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ON TRIGONOMETRIC FOURIER COEFFICIENTS

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1. Intrducition. In [1] we have proved the

THEOREM A. *Let $\{n_k\}$ be a sequence of positive integers and $\{a_k\}$ a sequence of non-negative real numbers satisfying*

$$n_{k+1} \geqq n_k(1+ck^{-\alpha}), \quad (c>0 \text{ and } 0 \leqq \alpha \leqq 1/2),$$

$$A_N^2 = 2^{-1}\sum_{k=1}^{N} a_k^2 \to +\infty \text{ and } a_N = O(A_N N^{-\alpha}), \quad \text{as } N \to +\infty.$$

Then for any sequence of real numbers $\{\alpha_k\}$ the trigonometric series $\sum a_k \cos(n_k x + \alpha_k)$ diverges a. e. and also is not a Fourier series.

This theorem was first proved by A. Zygmund for the case $\alpha = 0$, where $\{n_k\}$ has the *Hadamard* gap and the condition $a_N = O(A_N)$, as $N \to +\infty$, holds (c f. [2] p. 203).

The purpose of the present note is to prove the following

THEOREM B. *Let r, $1 \leqq r < 2$, be any given constants and (c, α) any pair of constants such that*

(1. 1) $$(c>0 \text{ and } 0 \leqq \alpha < 1) \text{ or } (c \geqq 1 \text{ and } \alpha = 1).$$

If a sequence of positive integers $\{n_k\}$ and a sequence of non-negative real numbers $\{a_k\}$ satisfy the conditions

(1. 2) $$n_{k+1} \geqq n_k(1+ck^{-\alpha}),$$

(1. 3) $$A_N^2 = 2^{-1}\sum_{k=1}^{N} a_k^2 \to +\infty \quad \text{and} \quad a_N = O(A_N^{2/(2-r)} N^{-\alpha}),^{1)} \text{ as } N \to +\infty,$$

1) If $r=1$, $\alpha=1$ and $\lim_{N\to\infty} a_N = 0$, the condition $a_N = O(A_N^2 N^{-1})$, as $N \to +\infty$, is impossible.

then for any $\{\alpha_k\}$ *the series* $\Sigma a_k \cos(n_k x+\alpha_k)$ *is not a Fourier series of a function of* $L_r(0, 2\pi)$.

REMARK. Putting $n_k = k$, then $n_{k+1} \geqq n_k(1+k^{-1})$, for all k.

If $1<r<2$ and $0\leqq\alpha<1$, there exists $\{a_k\}$ for which the conditions of Theorem B are satisfied and $\Sigma|a_k|^{r/(r-1)}<+\infty$. But if $1<r<2$ and $\alpha=1$, there exists $\{a_k\}$ which does not satisfy the conditions of Theorem B and $\Sigma|a_k|^{r/(r-1)} =+\infty$. (c f. Lemma 3).

On the conditions of Theorem B we can show the following

PROPOSITION. *Let* $1\leqq r<2$, $c>0$ *and* $0<\alpha\leqq 1/2$ *and let* $\{\varphi(n)\}$ *be any given sequence of positive numbers with* $\lim_{n\to\infty}\varphi(n)=+\infty$. *Then there exist* $\{n_k\}$ *and* $\{a_k\}$ *for which the conditions*

$$n_{k+1}\geqq n_k(1+ck^{-\alpha}),$$

$$A_N^2 = 2^{-1}\sum_{k=1}^{N} a_k^2 \to +\infty \text{ and } a_N = O(A_N^{2/(2-r)}\varphi(N)N^{-\alpha}), \text{ as } N\to +\infty,$$

are satisfied and the series $\Sigma a_k \cos n_k x$ *is the Fourier series of a function of* $L_r(0, 2\pi)$.

By the theorem of W. H. Young it is seen that the above proposition holds for $r=1$ and $n_k=k$, that is, $c=1$ and $\alpha=1$ (c f. [2] p. 183 (1. 5)).

