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ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES

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1. Introduction.

Let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x) \tag{1}$$

and let $s_n(x)$ and $\sigma_n^\alpha(x)$ $(\alpha>-1)$ denote the n-th partial sum and n-th (C,α) mean of Fourier series (1), respectively. If the series

$$\sum_{n=0}^{\infty}|\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)|$$

is convergent, we say that the series (1) is absolutely summable (C,α) or summable $|C,\alpha|$ at the point x.

We have

$$\{\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)\}=\frac{\tau_n^\alpha(x)}{n}$$

where

$$\tau_n^\alpha(x)=\frac{1}{A_n^\alpha}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}kA_k(x)$$

and

$$A_n^\alpha=\binom{n+\alpha}{n}.$$

For $f(x)\in L^p(1\leqq p<\infty)$ we define

$$\omega_p^{(1)}(t,f)=\sup_{0<h<t}\left\{\int_{-\pi}^{\pi}|f(x+h)-f(x-h)|^p dx\right\}^{1/p}$$

and

$$\omega_p^{(2)}(t,f)=\sup_{0<h<t}\left\{\int_{-\pi}^{\pi}|f(x+h)+f(x-h)-2f(x)|^p dx\right\}^{1/p}.$$

SUNOUCHI [7] proved the following theorems.

THEOREM A. *Let* $1<p\leqq 2$. *If*

$$\omega_p^{(1)}(t,f)=O\left\{\left(\log\frac{1}{t}\right)^{-\frac{1}{2}-\delta}\right\} \qquad (\delta>0),$$

then the series (1) *is summable* $|C,\alpha|$ *almost everywhere for* $\alpha>1/p$.

THEOREM B. Let $1<p\leqq 2$. *If*

$$\omega_p^{(1)}(t,f)=O\left\{\left(\log\frac{1}{t}\right)^{-\left(1-\frac{1}{p}+\frac{1}{2}+\delta\right)}\right\} \qquad (\delta>0),$$

then the series (1) *is summable* $|C,1/p|$ *almost everywhere.*

We prove the following theorems which generalize SUNOUCHI's theorems.

THEOREM I. *Let* $f(x)\in L^p$ $(1<p\leqq 2)$ *and let* $\{\mu_n\}$ $(n=1,2,3,\cdots)$ *be a monotonic non-increasing sequence tending to zero, and satisfying the condition*

$$\sum_{n=1}^{\infty}\frac{1}{n\left(\sum^{n}\mu_k\right)^2}<\infty. \tag{2}$$

If

$$\sum_{n=1}^{\infty}\mu_n\omega_p^{(2)}\left(\frac{1}{n},f\right)<\infty \tag{3}$$

then the series (1) *is summable* $|C,\alpha|$ *almost everywhere for* $\alpha>1/p$.

THEOREM II. *Let* $f(x)\in L^p$ $(1<p\leqq 2)$ *and let* $\{\rho_n\}(n=1,2,\cdots)$ *be a monotonic non-decreasing sequence such that* $\rho_n[\log(n+1)]^{-1/2}$ *is non-increasing and satisfying the condition :*

(4) $$\sum_{n=1}^{\infty} \frac{1}{n\rho_n^2 \log(n+1)} < \infty .$$

If

(5) $$\sum_{n=1}^{\infty} \frac{\rho_n \omega_p^{(2)}\left(\frac{1}{n}, f\right)}{n[\log(n+1)]^{\frac{1}{p}-\frac{1}{2}}} < \infty$$

then the Fourier series of $f(x)$ is $|C, 1/p|$ summable almost everywhere.

Using an equivalence theorem of LEINDLER [6] (Satz III), we get from Theorem I and Theorem II the following corollaries.

