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RESIDUALLY DECOMPOSABLE OPERATORS IN BANACH SPACES

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1. Introduction. In this paper we shall construct a certain spectral theory for closed linear operators on a Banach space.

These operators have a suitable spectral behaviour on subsets of their spectra but we must eliminate some residual part which do not offer information about the intimate structure of the considered objects, at least from our point of view.

It will be easy to see that this theory contains many examples of operators, bounded or not, having a functional calculus on their spectrum [1], [2], [3], [6], [8], [9].

A permanent model for our construction will be the theory of decomposable operators on a Banach space [7], [2].

Throughout this paper the sets of points will be taken in $\boldsymbol{C}_\infty = \boldsymbol{C} \cup \{\infty\}$ (the complex compactified plane).

We shall denote by $\mathfrak{X}$ a complex Banach space, by $B(\mathfrak{X})$ the algebra of bounded linear operators on $\mathfrak{X}$ and by $C(\mathfrak{X})$ the set of closed linear operators on $\mathfrak{X}$. If $T \in C(\mathfrak{X})$, we shall denote by D_T its domain of definition.

Many considerations are valid in a more general space, for example on a locally convex one endowed with a suitable structure, in particular on a Fréchet space, using a good definition for the spectrum of an operator [10].

2. The residual single valued extension property. The single valued extension property for bounded operators is a notion due to Dunford [3], [4].

Our definition is a generalization for closed operators, including the fact that only a part of their spectrum is considered.

DEFINITION 2.1. An open set $\Omega \subseteq \boldsymbol{C}_\infty$ is of *analytic uniqueness* of $T \in C(\mathfrak{X})$ if for any open $\omega \subseteq \Omega$ and analytic function $f_0 : \omega \rightarrow D_T$ verifying the equation $(\lambda I - T) f_0(\lambda) = 0 (\lambda \in \omega \cap \boldsymbol{C})$ it follows $f_0(\lambda) \equiv 0$ in ω.

PROPOSITION 2.1. *For any $T \in C(\mathfrak{X})$ there exists a unique maximal open set Ω_T of analytic uniqueness.*

PROOF. The family of open sets $\{\Omega_\nu\}$ of analytic uniqueness of T contains at least the set $\rho(T)$ (the resolvent set of T). Let us put

$$\Omega_T = \bigcup_\nu \Omega_\nu.$$

It is easy to see that Ω_T is an open set of analytic uniqueness. Indeed, if $\omega \subseteq \Omega_T$ is an open set and $f_0 : \omega \to D_T$ an analytic function verifying $(\lambda I - T)f_0(\lambda) = 0$ $(\lambda \in \omega \cap \boldsymbol{C})$ then for any $\lambda_0 \in \omega$ there is a neighbourhood V_{λ_0} completely contained in a set Ω_{ν_0}. Since Ω_{ν_0} is of analytic uniqueness then for the pair $(V_{\lambda_0}, f_0 | V_{\lambda_0})$ it follows $f_0 | V_{\lambda_0} \equiv 0$. The point $\lambda_0 \in \omega$ being arbitrarily chosen, we obtain $f_0 \equiv 0$ in ω. Obviously the set Ω_T is a maximal unique one with required property.

DEFINITION 2.2. We shall denote by $S_T = \complement\Omega_T$($\Omega_T$ given by the preceding proposition) and call it the *analytic residuum* of $T \in C(\mathfrak{X})$.

From the properties of Ω_T, it follows easily that S_T has no isolated points and if its interior is void then the set itself is void.

DEFINITION 2.3. An analytic function $f_x : \omega \to D_T$ verifying the equation $(\lambda I - T)f_x(\lambda) = x$ $(\lambda \in \omega \cap \boldsymbol{C})$ is called *T-associated* of $x \in \mathfrak{X}$ $(T \in C(\mathfrak{X})$ fixed).

For $T \in C(\mathfrak{X})$ and $x \in \mathfrak{X}$ fixed we shall denote by $\delta_T(x)$ the (open) set of points $\lambda_0 \in \boldsymbol{C}_\infty$ with the property that λ_0 has a neighbourhood where there exists a T-associated function of x.

Let us also put

$$\gamma_T(x) = \complement\delta_T(x),$$

$$\rho_T(x) = \delta_T(x) \cap \Omega_T,$$

$$\sigma_T(x) = \gamma_T(x) \cup S_T = \complement\rho_T(x).$$

It is easy to see that on $\rho_T(x)$ there is a unique T-associated function of x, denoted by $x(\cdot)$.

Therefore we can write:

$$\gamma_T(x) \subseteq \gamma_T(x) \cup S_T = \sigma_T(x) \subseteq \sigma(T)$$

and by complementing

$$\rho(T) \subseteq \delta_T(x) \cap \complement S_T = \rho_T(x) \subseteq \delta_T(x).$$

PROPOSITION 2.2. *Let $T \in C(\mathfrak{X})$, $x_0 \in \mathfrak{X}$ and $\lambda_0 \in \delta_T(x_0) \cap \boldsymbol{C}$. Then for any*

T-associated function f_{x_0} of x_0, defined in λ_0, we have the relations:

$$\text{(i)} \qquad \gamma_T(f_{x_0}(\lambda_0)) = \gamma_T(x_0),$$

$$\text{(ii)} \qquad \sigma_T(f_{x_0}(\lambda_0)) = \sigma_T(x_0).$$

PROOF. The second equality follows immediately from the first. Now, let $\mu_0 \in \delta_T(f_{x_0}(\lambda_0))$, therefore it has a neighbourhood V_{μ_0} and a function f: $V_{\mu_0} \to D_T$, T-associated with $f_{x_0}(\lambda_0)$, hence

$$(\mu I - T)f(\mu) = f_{x_0}(\lambda_0) \quad (\mu \in V_{\mu_0} \cap \boldsymbol{C}).$$

But $f_{x_0}(\lambda_0) \in D_T$ and $(\lambda_0 I - T)f(\mu) = f_{x_0}(\lambda_0) + (\lambda_0 - \mu)f(\mu)$ is an analytic function with values in D_T, therefore

$$(\mu I - T)(\lambda_0 I - T)f(\mu) = (\lambda_0 I - T)(\mu I - T)f(\mu) = (\lambda_0 I - T)f_{x_0}(\lambda_0) = x_0 .$$

Consequently $\mu_0 \in \delta_T(x_0)$.

