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NOTE ON THE RE-TOPOLOGIZATION OF A SPACE BY A SET OF OPERATORS

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Let $\{T_i\}$ $(i \in A)$ be a set of linear operators in a locally convex vector space E. In [2], M. Nagumo tried to adopt a new topology on E, which makes each T_i be continuous and makes it possible to extend T_i on all elements of E. His method is a generalization of a so-called Lax's negative norms. As the special cases of it we get Schwartz's distribution space and others. The purpose of this paper is to weaken the condition on E supposed by M. Nagumo. It may be said that our proof corresponds naturally to the technique used in the extension of differential operators.

As J. E. Roberts noted in [4], in particular taking a countable set of closed operators in a Hilbert space, we meet with a countably Hilbert space in the sense of Gel'fand [1]. We note here that the converse is also true, more precisely, that the topology of any countably Hilbert space can be constructed as the projective limit by a set of self-adjoint operators in a Hilbert space.

Let E be a separated locally convex vector space and $\{T_i\}$ $(i \in A)$ be a set of linear operators from E to E. We assume that $D(T_i)$, the domain of each T_i, is dense in E and that the set $\{T_i\}$ contains identity operator I. We shall from now onwards take the set of operators satisfying the following condition:

(C). $F = \bigcap_{i \in A} D(T_i')$ is total over E, i.e., the element x of E, for which $\langle f, x \rangle = 0$ for all f of F, is zero element. Here by T_i' we denote the transpose of T_i.

If the set $D(T_i')$ is total over E, T_i is a pre-closed operator. Therefore when the above condition is satisfied, all T_i are pre-closed. For the reflexive space E condition (C) means that F is dense in E', the dual space of E. Here the dual space of E is considered under the strong topology $\beta(E', E)$. We shall denote it by E'_β later.

Our main problem is to introduce a suitable topology in E and to make any T_i continuous from $D_i = D(T_i)$ to E. In that case, space E under such a topology we denote by $\widehat{E}$, and write completion of $\widehat{E}$ by $\widetilde{E}$. Then, under the some condition on $\{T_i\}$, T_i can be uniquely extended to continuous operators on $\widetilde{E}$.

THEOREM 1. *Suppose that E is a quasi-t-space, i.e., every strongly bounded set in E'_β is equi-continuous. Then, under the condition* (C), *we can give a new topology to E, in such a way that all T_i will be continuous from E with the primary topology into $\widehat{E}$, E with new topology.*

We remark here that bornological or barreled space is quasi-t-space. M. Nagumo proved this theorem for the bornological space.

PROOF. Since E is a quasi-t-space, natural embedding π from E to $E''=(E'_\beta)'_\beta$, the bidual of E, is a topological isomorphism [5, p. 71]. We introduce on F the coarsest convex topology which makes every T'_i to be a continuous operator from F to E'_β, and write F with this topology by $\widehat{F}$. That is, the topology on F is the projective topology with respect to $\{\{E'_\beta, T'_i\} i\in A\}$ [5, p. 84]. By the assumption that $\{T_i\}$ contains I, the topology of $\widehat{F}$ is finer than that of E'_β and so $\widehat{F}$ is separated. Any T''_i, the transpose of T'_i, the operator from E'' to $\widehat{F}'_\beta$, is strongly continuous. We restrict the domain of T''_i to $\pi(D_i)$. When we identify $\pi(D_i)$ to D_i, T''_i and T_i become algebraically equivalent. This fact is assured by that F is total over E. The space E, considered as a subspace of $\widehat{F}'_\beta$, is denoted by $\widehat{E}$. This space satisfies the condition of the theorem.

THEOREM 2. *In Theorem* 1, *let $\widehat{D}_i$ be the space D_i with relative topology induced by $\widehat{E}$, and assume that for any $i, j\in A$ there exists $k\in A$ satisfying $T_j(T_i)x=T_kx$ for any x of D_i. Then any T_i is a continuous operator from $\widehat{D}_i$ to $\widehat{E}$, therefore uniquely can be extended to a continuous operator on $\widetilde{E}$, which denotes the completion of $\widehat{E}$.*

PROOF. As $\widehat{F}$ has the projective limit topology with respect to the mappings $\{\{E'_\beta, T'_i\} i\in A\}$, $T'_i : \widehat{F}\to\widehat{F}$ are continuous if and only if $T'_j T'_i : \widehat{F}\to\widehat{E}'_\beta$ are continuous for all $j\in A$ [5, p. 84]. This latter condition is satisfied by the hypothesis. Thus all T'_i are continuous from $\widehat{F}$ to $\widehat{F}$, so that all T''_i are also continuous, a fortiori T''_i restricted to $\widehat{D}_i$.

The validity of last part of the theorem follows from the fact that D_i is dense in E and E is dense in $\widetilde{E}$, so $\widehat{D}_i$ dense in $\widetilde{E}$.

THEOREM 3. *If $\widehat{F}$ is reflexive and $\widehat{F}'_\beta$ is complete, $\widetilde{E}$ is the dual space* *of* $\widehat{F}$.

PROOF. $\widehat{E}$ is a linear subspace of $\widehat{F}'_\beta$, and by assumption $\widetilde{E}$ is also a subspace of $\widehat{F}'_\beta$. If $\widetilde{E} \neq \widehat{F}'_\beta$, there exists f, a non-zero element of $(\widehat{F}_\beta')' = \widehat{F}$, such that $<f, x> = 0$ for all x of $\widetilde{E}$. But this is impossible since $f \in \widehat{F} \subset E'$ and $E \subset \widetilde{E}$.

It must be remarked that if $\widehat{F}$ is bornological, its dual space $\widehat{F}'_\beta$ is complete [5, p. 104]. Fréchet space is bornological, and Fréchet space very well arises in application in the following form [3].

THEOREM 4. *Let E be a reflexive Banach space and A be a countable set. Then $\widehat{F}$ constructed above is a reflexive Frèchet space.*

PROOF. As the transpose of every T_i is a closed operator and $\{T_i\}$ contains I, $\widehat{F}$ is isomorphic to a closed subspace of the product $\prod_{i \in A} E'_i$, where $E'_i = E'_\beta$ [5, p. 88]. Since E'_β is a reflexive Banach space, $\prod_{i \in A} E'_i$ is reflexive and therefore $\widehat{F}$ is also reflexive Fréchet space.

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