COROLLARY I. *Let $f(x) \in L^p$ $(1 < p \leqq 2)$ and let $\mu(x)$ $(x \geqq 1)$ be a non-increasing function. If $\mu_n = \mu(n)$ satisfies the condition (2) and if for a certain $\beta(>0)$ $[\log(x+1)]^{-\gamma_1} \geqq x^\beta \mu(x) \geqq [\log(x+1)]^{-\gamma_2} (\gamma_1 < \gamma_2)$, then both conditions*

$$\int_0^1 \frac{\mu\left(\frac{1}{t}\right)}{t^2} \left(\int_0^{2\pi} |f(x+2t)+f(x-2t)-2f(x)|^p dx \right)^{1/p} dt < \infty$$

and

$$\sum_{n=1}^{\infty} \mu(n) E_n(f, p) < \infty \text{ [1]}$$

are sufficient for the summability $|C, \alpha|$ $(\alpha > 1/p)$ of the Fourier series (1) almost everywhere.

COROLLARY II. *Let $f(x) \in L^p$ $(1 < p \leqq 2)$ and let $\rho(x)$ $(x \geqq 1)$ be a non-decreasing function. If $\rho_n = \rho(n)$ satisfies the condition (4) and if for a certain $\alpha(>0)$*

$$[\log(x+1)]^{\frac{1}{p}-\frac{1}{2}-\gamma_1} \geqq x^{\alpha-1}\rho(x) \geqq [\log(x+1)]^{\frac{1}{p}-\frac{1}{2}-\gamma_2}$$

$(\gamma_1 < \gamma_2)$, furthermore if $\rho(n)[\log(n+1)]^{-\frac{1}{2}}$ is nonincreasing, then both conditions

$$\int_0^1 \frac{\rho\left(\frac{1}{t}\right)}{t|\log t|^{\frac{1}{p}-\frac{1}{2}}} \left(\int_0^{2\pi} |f(x+2t)+f(x-2t)-2f(x)|^p dx \right)^{1/p} dt < \infty$$

1) $E_n(f, p)$ denotes the best approximation of $f(x)$, in the sense of the metric of L^p, by trigonometric polynomials of order $(n-1)$.

and

$$\sum_{n=1}^{\infty} \frac{\rho(n)E_n(f,p)}{n[\log(n+1)]^{\frac{1}{p}-\frac{1}{2}}} < \infty$$

are sufficient for the summability $|C, 1/p|$ *of the Fourier series* (1) *almost everywhere.*

It is easy to see that in the case

$$\mu_n = n^{-1}[\log(n+1)]^{\varepsilon-\frac{1}{2}} \qquad \left(0<\varepsilon<\min\left(\frac{1}{2},\delta\right)\right)$$

Theorem I includes Theorem A and if

$$\rho_n = [\log(n+1)]^{\varepsilon} \qquad \left(0<\varepsilon<\frac{1}{2}\right)$$

then Theorem II implies Theorem B.

It is also easy to verify that if

$$\mu_n = \frac{[\log\log(n+2)]^{1/2+\varepsilon}}{n[\log(n+1)]^{1/2}} \qquad (\varepsilon>0)$$

and

$$\omega_p^{(2)}(t,f) = O\left\{\left(\log\frac{1}{t}\right)^{-1/2}\left(\log\log\frac{1}{t}\right)^{-3/2-\delta}\right\} \qquad \left(0<\varepsilon<\min\left(\frac{1}{2},\delta\right)\right)$$

or

$$\rho_n = [\log\log(n+2)]^{1/2+\varepsilon}$$

and

$$\omega_p^{(1)}(t,f) = O\left\{\left(\log\frac{1}{t}\right)^{\frac{1}{p}-\frac{3}{2}}\left(\log\log\frac{1}{t}\right)^{-\frac{3}{2}-\delta}\right\} \qquad \left(0<\varepsilon<\min\left(\frac{1}{2},\delta\right)\right)$$

then the conditions of Theorem I or Theorem II are satisfied, thus the series (1) is $|C,\alpha|$ $(\alpha>1/p)$ or $|C, 1/p|$ summable almost everywhere, respectively.