Conversely, if $\mu_0 \in \delta_T(x_0)$ it has a neighbourhood V_{μ_0} where there is a T-associated function $g_{x_0}(\mu)$ of x_0. If $\mu_0 = \lambda_0$ we take for g_{x_0} even the initial function f_{x_0}.

In these conditions we can define the following analytic function

$$h(\mu) = \begin{cases} -\dfrac{g_{x_0}(\mu) - f_{x_0}(\lambda_0)}{\mu - \lambda_0} & \mu \neq \lambda_0 \\ -f'_{x_0}(\lambda_0) & \mu = \lambda_0 \end{cases}$$

which verifies the equation $(\mu I - T)h(\mu) = f_{x_0}(\lambda_0)$ if $\mu \in V_{\mu_0} \cap \boldsymbol{C}$. Therefore $\mu_0 \in \delta_T(f_{x_0}(\lambda_0))$ and now both equalities are verified.

DEFINITION 2.4. For an arbitrary set $M \subseteq \boldsymbol{C}_\infty$ we shall denote

$$\breve{\mathfrak{X}}_T(M) = \{x \in \mathfrak{X}\,;\, \gamma_T(x) \subseteq M\},$$

$$\mathfrak{X}_T(M) = \{x \in \mathfrak{X}\,;\, \sigma_T(x) \subseteq M\}.$$

If $M \not\supseteq S_T$, the set $\mathfrak{X}_T(M)$ is void and if $M \supseteq S_T$ then $\breve{\mathfrak{X}}_T(M) = \mathfrak{X}_T(M)$.

Furthermore, the mapping $M \to \breve{\mathfrak{X}}_T(M)$ (or $\mathfrak{X}_T(M)$) preserves the inclusion relation.

PROPOSITION 2.3. *For every set $M \subseteq \boldsymbol{C}_\infty$ the vector sets $\mathfrak{X}_T(M \cup S_T)$ and $\breve{\mathfrak{X}}_T(M)$ are linear manifolds.*

PROOF. Indeed, it is straightforward to verify the following relations

$$\left.\begin{array}{l}\gamma_T(x)=\gamma_T(\alpha x)\\ \sigma_T(x)=\sigma_T(\alpha x)\end{array}\right\} \text{ any } x\in\mathfrak{X},\quad \alpha\neq 0,$$

$$\left.\begin{array}{l}\gamma_T(x+y)\subseteq\gamma_T(x)\cup\gamma_T(y)\\ \sigma_T(x+y)\subseteq\sigma_T(x)\cup\sigma_T(y)\end{array}\right\} \text{ any } x,y\in\mathfrak{X},$$

$$\gamma_T(0)=\emptyset\,,\quad \sigma_T(0)=S_T$$

and from these we obtain the assertions.

DEFINITION 2.5. A closed subspace $\mathfrak{Y}\subseteq\mathfrak{X}$ *is invariant for* $T\in C(\mathfrak{X})$ if $\mathfrak{Y}\subseteq D_T$ and $T\mathfrak{Y}\subseteq\mathfrak{Y}$.

Obviously, since T is closed, by Banach theorem it follows $T|\mathfrak{Y}\in B(\mathfrak{Y})$.

PROPOSITION 2.4. *Let* $T\in C(\mathfrak{X})$, $S_T\subseteq M_0\subseteq \boldsymbol{C}_\infty$ *and* $\mathfrak{Y}_0=\mathfrak{X}_T(M_0)$ *with the properties* :

1) $\mathfrak{Y}_0$ *closed in* $\mathfrak{X}$,
2) $\mathfrak{Y}_0\subseteq D_T$.

Then $\mathfrak{Y}_0$ *is invariant to* T *and* $\sigma(T|\mathfrak{Y}_0)\subseteq\overline{M_0\cap\sigma(T)}$.

PROOF. Let us prove that $\mathfrak{Y}_0$ is invariant to T.

Namely, if $y\in\mathfrak{Y}_0\subseteq D_T$ then we have $(\lambda I-T)y(\lambda)=y$ for any $\lambda\in\rho_T(y)\cap\boldsymbol{C}$ and the function $Ty(\lambda)=\lambda y(\lambda)-y$ is analitic, with values in D_T.

From this it follows $(\lambda I-T)Ty(\lambda)=Ty$, therefore $\sigma_T(Ty)\subseteq\sigma_T(y)\subseteq M_0$ and the space $\mathfrak{Y}_0$ is invariant to T.

For any $\lambda\in\complement(\overline{M_0\cap\sigma(T)})\cap\boldsymbol{C}$ we can define the linear operator

$$A_\lambda y=y(\lambda)\qquad (y\in\mathfrak{Y}_0)\,,$$

well defined since, by Proposition 2.2, $\sigma_T(y(\lambda))=\sigma_T(y)$. If $A_\lambda y=0$ then $y=(\lambda I-T)y(\lambda)=0$, therefore A_λ is injective. If $z\in\mathfrak{Y}_0$ and $y=(\lambda I-T)z$ then $A_\lambda y=A_\lambda(\lambda I-T)z=((\lambda I-T)z)(\lambda)=(\lambda I-T)z(\lambda)=z$(where the third equality is true since outside $M_0\supseteq S_T$ any T-associated function is uniquely determined for the elements of $\mathfrak{Y}_0$).

Therefore A_λ is also surjective.