2. Lemmas of Theorem B. The next lemma is well known.

LEMMA 1. *If* $f(x)\in L_r(0,2\pi)$, $r\geqq 1$, *and* $\sigma_n(x\,;f)$ *denotes the n-th* $(C,1)$ *mean of the Fourier series of* $f(x)$, *then* $\lim_{n\to\infty}\sigma_n(x\,;f)=f(x)$ *holds in the sense of* L_r*-norm.*

LEMMA 2. *For any trigonometric series* $\Sigma c_k\cos(kx+\gamma_k)$ *put*

$$D_0(x)=\sum_{k\leqq 2} c_k\cos(kx+\gamma_k) \text{ and } D_m(x)=\sum_{k=2^m+1}^{2^{m+1}} c_k\cos(kx+\gamma_k),\ (m\geqq 1).$$

Then there exists a constant C_0 *such that*

$$\int_0^{2\pi}\left\{\sum_{m=0}^{N} D_m(x)\right\}^4 dx \leqq C_0 \int^{2\pi}\left\{\sum_{m=0}^{N} D_m^2(x)\right\}^2 dx,\ (N\geqq 0),$$

and also the constant C_0 *does not depend on the series.*

Lemma 2 is a special case of Theorem (2. 1) on p. 224 in [3].

LEMMA 3. *If* $f(x) \in L_r(0, 2\pi)$, $1<r<2$, *and* $f(x) \sim \sum c_k \cos(kx+\gamma_k)$, *then there exists a constant* C_0 *such that*

$$\left(\sum |c_k|^{r/(r-1)}\right)^{(r-1)/r} \leqq C_0 \left\{\int_0^{2\pi} |f(x)|^r dx\right\}^{1/r}.$$

Lemma 3 is a part of the well known theorem of Hausdorff and Young.

A sequence of functions $\{f_n(x)\}$ defined over the interval $(0, 2\pi)$ is said to be *uniformly integrable* on the interval if the sequence $\int_0^{2\pi} |f_n(x)|\,dx$ is bounded and if $\lim_{n\to\infty}\int_{E_n} |f_n(x)|\,dx = 0$ for every sequence of measurable set $\{E_n\}$ satisfying $\lim_{n\to\infty}|E_n| = 0$[2] and $E_n \subset (0, 2\pi)$.

LEMMA 4. *If a uniformly integrable sequence of functions* $\{f_n(x)\}$ *defined on the interval* $(0, 2\pi)$ *converges in measure to* 0, *then we have* $\lim_{n\to\infty}\int_0^{2\pi} f_n(x)dx = 0$.

PROOF. Let $\varepsilon>0$, and set $E_n = \{x\,;\, x\in(0, 2\pi),\ |f_n(x)|>\varepsilon\}$. Since $\lim_{n\to\infty} f_n(x) = 0$, in measure, we have $\lim_{n\to\infty}|E_n| = 0$: hence, according to our hypothesis, $\lim_{n\to\infty}\int_{E_n}|f_n(x)|\,dx = 0$. Now we have $\int_{E_n^c}|f_n(x)|\,dx \leqq 2\pi\varepsilon$ and $\int_0^{2\pi}|f_n(x)|\,dx = \int_{E_n}|f_n(x)|\,dx + \int_{E_n^c}|f_n(x)|\,dx$, and this finishes the proof.

3. Preparations for the Proof of Theorem B. In this paragraph we assume that sequences $\{n_k\}$ and $\{a_k\}$ satisfy the conditions of Theorem B. First we put

2) For any measurable set E, $|E|$ denotes its Lebesgue measure.