2. We require the following known lemmas :

LEMMA 1. *If* $f(x)\in L^p$ $(1<p<\infty)$, *then*

$$\|f(x)-s_n(x)\|_p = O\left\{ \omega_p^{(2)}\left(\frac{1}{n}, f\right)\right\}.$$

(See, e. g. [9] p. 339 and [8] p. 226.)

LEMMA 2. (CHOW [3],Theorem I). *If $f(x) \in L^p$ $(1<p\leqq 2)$ then the series*

$$\sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(x)|^2}{n}$$

is convergent for almost all x, where $\alpha > 1/p$.

LEMMA 3. *If $f(x) \in L^p$ $(1<p\leqq 2)$, then the series*

$$\sum_{n=1}^{\infty} \frac{|\tau_n^{1/p}(x)|^2}{n[\log(n+1)]^{2-2/p}}$$

is convergent for almost all x.

This lemma follows from Theorem I of KOZIMA [4].

LEMMA 4. *Let $0<\alpha<1$ and $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_n \cdot n^{-1}$ is non-increasing and $\Delta\lambda_n = \lambda_n - \lambda_{n+1} = O\left\{\frac{\lambda_n}{n}\right\}$. If the series*

$$\sum_{n=1}^{\infty} \frac{\lambda_n |\tau_n^\alpha(x)|}{n}$$

is convergent, then the series $\sum_{n=0}^{\infty} \lambda_n A_n(x)$ is summable $|C, \alpha|$.

The proof of Lemma 4 runs similarly to that of Lemma 4 of CHOW [2].

LEMMA 5. (KOGBENTLIANTZ [5]). *If the series $\sum_{n=0}^{\infty} a_n$ is summable $|C,\alpha|$ $(\alpha>-1)$, then it is also summable $|C, \alpha+\beta|$ for any $\beta>0$.*

3. We prove the following lemmas :

LEMMA 6. *Let $f(x) \in L^p$ $(1<p\leqq 2)$ and let $\{u_n\}$ be a sequence of positive*

numbers. If

$$\sum_{n=1}^{\infty} u_n \omega_p^{(2)}\left(\frac{1}{n}, f\right) < \infty \tag{6}$$

then $\sum_{n=0}^{\infty} \overline{\lambda}_n A_n(x)$ *is the Fourier series of a function of class* L^p, *where* $\overline{\lambda}_0 = 1$ *and* $\overline{\lambda}_n = \sum_{k=1}^{n} u_k$ $(n = 1, 2, \cdots)$.

PROOF. Let us denote by $t_n(x)$ the n-th partial sum of the series $\sum_{n=0}^{\infty} \overline{\lambda}_n A_n(x)$, i. e., $t_n(x) = \sum_{k=0}^{n} \overline{\lambda}_k A_k(x)$, then

$$\|t_n(x) - f(x)\|_p = \|\overline{\lambda}_0 (A_0(x) - f(x)) + \sum_{k=1}^{n} \overline{\lambda}_k A_k(x)\|_p$$

$$= \|\sum_{k=0}^{n-1} (s_k(x) - f(x)) \Delta \overline{\lambda}_k + (s_n(x) - f(x)) \overline{\lambda}_n \|_p$$

$$\leqq \sum_{k=0}^{n-1} \|s_k(x) - f(x)\|_p |\Delta \overline{\lambda}_k| + \|s_n(x) - f(x)\|_p \overline{\lambda}_n$$

$$= \sum\nolimits_1 .$$

By Lemma 1 and (6) we have

$$\sum\nolimits_1 \leqq C_1 \sum_{k=1}^{n-1} \omega_p^{(2)}\left(\frac{1}{k}, f\right) u_k + C_2 \omega_p^{(2)}\left(\frac{1}{n}, f\right) \sum_{k=}^{n}$$

$$\leqq C_1 \sum_{k=1}^{\infty} \omega_p^{(2)}\left(\frac{1}{k}, f\right) u_k + C_2 \sum_{k=1}^{n} \omega_p^{(2)}\left(\frac{1}{k}, f\right) u_k$$

$$\leqq C_3 \sum_{k=1}^{\infty} \omega_p^{(2)}\left(\frac{1}{k}, f\right) u_k < C .$$

Hence

$$\|t_n(x)\|_p \leqq \|t_n(x) - f(x)\|_p + \|f(x)\|_p = O(1 .$$

From this it follows the statement of Lemma 6.