On the other hand

$$(\lambda I-T)A_\lambda y=(\lambda I-T)y(\lambda)=y$$

and

$$A_\lambda(\lambda I-T)y=((\lambda I-T)y)(\lambda)=(\lambda I-T)y(\lambda)=y.$$

Therefore, by Banach theorem, we conclude that $A_\lambda \in B(\mathfrak{X})$ and $A_\lambda=(\lambda I|\mathfrak{Y}_0-T|\mathfrak{Y}_0)^{-1}$, $\lambda \in \rho(T|\mathfrak{Y}_0)$ and this finishes the proof.

COROLLARY 1. *For any* $T\in C(\mathfrak{X})$ *we have the relation* $\bigcup_{x\in\mathfrak{X}}\sigma_T(x)=S_T\cup\bigcup_{x\in\mathfrak{X}}\gamma_T(x)=\sigma(T)$.

Indeed, we have obviously

$$\bigcup_{x\in\mathfrak{X}}\sigma_T(x)=S_T\cup\bigcup_{x\in\mathfrak{X}}\gamma_T(x)\subseteq\sigma(T)$$

and if $\lambda\in\sigma(T)\cap\complement\left(\bigcup_{x\in\mathfrak{X}}\gamma_T(x)\cup S_T\right)$ $(\lambda\in \boldsymbol{C})$ then in such a point we should be able to define the operator

$$A_\lambda x=x(\lambda) \qquad (x\in\mathfrak{X}).$$

As in the previous proposition, we should obtain $\lambda\in\rho(T)$ and this is preposterous.

The proof of the following proposition is similar with the proof of the preceding proposition.

However, we shall give it because its specificity.

PROPOSITION 2. 5. *Let* $T\in C(\mathfrak{X})$, $M_0\subseteq M\subseteq \boldsymbol{C}_\infty$ *and* $\mathfrak{Y}_0=\check{\mathfrak{X}}_T(M_0)$, $\mathfrak{Y}=\check{\mathfrak{X}}_T(M)$ *with the properties*

1) $\mathfrak{Y}_0$, $\mathfrak{Y}$ *closed in* $\mathfrak{X}$,

2) $\mathfrak{Y}\subseteq D_T$.

If $\tilde{\mathfrak{Y}}=\mathfrak{Y}/\mathfrak{Y}_0$ *then* T *induces naturally on* $\tilde{\mathfrak{Y}}$ *an operator* $\tilde{T}\in B(\tilde{\mathfrak{Y}})$ *with* $\sigma(\tilde{T})\subseteq\bar{M}$.

PROOF. As in the former proposition, for any $y\in D_T$ we have $\gamma_T(Ty)\subseteq\gamma_T(y)$, therefore the closed subspaces $\mathfrak{Y}_0$ and $\mathfrak{Y}$ are invariant to T.

Therefore, we have $T|\mathfrak{Y}\in B(\mathfrak{Y})$. Let $\tilde{\mathfrak{Y}}=\mathfrak{Y}/\mathfrak{Y}_0$ and $\tilde{T}$ be defined as $\tilde{T}\tilde{y}=\widetilde{(T|\mathfrak{Y})y}$ $(y\in\tilde{y}\in\tilde{\mathfrak{Y}})$. We shall define for any $\lambda\in\complement\bar{M}\cap\boldsymbol{C}$ the operator

$$\tilde{A}_\lambda\tilde{y}=\widetilde{f_y(\lambda)}$$

where $y\in\tilde{y}$ and f_y is T-associated with y. To see that T is coherently defined, let $y_1,y_2\in\tilde{y}$, thus $y_2=y_1+z$ with $z\in\mathfrak{Y}_0$. Therefore $f_z(\lambda)\in\mathfrak{Y}_0$ for any defined in λ, T-associated function of z, because by Proposition 2.2, $\gamma_T(f_z(\lambda)) = \gamma_T(z)$. Therefore

$$\widetilde{f_{y_2}(\lambda)} = \widetilde{f_{y_1+z}(\lambda)} = \widetilde{f_{y_1}(\lambda)} + \widetilde{f_z(\lambda)} = \widetilde{f_{y_1}(\lambda)}.$$

The second equality is true since

$$\widetilde{f_{\alpha_1 y_1+\alpha_2 y_2}(\lambda)} = \alpha_1 \widetilde{f_{y_1}(\lambda)} + \alpha_2 \widetilde{f_{y_2}(\lambda)}.$$

Indeed :

$$(\lambda I - T)[f_{\alpha_1 y_1+\alpha_2 y_2}(\lambda) - \alpha_1 f_{y_1}(\lambda) - \alpha_2 f_{y_2}(\lambda)] = 0$$

for any corresponding T-associated functions, therefore, by Proposition 2.2, $\gamma_T(f_{\alpha_1 y_1+\alpha_2 y_2}(\lambda) - \alpha_1 f_{y_1}(\lambda) - \alpha_2 f_{y_2}(\lambda)) = \gamma_T(0) = \emptyset \subseteq M_0$. In particular, $\widetilde{A}_\lambda$ is a linear mapping.

It is easy to see that $\widetilde{A}_\lambda$ is injective. For any $y \in \mathfrak{Y} \subseteq D_T$ we have the equality

$$\widetilde{f_{(\lambda I-T)y}(\lambda)} = \widetilde{(\lambda I - T) f_y(\lambda)} = \widetilde{y}$$

since $(\mu I - T)[f_{(\lambda I-T)y}(\mu) - (\lambda I - T) f_y(\mu)] = 0$
and from this is not difficult to see that $\widetilde{A}_\lambda$ is surjective.

Furthermore

$$(\lambda \widetilde{I} - \widetilde{T})\widetilde{A}_\lambda = \widetilde{A}_\lambda(\lambda \widetilde{I} - \widetilde{T}) = \widetilde{I}, \quad \text{hence } \lambda \in \rho(\widetilde{T})$$

and the proof is finished.

3. Invariant maximal spaces of a closed operator. We shall introduce the notion of invariant maximal space, corresponding to the notion of spectral maximal space in the theory of decomposable operators [7], [2].