(3.1) $$p(0)=0 \text{ and } p(k)=\max\{m\,;\, n_m \leqq 2^k\},^{3)} \quad k\geqq 1.$$

If $p(k)+1<p(k+1)$, then from the definition of $p(k)$ we have

$$2>n_{p(k+1)}/n_{p(k)+1}\geqq \prod_{m=p(k)+1}^{p(k+1)-1}(1+cm^{-\alpha}),$$

and this implies that

$$\begin{cases} 2\geqq 1+c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1), \text{ for } \alpha<1, \\ 5/2\geqq p(k+1)/\{p(k)+1\}, \quad \text{for } \alpha=1 \text{ and } k\geqq k_0. \end{cases}$$

Therefore we have

(3.2) $$p(k+1)-p(k)=O(p^{\alpha}(k)), \quad \text{as } k\to+\infty$$

and

(3.3) $$p(k+1)<3p(k), \quad \text{for } k\geqq k_0.$$

LEMMA 5. *For any given integers k, j, q and h satisfying*

$$\begin{cases} k\geqq j+3, \quad p(j)+1<h\leqq p(j+1) \\ p(k)+1<q\leqq p(k+1), \end{cases}$$

the total number of solutions (n_r, n_i) of the following equations

$$n_q-n_r=n_h\pm n_i, \quad \text{where } p(j)<i<h \text{ and } p(k)<r<q,$$

is at most $C_0 2^{j-k}p^{\alpha}(k)$, where C_0 does not depend on k, j, q and h.

PROOF. For any solutions (n_r, n_i) of the equations, we have

$$n_r=n_q-(n_h\pm n_i)>n_q-2^{j+2}>n_q(1-2^{j-k+2})\geqq n_q(1+2^{j-k+3})^{-1}.$$

Thus, if m_1 (or m_2) denotes the smallest (or the largest) index of n_r's satisfying either of the equations of the lemma, it is seen that

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3) For some k, $p(k)$ may be equal to $p(k+1)$.

$$1+2^{j-k+3}>n_q/n_{m_1}\geqq n_{m_2+1}/n_{m_1}$$

$$\geqq\prod_{k=m_1}^{m_2}(1+ck^{-\alpha})\geqq 1+c(m_2-m_1+1)p^{-\alpha}(k+1).$$

Hence, by (3. 3) we can prove the lemma 3.

Next, we put

$$(3.4)\quad \begin{cases} T_N(x)=\sum_{m=1}^{p(N+1)}\{1-n_m(n_{p(N+1)}+1)^{-1}\}a_m\cos(n_mx+\alpha_m), \\ C_N^2=2^{-1}\sum_{m=1}^{p(N+1)}a_m^2 \quad \text{and} \quad D_N^2=C_N^2-C_{N-1}^2, \end{cases}$$

that is, $T_N(x)$ is the $n_{p(N+1)}-$th $(C\ 1)-$mean of $\Sigma\ a_m\cos(n_mx+\alpha_m)$.

LEMMA 6. *We have*

$$\int_0^{2\pi}\{T_N^4(x)\}dx=O(C_N^{(8-2r)/(2-r)}),\quad \textit{as } N\to+\infty.$$

PROOF. If we put, for $k=0,1,2,\cdots,N$, and $N=1,2,\cdots$,

$$(3.5)\qquad \Delta_{k,N}(x)=\sum_{m=p(k)+1}^{p(k+1)}\{1-n_m(n_{p(N+1)}+1)^{-1}\}a_m\cos(n_mx+\alpha_m),$$

then by Lemma 2, it is sufficient to show that

$$(3.6)\qquad \int_0^{2\pi}\left\{\sum_{k=0}^{N}\Delta_{k,N}{}^2(x)\right\}^2dx=O(C_N^{(8-2r)/(2-r)}),\quad \text{as } N\to+\infty.$$

On the other hand we have, by (1. 3) and (3. 2),

$$\max_{k\leqq N}\max_x|\Delta_{k,N}(x)|\leqq\max_{k\leqq N}\sum_{m=p(k)+1}^{p(k+1)}|a_m|$$

$$=O(\max_{k\leqq N}C_k^{2/(2-r)}p^{-\alpha}(k)\{p(k+1)-p(k)\})=O(C_N^{2/(2-r)}),\ \text{as } N\to+\infty,$$

and hence

$$\text{(3.7)}\qquad \sum_{k=2}^{N}\sum_{j=k-2}^{k}\int_0^{2\pi}\Delta_{k,N}^2(x)\Delta_{j,N}^2(x)dx = O\left(C_N^{4/(2-r)}\sum_{k=2}^{N}\int_0^{2\pi}\Delta_{k,N}^2(x)\,dx\right)$$

$$= O\left(C_N^{4/(2-r)}\sum_{k=2}^{N}D_k^2\right) = O(C_N^{(8-2r)/(2-r)}),\quad \text{as } N\to+\infty.$$

