is strongly absolutely continuous on every finite interval. It follows from the reflexivility of X that $T(\xi)x$ is strongly differentiable at a.a. ξ and a fortiori $T(\xi)x \in D(A_0)$ for a.a. $\xi \geqq 0$.) Thus we have the following

COROLLARY 4.2. *If* $\{T(\xi);\ \xi \geqq 0\}$ *is a nonlinear semi-group of local type defined on a reflexive Banach space, then for each* $x \in \overline{D(A_0)}$

$$T(\xi)x = \lim_{\delta \to 0+} T_\delta(\xi) \quad \textit{for all } \xi \geqq 0,$$

and the convergence is uniform with respect to ξ *in every finite interval.*

We shall now prove Theorem 4.1.

PROOF. I. Fix $\delta > 0$. Since the map $x \to A_\delta x$ is Lipschitz continuous, uniformly in $x \in X$ (in fact, $\|A_\delta x - A_\delta y\| \leqq \delta^{-1}(e^{\omega\delta} + 1)\|x - y\|$ for $x, y \in X$), the equation

$$\begin{cases} (d/d\xi)\, u(\xi\,;\, x) = A_\delta u(\xi\,;\, x) & \text{for } \xi \geqq 0 \\ u(0\,;\, x) = x \end{cases}$$

has a unique solution $u(\xi\,;\, x) \in C^1([0, \infty)\,;\, X)$ for any $x \in X$. If we define $T_\delta(\xi)$ by

$$T_\delta(\xi)x = u(\xi\,;\, x) \quad \text{for } \xi \geqq 0,\ x \in X,$$

then $\{T_\delta(\xi);\ \xi \geqq 0\}$ is a nonlinear semi-group satisfying the condition (a) and its infinitesimal generator is A_δ.

We shall now prove that $\{T_\delta(\xi);\ \xi \geqq 0\}$ is of local type. Fix $x, y \in X$ and put

$$z(\xi) = T_\delta(\xi)x - T_\delta(\xi)y.$$

Clearly $z(\xi) \in C^1([0, \infty); X)$ and

$$\begin{cases} (d/d\xi)\, z(\xi) = A_\delta T_\delta(\xi)\, x - A_\delta T_\delta(\xi)\, y \\ z(0) = x - y. \end{cases}$$

Since $\|z(\xi)\|$ is absolutely continuous, $\|z(\xi)\|$ is differentiable at a.a. $\xi \geqq 0$. By Lemma 3.2, for a.a. $\xi \geqq 0$

$$\begin{aligned} \|z(\xi)\|\, [(d/d\xi)\, \|z(\xi)\|] &= \mathrm{Re}(z'(\xi), f_\xi) \\ &= \mathrm{Re}(A_\delta T_\delta(\xi)x - A_\delta T_\delta(\xi)y, f_\xi) \end{aligned}$$

for every $f_\xi \in F(z(\xi))$. Note that for each $u, v \in X$

$$\mathrm{Re}(A_\delta u - A_\delta v, f) \leqq \delta^{-1}(e^{\omega\delta} - 1)\, \|u - v\|^2$$

for all $f \in F(u - v)$. Hence

$$\|z(\xi)\|\, [(d/d\xi)\, \|z(\xi)\|] \leqq c_\delta \|z(\xi)\|^2 \qquad \text{for a.a. } \xi \geqq 0,$$

where $c_\delta = \delta^{-1}(e^{\omega\delta} - 1)$; and

$$\begin{aligned} \|z(\xi)\|^2 &= \|z(0)\|^2 + \int_0^\xi [(d/d\eta)\, \|z(\eta)\|^2]\, d\eta \\ &= \|z(0)\|^2 + 2\int_0^\xi \|z(\eta)\|\, [(d/d\eta)\, \|z(\eta)\|]\, d\eta \\ &\leqq \|z(0)\|^2 + 2c_\delta \int_0^\xi \|z(\eta)\|^2\, d\eta \end{aligned}$$

for any $\xi \geqq 0$. This leads the following inequality

$$\|z(\xi)\|^2 \leqq \|z(0)\|^2 \sum_{k=0}^{n} (2c_\delta \xi)^k / k! + [(2c_\delta)^{n+1}/n!] \int_0^\xi (\xi - \eta)^n\, \|z(\eta)\|^2\, d\eta$$

for all n and $\xi \geqq 0$. Letting $n \to \infty$, we get $\|z(\xi)\|^2 \leqq e^{2c_\delta \xi} \|z(0)\|^2$, i.e.,

$$\|T_\delta(\xi)x - T_\delta(\xi)y\| \leqq e^{c_\delta \xi} \|x - y\| \qquad \text{for all } \xi \geqq 0, \tag{4.3}$$

so that $\{T_\delta(\xi);\ \xi \geqq 0\}$ is of local type.

We shall show (b). Since $\|A_\delta x - A_\delta y\| \leqq \delta^{-1}(e^{\omega\delta}+1)\|x-y\|$ for all $x, y \in X$,

$$\begin{aligned}
&\|T_\delta(\xi+h)x - T_\delta(\xi+h)y - (T_\delta(\xi)x - T_\delta(\xi)y)\| \\
&= \|\int_\xi^{\xi+h} (A_\delta T_\delta(\eta)x - A_\delta T_\delta(\eta)y)\, d\eta\| \\
&\leqq \delta^{-1}(e^{\omega\delta}+1) \,|\int_\xi^{\xi+h} \|T_\delta(\eta)x - T_\delta(\eta)y\|\, d\eta| \\
&\leqq \delta^{-1}(e^{\omega\delta}+1)\, e^{c_\delta(\xi+|h|)}\|x-y\|\, |h|.
\end{aligned}$$

Hence we obtain (b).

II. Since $c_\delta = \delta^{-1}(e^{\omega\delta}-1) \to \omega$ as $\delta \to 0+$, there is a constant $c > 0$ such that $c_\delta \leqq c$ for $0 < \delta \leqq 1$. Hence by (4. 3) we obtain

$$\|T_\delta(\xi)x - T_\delta(\xi)y\| \leqq e^{c\xi}\|x-y\| \tag{4. 4}$$

for every $x, y \in X, \xi \geqq 0$ and $\delta \in (0, 1]$.

Let $\{\delta_n\}$ be a sequence such that $\delta_n \to 0+$. Put

$$T_n(\xi) = T_{\delta_n}(\xi) \ \textit{and}\ A_n = A_{\delta_n}(= \delta_n^{-1}(T(\delta_n) - I)).$$

Since $\lim_n A_{\delta n}x = A_0 x$ on $D(A_0)$ and $D(A_n) = X$, the assumptions in Theorem 2.1 are satisfied by taking $A = A_0$ and $D = D(A_0)$. Therefore for each $x \in \overline{D}_0$ we have

$$T(\xi)x = \lim_n T_{\delta_n}(\xi)x \quad \text{for each } \xi \geqq 0,$$

and the convergence is uniform with respect to ξ in every finite interval.

Q. E. D.

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