We shall denote by $\mathcal{I}_T$ the family of invariant subspaces of T (see Definition 2.5) and if F is a closed set in $\boldsymbol{C}_\infty$ we put :

$$\mathcal{I}_{T,F} = \{\mathfrak{Y} \in \mathcal{I}_T \,;\, \sigma(T|\mathfrak{Y}) \subseteq F\}\,.$$

DEFINITION 3.1. Whenever the family $\mathcal{I}_{T,F}$ is directed and has a maximal element $\mathfrak{X}_{T,F}$ (with respect to the inclusion relation) we shall call it a *maximal invariant space* of T (on F).

It is easy to see that a closed subspace $\mathfrak{Y} \subseteq \mathfrak{X}$ is a maximal invariant one of T if and only if for any $\mathfrak{Z} \in \mathcal{I}_T$ the relation $\sigma(T|\mathfrak{Z}) \subseteq \sigma(T|\mathfrak{Y})$ implies $\mathfrak{Z} \subseteq \mathfrak{Y}$.

Consequently our definition is a natural extension of the definition of spectral maximal spaces [7].

DEFINITION 3.2. A closed subspace $\mathfrak{Y} \subseteq \mathfrak{X}$ is called *T-absorbing* if for any $x \in \mathfrak{Y}$ the equation $(\lambda I - T)y = x$ has solutions y only in $\mathfrak{Y}$, for any $\lambda \in \sigma(T|\mathfrak{Y})$.

PROPOSITION 3.1. *Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be an invariant maximal space of $T \in C(\mathfrak{X})$. Then $\mathfrak{Y}$ is a T-absorbing subspace of $\mathfrak{X}$.*

PROOF. Since the proof is similar with one of [2], we shall only sketch it. If $\lambda_0 \in \sigma(T|\mathfrak{Y})$ and $x_0 \in \mathfrak{Y}$ the assumption that there exists $y_0 \notin \mathfrak{Y}$ with $(\lambda_0 I - T)y_0 = x_0$ leads to a contradiction. Indeed, if

$$\mathfrak{Y}_0 = \{x + \alpha y_0 \ ; \ x \in \mathfrak{Y} \ ; \ \alpha \in \boldsymbol{C}\}$$

then $\mathfrak{Y}_0 \subseteq D_T$ and $T\ \mathfrak{Y}_0 \subseteq \mathfrak{Y}_0$, therefore $\mathfrak{Y}_0 \in \mathcal{J}_T$. Furthermore if $\lambda \in \rho(T|\mathfrak{Y})$ the operator $\lambda I - T$ is injective and surjective (here the uniqueness of the representation for the elements of $\mathfrak{Y}_0$ is essential), therefore $\lambda \in \rho(T|\mathfrak{Y}_0)$. But this is a contradiction, since $\mathfrak{Y}_0 \supsetneqq \mathfrak{Y}$ which is maximal invariant.

COROLLARY 1. *Let $\mathfrak{Y}$ be an invariant maximal space of $T \in C(\mathfrak{X})$ and $f_0 : \omega \to D_T$, $f_0(\lambda) \not\equiv 0$ in ω, an analytic function verifying $(\lambda I - T)f_0(\lambda) = 0$.*
If ω is connected and $\omega \cap \sigma(T|\mathfrak{Y})$ contains an open set then $\omega \subseteq \sigma(T|\mathfrak{Y})$.

PROOF. Indeed, if $\omega \cap \sigma(T|\mathfrak{Y})$ contains an open set D then, by Proposition 3.1, $f_0(\lambda) \in \mathfrak{Y}$ $(\lambda \in D)$. By analytic prolongation we obtain easily that $f_0(\lambda) \in \mathfrak{Y}$ with all its derivatives, if $\lambda \in \omega$. Since $Tf_0(\lambda) = \lambda f_0(\lambda)$ and T is closed it follows that $Tf_0^{(k+1)}(\lambda) = \lambda f_0^{(k+1)}(\lambda) + (k+1)f_0^{(k)}(\lambda)$ and since $f_0(\lambda) \not\equiv 0$, for any $\lambda_0 \in \omega$ there exists an n_0 such that $f_0^{(n_0)}(\lambda_0) \neq 0$ and $Tf_0^{(n_0)}(\lambda_0) = \lambda_0 f_0^{(n_0)}(\lambda_0)$, thus $\lambda_0 \in \sigma(T|\mathfrak{Y})$.

COROLLARY 2. *If $\mathfrak{Y} \subseteq \mathfrak{X}$ is an invariant maximal space of $T \in C(\mathfrak{X})$ then $\sigma(T|\mathfrak{Y}) \subseteq \sigma(T)$.*

PROOF. Indeed, if $\lambda \in \rho(T) \cap \boldsymbol{C}$ then for any $x \in \mathfrak{Y}$ the element $y = R(\lambda, T)x$ is a solution of the equation $(\lambda I - T)y = x$. If $\lambda \in \sigma(T|\mathfrak{Y})$ then $y \in \mathfrak{Y}$ for any $x \in \mathfrak{Y}$ and being uniquely determined, we obtain easily that $R(\lambda, T)|\mathfrak{Y} = R(\lambda, T|\mathfrak{Y})$ and this is preposterous.

COROLLARY 3. *If $\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2$ are invariant maximal spaces of $T \in C(\mathfrak{X})$ then $\sigma(T|\mathfrak{Y}_1) \subseteq \sigma(T|\mathfrak{Y}_2)$.*

The proof is simlar with the preceding proof.

PROPOSITION 3.2. *Let $T \in C(\mathfrak{X})$ and $\{\mathfrak{X}_{T,F_\alpha}\}$ be a family of invariant maximal spaces of T. If the family of sets $\{F_\alpha\}$ is directed on the left (by the inclusion relation) and if $F = \bigcap_\alpha F_\alpha$ then there exists the invariant maximal space $\mathfrak{X}_{,F}$ and $\mathfrak{X}_{T,F} = \bigcap_\alpha \mathfrak{X}_{T,F_\alpha}$.*

PROOF. Let us put $\mathfrak{Y} = \bigcap_\alpha \mathfrak{X}_{T,F_\alpha}$. Obviously $\mathfrak{Y}$ is an invariant subspace of T. We shall show that $\sigma(T|\mathfrak{Y}) \subseteq F$.