Further, from the definitions of $\Delta_{k,N}(x)$ and D_k, we get

$$\text{(3.8)}\qquad \int_0^{2\pi}\Delta_{k,N}^2(x)\Delta_{j,N}^2(x)\,dx \leqq 8\pi D_k^2D_j^2 + 4\int_0^{2\pi}V_{k,N}(x)V_{j,N}(x)dx,$$

where

$$\begin{cases} V_{k,N}(x) = \displaystyle\sum_{q=p(k)+2}^{p(k+1)}\sum_{r=p(k)+1}^{q-1} b_{q,N}b_{r,N}\cos(n_qx+\alpha_q)\cos(n_rx+\alpha_r), \\ b_{m,N} = \{1-n_m(n_{p(N+1)}+1)^{-1}\}a_m. \end{cases}$$

Applying Lemma 5 to $V_{k,N}(x)V_{j,N}(x)$, $k\geqq j+3$, we obtain

$$\left|\int_0^{2\pi}V_{k,N}(x)V_{j,N}(x)dx\right|$$

$$\leqq C_0 2^{j-k}p^{\alpha}(k)\sum_{q=p(k)+2}^{p(k+1)}|a_q|(\max_{p(k)<r<q}|a_r|)\sum_{h=p(j)+2}^{p(j+1)}|a_h|(\max_{p(j)<i<h}|a_i|).$$

Since (1.3) and (3.2) imply that

$$\sum_{q=p(k)+2}^{p(k+1)}|a_q|(\max_{p(k)<r<q}|a_r|)$$

$$= O\left(\sum_{q=p(k)+2}^{p(k+1)}|a_q|^2\right)^{1/2}\{p(k+1)-p(k)\}^{1/2}C_k^{2/(2-r)}p^{-\alpha}(k)$$

$$= O(D_kC_k^{2/(2-r)}p^{-\alpha/2}(k)),\quad \text{as } k\to+\infty,$$

we have

$$\sum_{k=3}^{N}\sum_{j=1}^{k-3}\left|\int_0^{2\pi}V_{k,N}(x)V_{j,N}(x)dx\right|$$

$$= O\left(C_N^{4/(2-r)}\sum_{k=3}^{N}D_kp^{\alpha/2}(k)\sum_{j=1}^{k-3}2^{j-k}p^{-\alpha/2}(j)D_j\right),\quad \text{as } N\to+\infty.$$

On the other hand from (3.3) it is seen that $p(k) < 3^{j-k}p(j)$, for $k_0 < j < k$, and consequently

$$\sum_{j=}^{k-3} 2^{j-k}p^{-\alpha/2}(j)D_j = O(p^{-\alpha/2}(k)\sum_{j=1}^{k-3}(2/3^{\alpha/2})^{j-k}D_j)$$

$$= O\left(p^{-\alpha/2}(k)\left\{\sum_{j=1}^{k-3}(2/3^{\alpha/2})^{j-k}D_j^2\right\}^{1/2}\right), \text{ as } k \to +\infty.$$

Therefore, we have

$$\sum_{=3}^{N}\sum_{j=0}^{k-3}\left|\int_0^{2\pi} V_{k,N}(x)V_{j,N}(x)dx\right|$$

$$= O\left(C_N^{4/(2-r)}\sum_{k=3}^{N} D_k\left\{\sum_{j=1}^{k-3}(2/3^{\ /2})^{j-k}D_j^2\right\}^{1/2}\right)$$

$$= O\left(C_N^{4/(2-r)}\left(\sum_{k=3}^{N} D_k^2\right)^{1/2}\left\{\sum_{k=3}^{N}\sum_{j=1}^{k-3}(2/3^{\alpha/2})^{j-k}D_j^2\right\}^{1/2}\right)$$

$$= O\left(C_N^{(6-r)/(2-r)}\left\{\sum_{j=1}^{N} D_j^2\sum_{k=j+3}^{N}(2/3^{\alpha/2})^{j-k}\right\}^{1/2}\right) = O(C_N^{(8-2r)/(2-r)}),$$

$$\text{as } N \to +\infty.$$

Combining (3.7), (3.8) and the above relation we can obtain (3.6).