LEMMA 7. *Let* $f(x) \in L^p$ $(1 < p \leqq 2)$ *and let* $\{v_n\}$ *be a sequence of positive numbers. If*

$$\sum_{n=1}^{\infty} v_n \omega_p^{(2)}\left(\frac{1}{n}, f\right)[\log(n+1)]^{1-1/p} < \infty ,$$

then the series $\sum_{n=0}^{\infty} l_n A_n(x)$ *is the Fourier series of a function of class* L^p, *where* $l_0 = 1$ *and* $l_n = \sum_{k=1}^{n} v_k[\log(k+1)]^{1-1/p}$ $(n = 1, 2, \cdots)$

PROOF. It runs similarly to the proof of Lemma 6.

LEMMA 8. *Let* $f(x) \in L^p$ $(1 < p \leqq 2)$ *and let* $\{\kappa_n\}$ *be a sequence of positive numbers, such that* κ_n/n *is non-increasing and* $\Delta\kappa_n = O(\kappa_n/n)$. *If*

$$\sum_{n=1}^{\infty} \frac{\kappa_n^2}{n} < \infty$$

then the series

$$\sum_{n=0}^{\infty} \kappa_n A_n(x) \tag{7}$$

is summable $|C, \alpha|$ *almost everywhere, for any* $\alpha > 1/p$.

If

$$\sum_{n=1}^{\infty} \frac{\kappa_n^2[\log(n+1)]^{2-2/p}}{n} < \infty$$

then the series (7) *is summable* $|C, 1/p|$ *almost everywhere.*

PROOF. Let $1/p < \alpha' < 1$. Applying Schwarz's inequality we have

$$\sum_{n=1}^{\infty} \frac{\kappa_n |\tau_n^{\alpha'}(x)|}{n} \leqq \left(\sum_{n=1}^{\infty} \frac{\kappa_n^2}{n}\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{|\tau_n^{\alpha'}(x)|^2}{n}\right)^{1/2}.$$

From this inequality, by Lemma 2 and Lemma 4, we get that the series (7) is summable $|C, \alpha'|$ almost everywhere, and by Lemma 5, we get that the series (7) is summable $|C, \alpha|$ almost everywhere, for any $\alpha > 1/p$.

The proof of the second statement follows the same lines as that of the first statement. Applying Schwarz's inequality we have

$$\sum_{n=1}^{\infty} \frac{\kappa_n |\tau_n^{1/p}(x)|}{n} \leqq \left(\sum_{n=1}^{\infty} \frac{|\tau_n^{1/p}(x)|^2}{n[\log(n+1)]^{2-2/p}}\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{\kappa_n^2[\log(n+1)]^{2-2/p}}{n}\right)^{1/2}.$$

From this inequality, by Lemma 3 and Lemma 4 we obtain the statement.

4. Proof of Theorem I. Let $\overline{\lambda}_0 = 1$ and $\overline{\lambda}_n = \sum_{k=1}^{n} \mu_k$ $(n = 1, 2, \cdots)$. By condition (3) and Lemma 6 we have that $\sum_{n=0}^{\infty} \overline{\lambda}_n A_n(x)$ is the Fourier series of a function in L^p.

Let now $\kappa_n = \overline{\lambda}_n^{-1}$ $(n = 0, 1, \cdots)$. By condition (2) $\{\kappa_n\}$ satisfies the conditions of Lemma 8, so we have that the series (1) is summable $|C, \alpha|$ almost everywhere, for any $\alpha > 1/p$, as it was stated.

