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ESTIMATES FOR THE MAXIMAL FUNCTION OF HARDY-LITTLEWOOD AND THE MAXIMAL HILBERT TRANSFORM WITH WEIGHTED MEASURES

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Introduction. When we consider the Hilbert transform $\tilde{f}(x) = \text{v.p.}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt$ of a function f, we have to treat the function $\tilde{f}_\varepsilon(x) = \int_{|x-t|\geqq\varepsilon}\frac{f(t)}{x-t}\,dt$. In particular, it is interesting to estimate the maximal Hilbert transform $\tilde{f}^*(x) = \sup_\varepsilon|\tilde{f}_\varepsilon(x)|$ by the measure $m(e) = \int_e \frac{1}{\delta+|t|^\alpha}\,dt$, where $\delta = 0$ or 1 and $0 \leqq \alpha < 1$. S. Koizumi [4] shows that the operator $f \to \tilde{f}$ is of weak type $(1,1)$ with respect to the measure $m(e)$ where $\delta = 1$. He says also that the operator $f \to \tilde{f}^*$ is of weak type $(1,1)$ with respect to the same measure and the proof is carried over by the same method. However the latter proposition does not seem to be proved as the former proposition[1]. The purpose of this paper is to give the complete proof of this proposition.

We estimate the maximal function of Hardy-Littlewood with respect to the measure $m(e)$ in §1 and then $\tilde{f}^*$ in §2 with the same measure.

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1. Maximal function of Hardy-Littlewood.

For a non-negative locally integrable function f on $(-\infty, +\infty)$, the maximal function is defined by

$$(\Theta f)(x) = \sup_\varepsilon \frac{1}{\varepsilon}\int_{|x-t|\leqq\varepsilon} f(t)\,dt$$

where dt is the Lebesgue measure, dm is a measure on $(-\infty, +\infty)$ defined by

$$(1.1)\qquad m(e) = \int_e \frac{1}{\delta+|t|^\alpha}\,dt, \quad 0 \leqq \alpha < 1, \ \delta = 0 \text{ or } 1,$$

[1] For example, see Y. M. Chen [1], in particular, p. 243 footnote.

and L_m^p represents the set of all functions such that $\int_{-\infty}^{\infty} |f(t)|^p dm < +\infty$.

In the following, c denotes a constant depending only on α, and may be different in each occurence.

THEOREM 1. *If a non-negative function f belongs to L_m^1, we have for any $\lambda > 0$*

$$m(\{x\,;\,(\Theta f)(x) > \lambda\}) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty} f(t)dm(t).$$

To prove this theorem, it is sufficient to show the following proposition on θf:

$$(\theta f)(x) = \sup_{-\infty<\xi<x} \frac{1}{x-\xi}\int_{\xi}^{x} f(t)dt.$$

PROPOSITION 1. *If a non-negative function f belongs to L_m^1, we have for any $\lambda > 0$.*

$$m(\{x\,;\,(\theta f)(x) > \lambda\}) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty} f(t)dm(t).$$

PROOF. Since

$$\frac{1}{x-\xi}\int_{\xi}^{x} f(t)dt = \frac{1}{m(\xi, x)}\int_{\xi}^{x} \frac{m(\xi, x)(\delta + |t|^{\alpha})}{x-\xi} f(t)dm(t)$$

where $m(\xi, x)$ means the m-measure $m([\xi, x])$ of the interval $[\xi, x]$, let us estimate

$$I = \frac{m(\xi, x)(\delta + |t|^{\alpha})}{x-\xi}$$

under the condition $\xi < t \leqq x$.