LEMMA 7. *There exists a positive constant C such that*

$$C_N^{-2}\int_E T_N^2(x)dx \leqq C\left\{\int_E |T_N(x)|^r dx\right\}^{\frac{2}{4-r}}.$$

holds for any measurable set E and $N = 1, 2, \cdots$.

PROOF. We have, by the Hölder inequality,

$$\int_E T_N^2(x)dx \leqq \left\{\int_E |T_N(x)|^r dx\right\}^{\frac{2}{4-r}}\left\{\int_0^{2\pi} T_N^4(x)dx\right\}^{\frac{2-r}{4-r}}.$$

Therefore, by Lemma 6 we can complete the proof.

4. **Proof of Theorem B.** Suppose, on the contrary, that the given series

$\Sigma a_k \cos(n_k x+\alpha_k)$, for some $\{\alpha_k\}$, is the Fourier series of a function $f(x) \in L_r(0,2\pi)$. Then by the Riemann-Lebesgue lemma, we have

$$a_N \to 0, \qquad \text{as } N\to +\infty. \tag{4.1}$$

If $r=1$, (1.3), (3.2) and (4.1) imply that

$$D_N^2 = o(\max_{p(N)<m\leqq p(N+1)} |a_m|)\{p(N+1)-p(N)\}=o(C_N^2), \text{ as } N\to +\infty,$$

and if $1<r<2$, (1.3), (3.2) and Lemma 3 imply that

$$\begin{aligned} D_N^2 &\leqq (\max_{p(N)<m\leqq p(N+1)} |a_m|^{2-r})\left(\sum_{m=p(N)+1}^{p(N+1)} |a_m|^r\right) \\ &= O(C_N^2 p^{-\alpha(2-r)}(N)\left(\sum_{m=p(N)+1}^{p(N+1)} |a_m|^{\frac{r}{r-1}}\right)^{r-1} \{p(N+1)-p(N)\}^{2-r} \\ &= o(C_N^2), \qquad \text{as } N\to +\infty. \end{aligned}$$

Therefore, it is seen that

$$\lim_{N\to\infty} C_N/C_{N-1}=1. \tag{4.2}$$

Putting

$$B_N^2 = 2^{-1}\sum_{m=1}^{p(N+1)} \{1-n_m(n_{p(N+1)}+1)^{-1}\}^2 a_m^2, \tag{4.3}$$

we have

$$B_N^2=(2\pi)^{-1}\int_0^{2\pi} T_N^2(x)\, dx \tag{4.4}$$

and

$$B_N^2 > C_{N-1}^2/4, \quad \text{if } p(N+1)>p(N). \tag{4.5}$$

Therefore, we have, by (4.2) and (4.5),

$$C_N \geqq B_N \geqq C_N/3, \quad \text{for } N\geqq N_0, \tag{4.6}$$

and consequently, by Lemma 7, for any set $E\subset (0,2\pi)$ and $N=1,2,\cdots,$

$$(4.7) \qquad \int_E \{T_N(x)/B_N\}^2 dx \leqq C' \left\{ \int_E |T_N(x)|^r dx \right\}^{2/(4-r)},$$

for some constant C' which does not depend on E and N. Since $T_N(x)$ is the $n_{p(N+1)}$-th $(C, 1)$-mean of the Fourier series of $f(x)$, we have, from Lemma 1 and the Minkowski inequality,

$$(4.8) \qquad \lim_{N\to\infty} \int_E |T_N(x)|^r dx = \int_E |f(x)|^r dx, \text{ uniformly in } E \subset (0.2\pi).$$

From (4.7) and (4.8) it is seen that $\{T_N(x)/B_N\}^2$ is uniformly integrable over the interval $(0, 2\pi)$. Further $T_N^2(x)/B_N^2 \to 0$, in measure, as $N \to +\infty$. Therefore by Lemma 4, we have

$$\lim_{N\to\infty} \int_0^{2\pi} \{T_N(x)/B_N\}^2 dx = 0,$$

and this contradicts with (4.4).