For, let $\lambda_0 \in \complement F \cap \boldsymbol{C}$. The operator $(\lambda_0 I - T)|\mathfrak{Y}$ is injective since if $(\lambda_0 I - T)x_0 = 0$ there exists an index α_0 such that $\lambda_0 \notin F_{\alpha_0}$ and from $\sigma(T|\mathfrak{X}_{T,F_{\alpha_0}}) \subseteq F_{\alpha_0}$ it follows $x_0 = 0$.

The operator $(\lambda_0 I - T)|\mathfrak{Y}$ is also surjective since if $y_0 \in \mathfrak{Y}$ then for indexes β with $\lambda_0 \notin F_\beta$ we have $z_0 = R(\lambda_0, T|\mathfrak{X}_{T,F_\beta})y_0 \in \mathfrak{X}_{T,F_\beta}$ and the solution z_0 is unique for $(\lambda_0 I - T)z_0 = y_0$. Indeed, if $(\lambda_0 I - T)z_{\beta_1} = (\lambda_0 I - T)z_{\beta_2} = y_0$ then if $F_\beta \subseteq F_{\beta_1} \cap F_{\beta_2}$ we have $(\lambda_0 I - T)z_\beta = y_0$, therefore $(\lambda_0 I - T)(z_{\beta_1} - z_\beta) = 0 = (\lambda_0 I - T)(z_{\beta_2} - z_\beta)$ on the spaces $\mathfrak{X}_{T,F_{\beta_1}}$ and $\mathfrak{X}_{T,F_{\beta_2}}$, thus by injectivity, $z_{\beta_1} = z_\beta = z_{\beta_2}$. On the other hand, let us remark the equality $\mathfrak{Y} = \bigcap_{\substack{\beta \\ \lambda_0 \notin F_\beta}} \mathfrak{X}_{T,F_\beta}$, since the family $\{F_\alpha\}$ is directed by the inclusion relation. Consequently $z_0 \in \mathfrak{Y}$. A Banach theorem gives us $\lambda_0 \in \rho(T|\mathfrak{Y})$. Now, if $\mathfrak{Z}$ is an invariant subspace of T with $\sigma(T|\mathfrak{Z}) \subseteq F$ then $\mathfrak{Z} \subseteq \bigcap_\alpha \mathfrak{X}_{T,F_\alpha}$, therefore we have indeed $\mathfrak{Y} = \mathfrak{X}_{T,F_\alpha}$.

For every operator $A \in B(\mathfrak{X})$ we shall denote by $\mathcal{F}(A)$ the family of complex valued functions analytic in a neighbourhood of $\sigma(A)$.

It is known that for any $f \in \mathcal{F}(A)$ there exists an operator $f(A) \in B(\mathfrak{X})$ given by the formula :

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)\ R(\lambda, A) d\lambda$$

where Γ is a suitable system of curves in $\rho(A)$ (see [5], Ch. VII).

PROPOSITION 3.3 *Let $T \in C(\mathfrak{X})$, $A \in B(\mathfrak{X})$ and $\mathfrak{Y} \subseteq \mathfrak{X}$ a maximal invariant space of T. If $AT \subseteq TA$ and $R(\lambda_0, A)T \subseteq TR(\lambda_0, A)$ for a $\lambda_0 \in \rho(A) \cap \boldsymbol{C}$ then the subspace $\mathfrak{Y}$ is invariant to $f(A)$, for any $f \in \mathcal{F}(A)$.*

PROOF. Since $(\lambda_0 - \lambda)^{-1} \in \rho(R(\lambda_0, A))$, if $\lambda \in \rho(A)$, then we have the equality $R(\lambda, A) = (\lambda - \lambda_0)^{-1}[R(\lambda_0, A) - (\lambda_0 - \lambda)^{-1}\ I]^{-1} R(\lambda_0, A)$ and it will be easy to see that $TR(\lambda, A)x = R(\lambda, A)Tx$ for $x \in D_T$ (because from $A_T D \subseteq D_T$ and $R(\lambda_0, A)D_T \subseteq D_T$ we have $R(\lambda_0, T)D_T = D_T$).

Now if $\lambda \in \rho(A) \cap \boldsymbol{C}$, then $\mathfrak{Y}_\lambda = R(\lambda, A)\ \mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$ and for any $y \in \mathfrak{Y}_\lambda (\subseteq D_T)$ we have $Ty = TR(\lambda, A)x = R(\lambda, A)Tx \in \mathfrak{Y}_\lambda$ (where $x \in \mathfrak{Y} \subseteq D_T$).

Consequently $\mathfrak{Y}_\lambda \in \mathcal{I}_T$. The relation $Ty = R(\lambda, A)T(\lambda I - A)y$ shows that $T|\mathfrak{Y}_\lambda = [R(\lambda, A)|\mathfrak{Y}][T|\mathfrak{Y}][(\lambda I - A)|\mathfrak{Y}_\lambda]$ and since $(\lambda I - T)|\mathfrak{Y}_\lambda : \mathfrak{Y}_\lambda \to \mathfrak{Y}_\lambda$ is bounded, we obtain $\sigma(T|\mathfrak{Y}_\lambda) = \sigma(T|\mathfrak{Y})$, therefore $\mathfrak{Y}_\lambda \subseteq \mathfrak{Y}$.

So $R(\lambda, A)\mathfrak{Y} \subseteq \mathfrak{Y}$ for any $\lambda \in \rho(A)$ and if $f \in \mathcal{F}(A)$ the approximation of the integral with finite sums leads to the desired result.

PROPOSITION 3.4. *With the conditions of Proposition* 2.4, *the space* $\mathfrak{X}_T(M_0)$ *is invariant maximal of* T.