(1) Case $0 \leqq \xi < x$. Since $1/(\delta + S^{\alpha})$ is decreasing in $0 < S < +\infty$,

$$I = \frac{\delta + t^{\alpha}}{x-\xi}\int_{\xi}^{x} \frac{dS}{\delta + S^{\alpha}} \leqq \frac{\delta + t^{\alpha}}{x}\int_{0}^{x} \frac{dS}{\delta + S^{\alpha}} \leqq \frac{\delta + x^{\alpha}}{x}\int_{0}^{x} \frac{dS}{\delta + S^{\alpha}}$$

(2) Case $\xi < x \leqq 0$. Similarly to (1), we have

$$I \leqq \frac{\delta+|\xi|^{\alpha}}{|\xi|}\int_{0}^{|\xi|}\frac{dS}{\delta+S^{\alpha}}$$

(3) Case $\xi<0<x$. Setting $N=\mathrm{Max}(|\xi|, x)$,

$$I=\frac{\delta+|t|^{\alpha}}{x-\xi}\left(\int_{\xi}^{0}+\int_{0}^{x}\right)\frac{dS}{\delta+|S|^{\alpha}} \leqq \frac{2(\delta+N^{\alpha})}{N}\int_{0}^{N}\frac{dS}{\delta+S^{\alpha}}.$$

Therefore in any cases,

$$\frac{m(\xi,x)(\delta+|t|^{\alpha})}{x-\xi} \leqq \frac{2(\delta+N^{\alpha})}{N}\int_{0}^{N}\frac{dS}{\delta+S^{\alpha}} \leqq c. \tag{1.2}$$

Consequently we get

$$(\theta f)(x) \leqq c \sup_{-\infty<\xi<x}\frac{1}{m(\xi,x)}\int_{\xi}^{x} f(t)dm(t).$$

Thus we shall have Proposition 1, if we show the next proposition on Λf:

$$(\Lambda f)(x)=\sup_{-\infty<\xi<x}\frac{1}{m(\xi,x)}\int_{\xi}^{x} f(t)dm(t).$$

PROPOSITION 2. *If a non-negative function f belongs to L_m^1, we have for any $\lambda>0$*

$$m(\{x;\ (\Lambda f)(x)>\lambda\}) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty} f(t)dm(t).$$

This is a consequence of the Theorem 2.1. in [2].

REMARK. Theorem 1 is false for $\alpha>1$. Let $f(t)=t^{\beta}$ for $t>1$ and $f(t)=0$ for $t\leqq 1$, where $0<\beta<\alpha-1$, then we find that $(\Theta f)(x)=+\infty$ for all x. Thus we get that $m(\{x;\ (\Theta f)(x)>\lambda\})=\text{constant}\ (\delta=1),\ =\infty\ (\delta=0)$, which is impossible.

2. Maximal Hilbert transform.

THEOREM 2. *If a function f belongs to L_m^1, we have for any $\lambda>0$*

$$m(\{x;\tilde{f}^*(x)>\lambda\})\leqq\frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t).$$

In order to prove this theorem, we start with the following lemma which is stated in [3] when $\delta=0$ and, when $\delta=1$, it can be shown similarly.

LEMMA 1. *For any $f\in L_m^1$ and $\lambda>0$, we get the following decomposition*:

$$f(t)=v(t)+\sum_n w_n(t)=v(t)+w(t), \tag{2.1}$$

$$\operatorname{supp} w_n\subset I_n, \tag{2.2}$$

(2.3) *I_n do not contain the origin in their insides and are mutually disjoint,*

$$|v(t)|\leqq c\lambda, \tag{2.4}$$

$$\int_{-\infty}^{\infty}|v(t)|\,dm(t)+\sum_n\int_{-\infty}^{\infty}|w_n(t)|\,dm(t)\leqq c\int_{-\infty}^{\infty}|f(t)|\,dm(t), \tag{2.5}$$

$$\sum_n m(I_n)\leqq\frac{1}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t), \tag{2.6}$$

$$\int_{-\infty}^{\infty}w_n(t)dt=0. \tag{2.7}$$

Corresponding the above decomposition, we get $\tilde{f}^*(x)\leqq\tilde{v}^*(x)+\tilde{w}^*(x)$. We see $v\in L_m^2$ by virtue of (2.4) and (2.5). By [5] we get

$$\int_{-\infty}^{\infty}\tilde{v}^*(x)^2dm(x)\leqq c\int_{-\infty}^{\infty}|v(t)|^2dm(t).$$

So we get, taking (2.4) in consideration,

$$m\left(\left\{x;\ \tilde{v}^*(x)>\frac{\lambda}{2}\right\}\right)\leqq\frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t).$$

Next we turn our attention to $\tilde{w}^*(x)$. We denote by a_n the center of I_n in the Lemma 1, and I_n^* the interval which is obtained by magnifying I_n two times with center a_n and set $Q^*=\bigcup_n I_n^*$. Let us investigate $\tilde{w}_n(x)$ for x in

$\complement Q^*$.