5. Lemmas of the Proposition. First we prove the

LEMMA 8. *If* $\sum_{k=1}^{\infty} b_k \cos kx$ $(b_1 \neq 0)$ *converges in* L_1*-norm, then the series* $\Sigma\, b_k B_k^{-1} \cos kx$ *is the Fourier series of a function of* $L_r(0, 2\pi)$, *for any* r, $1 \leqq r < 2$, *where* $B_N = \left(2^{-1} \sum_{k=1}^{N} b_k^2\right)^{1/2}$.

PROOF. It is sufficient to consider the case $B_N \to +\infty$, as $N \to +\infty$, and $1 < r < 2$. Putting $S_N(x) = \sum_{k=1}^{N} b_k \cos kx$, we have, by the Hölder inequality,

$$(5.1) \qquad \|S_N\|_r \leqq \|S_N\|_1^{\frac{2-r}{r}} \|S_N\|_2^{\frac{2r-2}{r}} = O(B_N^{2-\frac{2}{r}}), \qquad \text{as } N \to +\infty.$$

By the partial summation, it is seen that

$$\sum_{k=M}^{N} b_k B_k^{-1} \cos kx = S_N(x) B_N^{-1} - S_{M-1}(x) B_M^{-1} + \sum_{k=M}^{N-1} S_k(x)(B_k^{-1} - B_{k+1}^{-1}),$$

and hence, by the Minkowski inequality and (5.1),

$$\|\sum_{k=M}^{N} b_k B_k^{-1}\cos kx\|_r \leqq \|S_N\|_r B_N^{-1} + \|S_{M-1}\|_r B_M^{-1} + \sum_{k=M}^{N-1} \|S\|_{kr}(B_k^{-1} - B_{k+1}^{-1})$$

$$= o(1) + O\left(\sum_{k=M}^{N-1} b_k^2 B_k^{-1-\frac{2}{r}}\right) = o(1), \quad \text{as } M \text{ and } N \to +\infty.$$

Therefore, the series $\Sigma\ b_k B_k^{-1} \cos kx$ converges in L_r-norm.

LEMMA 9. *Let $\{\rho_j\}$ be a sequence of positive numbers such that $\{\rho_j^{-1}\}$ is convex, $\rho_j \leqq \log j$ for $j \geqq 1$, and $\rho_j \uparrow +\infty$, as $j \to +\infty$. Then there exists a sequence $\{\varepsilon_j\}$, $\varepsilon_j = 0$ or 1, satisfying*

$$\sum \rho_j^2 j^{-1} \varepsilon_j < +\infty \quad \textit{and} \quad \sum \rho_j^3 j^{-1} \varepsilon_j = +\infty.$$

PROOF. Since $\{\rho_j^{-1}\}$ is positive, convex and non-increasing, $j(\rho_j^{-1} - \rho_{j+1}^{-1}) \to 0$, as $j \to +\infty$, there exists a positive number c_0 such that

$$0 < p_j = c_0 j(\rho_j^{-1} - \rho_{j+1}^{-1})\rho_j^{-2} < 1, \text{ for } j \geqq 1.$$

Therefore, we can take a probability space $(\Omega, \mathcal{F}, P)$ and asequence of independent random variables $\{X_j(\omega)\}$on it with the following probability distributions ;

$$X_j(\omega) = \begin{cases} 1, & \text{with probability } p_j, \\ 0, & \text{with probability } 1 - p_j. \end{cases}$$