5. Proof of Theorem II. Let $v_n = \dfrac{\rho_n}{n\sqrt{\log(n+1)}}$ $(n = 1, 2, \cdots)$. By condition (5) and Lemma 7 we have that $\sum_{n=0}^{\infty} l_n A_n(x)$ is the Fourier series of a function in L^p, where $l_0 = 1$ and $l_n = \sum_{k=1}^{n} \dfrac{\rho_k}{k[\log(k+1)]^{1/p-1/2}}$.

Let now $\kappa_n = l_n^{-1}$ $(n = 0, 1, \cdots)$. Since

$$\Delta\kappa_n = \kappa_n - \kappa_{n+1} = \frac{1}{l_n} - \frac{1}{l_{n+1}}$$

$$= \frac{\rho_{n+1}}{(n+1)[\log(n+2)]^{1/p-1/2} l_n \cdot l_{n+1}}$$

$$\leqq \frac{\rho_{n+1}}{(n+1)[\log(n+2)]^{1/p-1/2} \cdot l_n \sum_{k=1}^{n+1} \dfrac{\rho_k[\log(k+1)]^{1-1p}}{\sqrt{\log(k+1)} \cdot k}}$$

$$\leqq \frac{\rho_{n+1}}{(n+1)[\log(n+2)]^{1/p-1/2} \cdot l_n \dfrac{\rho_{n+1}[\log(n+2)]^{1-1/p}}{\sqrt{\log(n+2)}}}$$

$$= \frac{1}{(n+1)l_n} < \frac{\kappa_n}{n}$$

and since on the other hand

$$\sum_{n=1}^{\infty} \frac{\kappa_n^2[\log(n+1)]^{2-2/p}}{n} = \sum_{n=1}^{\infty} \frac{[\log(n+1)]^{2-2/p}}{n l_n^2}$$

$$\leqq \sum_{n=1}^{\infty} \frac{[\log(n+1)]^{2-2/p}}{n\left(\frac{\rho_n}{\sqrt{\log(n+1)}} \sum_{k=1}^{n} \frac{[\log(k+1)]^{1-1/p}}{k}\right)^2}$$

$$\leqq C \sum_{n=1}^{\infty} \frac{[\log(n+1)]^{2-2/p}}{n\rho_n^2[\log(n+1)]^{3-2/p}}$$

$$= C \sum_{n=1}^{\infty} \frac{1}{n\rho_n^2 \log(n+1)} < \infty,$$

the sequence $\{\kappa_n\}$ satisfies the conditions of Lemma 8. By using of Lemma 8 we have that the series (1) is summable $|C, 1/p|$ almost everywhere.

This completes the proof of Theorem II.

REFERENCES

[1] H. C. CHOW, Some new criteria for the absolute summability of a Fourier series and its conjugate series, J. London Math. Soc., 30(1955), 133-142.

[2] H. C. CHOW, Theorems on power series and Fourier series, Proc. London Math. Soc., 1(1951), 206-216.

[3] H. C. CHOW, An extension of a theorem of Zygmund and its application, J. London Math. Soc., 29(1954), 189-197.

[4] M. KOZIMA, On the absolute summability factors of power series, Tôhoku Math. J., 18(1966), 60-70.

[5] E. KOGBENTLIANTZ, Sur les séries absolument sommables par la méthode des moyennes arithmêtiques, Bull. Sci. Math., 49(1925), 234-256.

[6] L. LEINDLER, Über verschiedene Konvergenzarten trigonometrischer Reihen III, Acta Sci. Math., 27(1966), 205-215.

[7] G. SUNOUCHI, Some criteria for the absolute summability of a Fourier series, Tôhoku Math. J., 19(1967), 311-314.

[8] S. B. STECKIN, On the order of the best approximations of continuous functions, Izv. Akad. Nauk SSSR., 15(1951), 219-242.

[9] A. F. TIMAN, Theory of approximation of functions of a real variable, Moscow, 1960.

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