Setting $B(x, \varepsilon)=\{t;\ |x-t|\leqq\varepsilon\}$, we get by (2.7)

$$\tilde{w}_{n\varepsilon}(x)=\int_{|x-t|\geqq\varepsilon}\frac{w_n(t)}{x-t}\,dt$$

$$=\begin{cases}\displaystyle\int_{-\infty}^{\infty}\left(\frac{1}{x-t}-\frac{1}{x-a_n}\right)w_n(t)dt, & \text{if } I_n\cap B(x,\varepsilon)=\emptyset\\ \displaystyle\int_{|x-t|\geqq\varepsilon}\frac{w_n(t)}{x-t}\,dt, & \text{if } I_n\cap B(x,\varepsilon)\neq\emptyset .\end{cases}$$

If $I_n\cap B(x,\varepsilon)\neq\emptyset$, $I_n\subset B(x, 3\varepsilon)$, so that

$$\int_{|x-t|\geqq\varepsilon}\frac{|w_n(t)|}{|x-t|}\,dt\leqq\frac{1}{\varepsilon}\int_{-\infty}^{\infty}|w_n(t)|\,dt=\frac{1}{\varepsilon}\int_{|x-t|\leqq 3\varepsilon}|w_n(t)|\,dt\,,$$

and that we find

$$|\tilde{w}_{n\varepsilon}(x)|\leqq\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt+\frac{1}{\varepsilon}\int_{|x-t|\leqq 3\varepsilon}|w_n(t)|\,dt\,.$$

Summing up with respect to n, we get

$$|\tilde{w}_{\varepsilon}(x)|\leqq\sum_n\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt+\frac{1}{\varepsilon}\int_{|x-t|\leqq 3\varepsilon}|w(t)|\,dt$$

$$\leqq\sum_n\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt+3(\Theta w)(x)\,,$$

where Θw stands for $\Theta(|w|)$. Consequently,

$$w^*(x)\leqq\sum_n\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt+3(\Theta w)(x) \tag{2.8}$$

for $x\in\complement Q^*$.

Hence

$$\int_{\complement Q^*}\left(\sum_n\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt\right)dm(x) \tag{2.9}$$

$$\leqq \sum_n \int_{CI_n^*} dm(x) \int_{-\infty}^{\infty} \left| \frac{1}{x-t} - \frac{1}{x-a_n} \right| |w_n(t)| \, dt$$

By virtue of the following Lemma 2, the last sum does not exceed

$$c \sum_n \int_{-\infty}^{\infty} |w_n(t)| \, dm(t) = c \int_{-\infty}^{\infty} |w(t)| \, dm(t) .$$

LEMMA 2. *For a function g whose support is contained in an interval $[a-k, a+k]$ not containing the origin in its inside, it holds that*

$$\int_{|x-a| \geqq 2k} dm(x) \int_{-\infty}^{\infty} \left| \frac{1}{x-t} - \frac{1}{x-a} \right| |g(t)| \, dt \leqq c \int_{-\infty}^{\infty} |g(t)| \, dm(t) .$$

PROOF.

$$\int_{|x-a| \geqq 2k} dm(x) \int_{-\infty}^{\infty} \left| \frac{1}{x-t} - \frac{1}{x-a} \right| |g(t)| \, dt$$

$$= \int_{-k}^{k} |g(a+t)| \, dt \int_{|x| \geqq 2k} \left| \frac{1}{x-t} - \frac{1}{x} \right| \frac{dx}{\delta + |x+a|^{\alpha}}$$

$$= \int_{-k}^{k} |g(a+t)| \, dt \left(\int_{(|x| \geqq 2k) \cap (|x+a| \leqq |a|/2)} + \int_{(|x| \geqq 2k) \cap (|x+a| > |a|/2)} \right)$$

$$\left| \frac{1}{x-t} - \frac{1}{x} \right| \frac{dx}{\delta + |x+a|^{\alpha}}$$