PROOF. Indeed, if $\mathfrak{Z} \in \mathcal{I}_T$ and $\sigma(T|\mathfrak{Z}) \subseteq M_0 \supseteq S_T$ then $(\lambda I|\mathfrak{Z} - T|\mathfrak{Z})^{-1}$ exists outside M_0, hence $\mathfrak{Z} \subseteq \mathfrak{X}_T(M_0)$.

4. **Residually decomposable operators**. In this paragraph we shall define the notion of residually decomposable operator, a natural generalization of the notion of decomposable (bounded) operator [7]. Our definition is valid for any closed operator. Let $S \subseteq \boldsymbol{C}_\infty$ be a closed set.

DEFINITION 4.1. A family of open sets $\{G_j\}_{j=1}^n \cup \{G_S\}$ is an S-covering of the closed set $\Delta \subseteq \boldsymbol{C}_\infty$ if :

$$(\mathcal{H}_1) \qquad \bigcup_{j=1}^n G_j \cup G_S \supseteq \Delta \cup S,$$

$$(\mathcal{H}_2) \qquad \overline{G}_j \cap S = \emptyset \qquad (j = 1, \cdots, n).$$

DEFINITION 4.2. An operator $T \in C(\mathfrak{X})$ is called *S-residually decomposable* if :

(δ_1) For any closed set $F \subseteq \boldsymbol{C}_\infty$ with $F \cap S = \emptyset$ the family $\mathcal{I}_{T,F}$ is directed and has a maximal element $\mathfrak{X}_{T,F}$,

(δ_2) For any S-covering $\{G_j\}_{j=1}^n \cup \{G_S\}$ of the set $\sigma(T)$ there exist the invariant subspaces $\{\mathfrak{X}_j\}_{j=1}^n$ of T with the properties :

(δ_2') $\quad \sigma(T|\mathfrak{X}_j) \subseteq G_j \qquad (j = 1, \cdots, n)$,

(δ_2'') $\quad$ every $x \in \mathfrak{X}$ has a decomposition of the form :

$$x = x_1 + x_2 + \cdots + x_n + x_S$$

where $x_j \in \mathfrak{X}_j (j = 1, \cdots, n)$ and $\sigma_T(x_S) \subseteq \overline{G}_S$.

First of all we must observe that we may always suppose $S \subseteq \sigma(T)$ since it is straightforward to see that any S-residually decomposable operator is $S\cap\sigma(T)$-residually decomposable and conversely.

PROPOSITION 4.1. *If $T\in C(\mathfrak{X})$ is $\emptyset$-residually decomposable then $T\in B(\mathfrak{X})$ and T is decomposable* [7].

PROOF. If $S=\emptyset$ then any covering $\{G_j\}_{j=1}^n$ of $\sigma(T)$ is an S-covering of $\sigma(T)$. Corresponding to this covering there exist the invariant subspaces $\mathfrak{X}_j$ $(j=1,\cdots,n)$ such that any $x\in\mathfrak{X}$ has a decomposition of the form $x=x_1+\cdots+x_n+x_S$ with $x_j\in\mathfrak{X}_j$ $(j=1,\cdots,n)$ and $\sigma_T(x_S)=\emptyset$. The mapping $\lambda\to x_S(\lambda)$ is analytic in the whole complex plane (by(δ_2'')). From this it follows easily $x_S=0$. Indeed, if $\lambda_k\to\infty$ then $y_k=\dfrac{x_S(\lambda_k)}{\lambda_k}\to 0$ and $Ty_k=x_S(\lambda_k)-\dfrac{x_S}{\lambda_k}\to x_S(\infty)$ and T being closed it follows $x_S(\infty)=0$. By Liouville theorem we have $x_S(\lambda)\equiv 0$, thus $x_S=0$.

If $F_j=\sigma(T|\mathfrak{X}_j)\subseteq G_j$ then, by (δ_1), there exist $\mathfrak{X}_{T,F_j}\supseteq\mathfrak{X}_j$ $(j=1,\cdots,n)$ which are maximal spectral [7], and $\sum_{j=1}^{n}\mathfrak{X}_{T,F_j}=\mathfrak{X}$, therefore T is decomposable.

LEMMA 4.1. *Let $T\in C(\mathfrak{X})$ be an operator S-residually decomposable and G an open set with the property $G\cap(\sigma(T)\backslash S)\neq\emptyset$. Then there exists an invariant maximal space $\mathfrak{Y}\neq\{0\}$ such that $\sigma(T|\mathfrak{Y})\subseteq G\cap(\sigma(T)\backslash S)$.*

PROOF. With no loss of generality we may assume that $\overline{G}\cap S=\emptyset$.

Then we choose an open set G_1 such that $G\subseteq\overline{G}\subseteq G_1$, $\overline{G}_1\cap S=\emptyset$ and let us consider the S-covering $\{G_1,G_S\}$ where $G_S=\complement\overline{G}$. Then there exists an invariant subspace $\mathfrak{Y}$ with $\sigma(T|\mathfrak{Y})\subseteq G_1$. If $\mathfrak{Y}=\{0\}$ then any $x\in\mathfrak{X}$ would have the form $x=x_S$, and since $\sigma_T(x_S)\subseteq\overline{G}_S$ we should obtain, by Corollary 1 of Proposition 2.4, that $\sigma(T)\subseteq\overline{G}_S$. This is preposterous since we have, by our assumption, that $G\cap\sigma(T)\neq\emptyset$ and $\overline{G}_S\subseteq\complement G$.

Thus necessarily $\mathfrak{Y}\neq\{0\}$.

PROPOSITION 4.2. *If $T\in C(\mathfrak{X})$ is an operator S-residually decomposable then $S_T\subseteq S$. Moreover for any open G, $\overline{G}\subseteq\sigma(T)$, $G\cap S=\emptyset$, we have $\sigma(T/\mathfrak{X}_{T,\overline{G}})=\overline{G}$.*

PROOF. Let $f:\omega\to D_T$ be ($\omega\subseteq\complement S$) an analytic function which verifies $(\lambda I-T)f(\lambda)\equiv 0$ if $\lambda\in\omega\cap\boldsymbol{C}$. We shall show that $f(\lambda)\equiv 0$ for $\lambda\in\omega$. We can suppose with no loss of generality that $\omega\subseteq\sigma(T)\backslash S$.