Since $\Sigma\ [E\{(\rho_j^r j^{-1} X_j)^2\} - \{E(\rho_j^r j^{-1} X_j)\}^2] \leqq \Sigma \rho_j^{2r} j^{-2} < +\infty$, for $r = 2$ and 3, we have, by the well known theorem of Khintchine and Kolmogorov,

$$P\left[\sum_{j=1}^{\infty} \{\rho_j^r j^{-1} X_j - E(\rho_j^r j^{-1} X_j)\} \text{ converges}\right] = 1, \ (r = 2, 3). \tag{5.2}$$

On the other hand it is easily seen that

$$\sum_{j=1}^{\infty} E(\rho_j^r j^{-1} X_j) \begin{cases} < +\infty, & \text{if } r=2, \\ = +\infty, & \text{if } r=3. \end{cases} \tag{5.3}$$

By (5.2) and (5.3), we can take a point $\omega_0 \in \Omega$ such that

$$\sum \rho_j^2 j^{-1} X_j(\omega_0) < +\infty \text{ and } \sum \rho_j^3 j^{-1} X_j(\omega_0) = +\infty.$$

Putting $\varepsilon_j = X_j(\omega_0)$, we can prove the lemma.

6. Proof of the Proposition. I. First let us put

$$(6.1)\qquad \begin{cases} q(j) = [j^{1/\alpha}], \\ l(j) = \min\{[q^{\alpha}(j)/c],\ q(j+1)-q(j)\}, \\ j_0 = \min\{j\,;\,l(j) \geqq 1\}.^{4)} \end{cases}$$

Since $q(j+1)-q(j) \sim \alpha^{-1}j^{(1-\alpha)/\alpha}$ and $q^{\alpha}(j) \sim j$, as $j \to +\infty$,[5] we have

$$(6.2)\qquad l(j) \sim \begin{cases} j/c, & \text{if } 0<\alpha<1/2, \\ j\ \min\ (2, 1/c), & \text{if } \alpha=1/2. \end{cases}$$

Next we put

$$n_1 = 1 \text{ and } n_{k+1} = [n_k(1+ck^{-\alpha})+1], \quad \text{for } k+1 \leqq q(j_0),$$

and if $n_{q(j)}$, $j \geqq j_0$, is defined, then we put

$$n_{q(j)+l} = \begin{cases} n_{q(j)}(1+l), & \text{if } 1 \leqq l \leqq l(j), \\ [n_{q(j)+l-1}\{1+cq^{-\alpha}(j)\}+1], & \text{if } l(j)<l \leqq q(j+1)-q(j). \end{cases}$$

Then (6.2) and $q^{\alpha}(j) \sim j$, as $j \to +\infty$, imply that $n_{k+1} \geqq n_k(1+ck^{-\alpha})$.

II. It is well known that we can take a sequence $\{\rho(j)\}$ such that $0<\rho(j) <\min\ \{\varphi^{1/2}(j),\ \log j\}$, $\{1/\rho(j)\}$ is convex and $\rho(j) \uparrow +\infty$, as $j \to +\infty$. On the other hand there exists an integrable function $f(x)$ such that $f(x) \sim \sum_{k=1}^{\infty} c_k \cos kx$ and

$$(6.3)\qquad c_n \geqq \{\rho([n^{1/2}])\}^{-1/2}, \quad \text{for all } n \geqq 1.$$

Further, by Lemma 9 we can take a sequence $\varepsilon_j (\varepsilon_j = 0 \text{ or } 1)$ for which

$$(6.4)\qquad \sum \rho^2(j)j^{-1}\varepsilon_j < +\infty \text{ and } \sum \rho^3(j)j^{-1}\varepsilon_j = +\infty.$$

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4) For real number x, $[x]$ denotes the integral part of x.
5) For two sequences $\{d_k\}$ and $\{e_k\}$, $d_k \sim e_k$, as $k \to +\infty$, means that $\lim_{k\to\infty} d_k/e_k = 1$.