Let us assume the contrary. We shall choose the connected open sets G

and G_1 with $G\subseteq\overline{G}\subseteq G_1\subseteq\omega$ and we suppose $f(\lambda)\not\equiv 0$ in G_1. Let now the invariant maximal space $\mathfrak{X}_{T,\overline{G}}$. We have the equality $\sigma(T|\mathfrak{X}_{T,\overline{G}})=\overline{G}$. Indeed, if $\overline{G}\backslash\sigma(T|\mathfrak{X}_{T,\overline{G}})\neq\emptyset$ then this set contains a non-void open set and if F is a closed subset of $\overline{G}\backslash\sigma(T|\mathfrak{X}_{T,\overline{G}})$ with $\overset{\circ}{F}\neq\emptyset$ ($\overset{\circ}{F}$ is the interior of F), on account of Lemma 4.1, we have $\mathfrak{X}_{T,F}\neq\{0\}$. On the other hand $\sigma(T|\mathfrak{X}_{T,F})\subseteq F\subseteq\overline{G}$, thus $\mathfrak{X}_{T,F}\subseteq\mathfrak{X}_{T,\overline{G}}$. By Corollary 3 of Proposition 3.1, we have $\sigma(T|\mathfrak{X}_{T,F})\subseteq\sigma(T|\mathfrak{X}_{T,\overline{G}})$ and this fact is impossible. Thus $\sigma(T|\mathfrak{X}_{T,\overline{G}})=\overline{G}$ and obviously $\overset{\circ}{G}\neq\emptyset$, therefore by Corollary 1 of Proposition 3.1, we have $G_1\subseteq\sigma(T|\mathfrak{X}_{T,\overline{G}})=\overline{G}$ and this is a contradiction. Consequently $f(\lambda)\equiv 0$ in ω and, by Proposition 2.1, $S_T\subseteq S$.

Till now we have introduced, for an operator $T\in C(\mathfrak{X})$ and a closed set $F\subseteq \boldsymbol{C}_\infty$, some types of linear manifolds : $\mathfrak{X}_T(F\cup S_T)$, $\breve{\mathfrak{X}}_T(F)$ and $\mathfrak{X}_{T,F}$. We have the following obvious inclusions : $\mathfrak{X}_{T,F}\subseteq\breve{\mathfrak{X}}_T(F)$ and $\mathfrak{X}_{T,F}\subseteq\mathfrak{X}_T(F)=\breve{\mathfrak{X}}_T(F)$ when $S_T=\emptyset$.

A natural question is: which is the "true" relation among these linear manifolds when the operator T is an S-residually decomposable one, for a certain closed $S\subseteq\boldsymbol{C}_\infty$.

THEOREM 4.1. *Let $T\in C(\mathfrak{X})$ be an operator S-residually decomposable with $S_T=\emptyset$. Then for any compact $F\subseteq\boldsymbol{C}$, $F\cap S=\emptyset$ we have $\mathfrak{X}_{T,F}=\mathfrak{X}_T(F)$.*

PROOF. Let $F\subseteq\boldsymbol{C}$ be compact with $F\cap S=\emptyset$. We shall choose the open sets G_1 an G_S so that $\overline{G}_1\cap S=\emptyset$, $\overline{G}_S\cap F=\emptyset$, $G_1\supseteq F$, $\overline{G}_S\supseteq S$ and $G_1\cup G_S\supseteq\sigma(T)$. So the system $\{G_1, G_S\}$ forms an S-covering of $\sigma(T)$. Then there exists an invariant subspace $\mathfrak{X}_1$ with $\sigma(T|\mathfrak{X}_1)\subseteq G_1$ such that any $x\in\mathfrak{X}_T(F)$ has the representation $x=x_1+x_S$ with $x_1\in\mathfrak{X}_1$ and $\sigma_T(x_S)\subseteq\overline{G}_S$. We shall take a suitable contour Γ_0 round F, which separates the sets F and G_S, in $\complement F\cap\complement\overline{G}_S\cap\boldsymbol{C}$. In a neighbourhood of Γ_0 the T-associated functions $x(\lambda)$, $x_S(\lambda)$and $x_1(\lambda)$ exist, thus we can write

$$\frac{1}{2\pi i}\int_{\Gamma_0}x(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\Gamma_0}x_1(\lambda)d\lambda+\frac{1}{2\pi i}\int_{\Gamma_0}x_S(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\Gamma_0}x_1(\lambda)d\lambda$$

since $x_S(\lambda)$ is analytic in the domain delimited by Γ_0.

Because $\mathfrak{X}_1$ can be supposed invariant maximal, it is T-absorbing (Proposition 3.1), therefore $x_1(\lambda)\in\mathfrak{X}_1$ for any λ, thus $\frac{1}{2\pi i}\int_{\Gamma_0}x_1(\lambda)d\lambda\in\mathfrak{X}_1$.

On the other hand, since $S_T=\emptyset$, the function $x(\lambda)$ is analytic outside the domain delimited by Γ_0, thus we have :

$$\frac{1}{2\pi i}\int_{\Gamma_0}x(\lambda)d\lambda=\lim_{\lambda\to\infty}\lambda x(\lambda)=x.$$

Indeed as in Proposition 4.1, we have $x(\infty)=0$, thus in a suitable neighbourhood of ∞ we can write

$$x(\mu)=\sum_{k=0}^{\infty}\frac{x_k}{\mu^{k+1}}$$

(since $x(\infty)=0$).

From the relation (T being closed)

$$0=Tx(\infty)=\lim_{\mu\to\infty}\mu x(\mu)-x$$

it follows $\lim\limits_{\mu\to\infty}\mu x(\mu)=x$.