Using the above defined quantities, we put b_k as follows : If $k=q(j)+l$, for $j \geqq j_0$, $0 \leqq l \leqq l(j)$ and $\varepsilon_j=1$, then

$$b_k=\rho^2(j)j^{-1}[1-(l+1)\{l(j)+1\}^{-1}]c_{l+1}, \tag{6.5}$$

and if otherwise, then

$$b_k=k^{-2}. \tag{6.5'}$$

Then it is seen that if $j \geqq j_0$ and $\varepsilon_j=1$,

$$\sum_{l=0}^{l(j)} b_{q(j\ +} \cos n_{q(j)+l}x=\rho^2(j)j^{-1}\varepsilon_j\sigma_{l(j)}(n_{q(j)}x\,;f),$$

where $\sigma_n(x;f)$ denotes the n-th $(C,1)$-mean of the Fourier series of $f(x)$. Therefore, putting $S_n(x)=\sum_{l=1}^{n} c_l \cos lx$ we have

$$\begin{aligned}
&\max_{m\leqq l(j)} \int_0^{2\pi} \left|\sum_{l=0}^{m} b_{q(j)+l}\cos n_{q(j)+l}\,x\right| dx \\
&\leqq j^{-1}\rho^2(j) \max_{m\leqq l(j)} \int_0^{2\pi} \left|\sum_{l=1}^{m+1} [1-l\{l(j)+1\}^{-1}]c_l \cos lx\right| dx \\
&\leqq j^{-1}\rho^2(j) \max_{m\leqq l(j)} \left[\int_0^{2\pi} |S_{m+1}(x)|\, dx+\{l(j)+1\}^{-1}\sum_{l=0}^{m}\int_0^{2\pi} |S_l(x)|\, dx\right. \\
&=O(j^{-1}\rho^2(j)\log l(j))=o(1), \qquad \text{as } j\to+\infty,
\end{aligned}$$

and, if $\varepsilon_j=1$, we have, by Lemma 1,

$$\int_0^{2\pi} \left|\sum_{l=0}^{l(j)} b_{q(j)+l}\cos n_{q(j)+l}x\right| dx<\rho^2(j)j^{-1}C_0, \text{ for some } C_0>0.$$

Hence, by (6.4) and (6.5'),

$$\sum b_k \cos n_k x \quad \text{converges in } L_1\text{-norm.} \tag{6.6}$$

Further, we have, by (6.2), (6.3) and (6.4),

$$(6.7)\qquad 2B^2_{q(m)+l(m)} = \sum_{k=1}^{q(m)+l(m)} b_k{}^2 \geqq \sum_{j=j_0}^{m} \sum_{l=0}^{l(i)} b^2_{q(j)+l}$$

$$\geqq \sum_{j=j_0}^{m} \rho^4(j) j^{-2}\varepsilon_j \sum_{l=0}^{l(j)} [1-(l+1)\{l(j)+1\}^{-1}]^2 c^2_{l+1}$$

$$\geqq \beta \sum_{j=j_0}^{m} \rho^3(j) j^{-1}\varepsilon_j \to +\infty, \qquad \text{as } m\to +\infty,$$

and since $q^\alpha(j)\sim j$, as $j\to+\infty$,

$$(6.8)\qquad b_k = O(\rho^2(k)k^{-\alpha}) = O(\varphi(k)k^{-\alpha}), \qquad \text{as } k\to+\infty.$$

III. Putting $a_k = b_k B_k{}^{-1}$, then Lemma 8 and (6.6) imply that $\Sigma\ a_k \cos n_k x$ is the Fourier series of a function of $L_r(0, 2\pi)$, $1\leqq r<2$, and by (6.7) and (6.8),

$$\begin{cases} A_N^2 = 2^{-1}\sum_{k=1}^{N} a_k{}^2 = 2^{-1} \sum_{k=1}^{N} b_k{}^2 B_k^{-2} \to +\infty, \\ a_N = o(b_N) = O(\varphi(N)N^{-\alpha}) = O(A_N^{2/(2-r)}\varphi(N)N^{-\alpha}), \qquad \text{as } N\to+\infty. \end{cases}$$

Thus, we can complete the proof of the proposition.

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