Hence $x\in\mathfrak{X}_1$ and $x\in\mathfrak{X}_T(F)$ being arbitrarily chosen, we obtain

$$\mathfrak{X}_T(F)\subseteq\bigcap_{\substack{G_1\supseteq F\\ \overline{G}_1\cap S=\phi}}\mathfrak{X}_{T,\overline{G}_1}.$$

Since the family $\{\overline{G}_1;\ \overline{G}_1\supseteq F,\ \overline{G}_1\cap S=\emptyset\}$ is directed, on account of Propo... 3.2, we shall have

$$\bigcap_{\substack{\overline{G}_1\supseteq F\\ \overline{G}_1\cap S=\phi}}\mathfrak{X}_{T,\overline{G}_1}=\mathfrak{X}_{T,F}.$$

Because the inclusion $\mathfrak{X}_{T,F}\subseteq\breve{\mathfrak{X}}_T(F)=\mathfrak{X}_T(F)$ is obvious, the proof is finished.

The linear manifolds $\breve{\mathfrak{X}}_T(F)$ with $F\subseteq\boldsymbol{C}_\infty$ closed are generally not closed in $\mathfrak{X}$. However, it seems to be interesting the following result :

THEOREM 4.2. *Let $T\in B(\mathfrak{X})$ be an operator S-residually decomposable with $S\subseteq\boldsymbol{C}$. Then for every $F\subseteq\boldsymbol{C}$ compact with $F\cap S=\emptyset$ we have the relation* :

$$\breve{\mathfrak{X}}_T(F)=\mathfrak{X}_{T,F}\dot{+}\breve{\mathfrak{X}}_T(\emptyset)$$

(*where* "$\dot{+}$" *denotes the direct sum between two linear not necessarily closed manifolds*).

PROOF. Let us remark that if $y\in\mathfrak{X}_{T,F}\cap\breve{\mathfrak{X}}_T(\emptyset)$ then we can define the T-associated function :

$$f_y(\lambda)=\begin{cases}(\lambda I|\mathfrak{X}_{T,F}-T|\mathfrak{X}_{T,F})^{-1}y & \lambda\in\complement F\cap\boldsymbol{C}\\ y(\lambda) & \lambda\in\complement S_T\cap\boldsymbol{C}\end{cases}$$

because $\sigma_T(y)=S_T$. Since $\lim_{\lambda\to\infty} f_y(\lambda)=0$, by Liouville theorem it follows $f_y\equiv 0$, thus $y=0$.

Now, let us consider two systems Γ_1 and Γ_2 which surround and separate F, S respectively in $\rho_T(x)$, where $x\in\breve{\mathfrak{X}}_T(F)$ is arbitrary. Also, let Γ be another system which surrounds Γ_1 and Γ_2 (this construction is possible since F and S are compact in $\boldsymbol{C}$).

We shall have the equality :

$$x=\frac{1}{2\pi i}\int_{\Gamma} x(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\Gamma_1} x(\lambda)d\lambda+\frac{1}{2\pi i}\int_{\Gamma_2} x(\lambda)d\lambda.$$

Let us denote by $x_F=\frac{1}{2\pi i}\int_{\Gamma_1} x(\lambda)d\lambda$ and $x_S=\frac{1}{2\pi i}\int_{\Gamma_2} x(\lambda)d\lambda$.

Obviously, the elements x_F and x_S do not depend by the particular choosing of the systems Γ_1 and Γ_2. We have $\gamma_T(x_F)\subseteq F$ and $\gamma_T(x_S)\subseteq S$.

Indeed, if $\mu_0\in \complement F$ then there exists a system Γ_1' "into" Γ_1 such that μ_0 be "outside" Γ_1', thus for μ in a neighbourhood of μ_0 we shall have

$$(\mu I-T)\frac{1}{2\pi i}\int_{\Gamma_1'}\frac{x(\lambda)}{\mu-\lambda}d\lambda=\frac{1}{2\pi i}\int_{\Gamma_1'} x(\lambda)d\lambda+\frac{1}{2\pi i}\int_{\Gamma_1'}\frac{x}{\mu-\lambda}\,d\lambda$$

$$=\frac{1}{2\pi i}\int_{\Gamma_1'} x(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\Gamma_1} x(\lambda)d\lambda=x_F.$$

Analogusly we have $\gamma_T(x_S)\subseteq S$.

On the other hand $\gamma_T(x_S)=\gamma_T(x-x_F)\subseteq F$, so $\gamma_T(x_S)=\emptyset$, thus $x_S\in\mathfrak{X}_T(\emptyset)$.

Let now $\{G_1,G_S\}$ an S-covering of the set $\sigma(T)$ with $G_1\supseteq F$, $G_1\cap S=\emptyset$, $\overline{G}_S\cap F=\emptyset$.

Corresponding to this covering there exists a maximal invariant space $\mathfrak{X}_1$ and thus the element x has the form $x=y_1+y_S$ where $y_1\in\mathfrak{X}_1$ and $\sigma_T(y_S)\subseteq\overline{G}_S$.

Since the system Γ_1 can be chosen in $\complement F\cap\complement\overline{G}_S$, then in a neighbourhood of Γ_1 there exist the functions $x(\lambda), y_1(\lambda), y_S(\lambda)$ and $y_S(\lambda)$ is analytic in the domain delimited by Γ_1. From $x(\lambda)=y_1(\lambda)+y_S(\lambda)$ we have

$$x_F=\frac{1}{2\pi i}\int_{\Gamma_1} y_1(\lambda)d\lambda\in\mathfrak{X}_1$$

because $\mathfrak{X}_1$ is T-absorbing.

As in the preceding theorem, we shall obtain $x_F\in\bigcap_{\overline{G}_1\supseteq F,\,\overline{G}_1\cap S=\phi}\mathfrak{X}_1=\mathfrak{X}_{T,F}$ and this finishes our proof.

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