For $|t| \leqq k$ and $x \in (|x| \geqq 2k) \cap (|x+a| > |a|/2)$, $|a+t| \leqq |a| + |t| < 2|a| \leqq 4|x+a|$, so that $1/(\delta + |x+a|^{\alpha}) < 4/(\delta + |t+a|^{\alpha})$. We get

$$\int_{(|x| \geqq 2k) \cap (|x+a| > |a|/2)} \left| \frac{1}{x-t} - \frac{1}{x} \right| \frac{dx}{\delta + |x+a|^{\alpha}} \leqq \frac{4}{\delta + |t+a|^{\alpha}} \int_{|x| \geqq 2k} \left| \frac{1}{x-t} - \frac{1}{x} \right| dx$$

$$\leqq \frac{c}{\delta + |t+a|^{\alpha}} .$$

While, if $(|x| \geqq 2k) \cap (|x+a| \leqq |a|/2) \neq \emptyset$, then $k \leqq 3|a|/4$, so that $|a+t| < 2|a|$, and $|x-t| \geqq k$ and $|x| \geqq |a|/2$ for $|t| \leqq k$ and

$x \in (|x| \geqq 2k \cap (|x+a| \leqq |a|/2)$. So we get $|1/(x-t)-1/x| \leqq 2/|a|$. Consequently, by (1.2)

$$\int_{(|x|\geqq 2k)\cap(|x+a|\leqq|a|/2)} \left|\frac{1}{x-t}-\frac{1}{x}\right| \frac{dx}{\delta+|x+a|^{\alpha}} \leqq \frac{2}{|a|}\int_{|x+a|\leqq|a|/2} \frac{dx}{\delta+|x+a|^{\alpha}}$$

$$=\frac{2}{(|a|/2)}\int_0^{|a|/2} \frac{1}{\delta+|x|^{\alpha}}dx \leqq \frac{c}{\delta+(|a|/2)^{\alpha}} < \frac{c}{\delta+(2|a|)^{\alpha}} < \frac{c}{\delta+|a+t|^{\alpha}}.$$

Thus we get

$$\int_{|x-a|\geqq 2k} dm(x)\int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a}\right||g(t)|\,dt \leqq c\int_{-k}^{k}|g(a+t)|\frac{1}{\delta+|a+t|^{\alpha}}\,dt$$

$$=c\int_{-\infty}^{\infty}|g(t)|\,dm(t). \qquad \text{(q. e. d.)}$$

Since $\tilde{w}^*(x)$ satisfies the inequality (2.8) for $x \in \complement Q^*$,

$$\left\{x \in \complement Q^*;\ w^*(x) > \frac{\lambda}{2}\right\}$$

$$\subset \left\{x \in \complement Q^*;\ \sum_n \int_{-\infty}^{\infty}\left|\frac{1}{x-t}-\frac{1}{x-a_n}\right||w_n(t)|\,dt > \frac{\lambda}{4}\right\}$$

$$\cup \left\{x \in \complement Q^*;\ (\Theta w)(x) > \frac{\lambda}{12}\right\} \equiv E_1 \cup E_2, \quad \text{say}.$$

From (2.9), we get

$$m(E_1) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty}|w(t)|\,dm(t) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t).$$

While, from Theorem 1,

$$m(E_2) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty}|w(t)|\,dm(t) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t).$$

So that, if we show that $m(Q^*) \leqq \frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t)$, then we get

$$m\left(\left\{x;\ \tilde{w}^*(x)>\frac{\lambda}{2}\right\}\right)\leqq\frac{c}{\lambda}\int_{-\infty}^{\infty}|f(t)|\,dm(t)$$

and the proof of Theorem 2 is completed. To prove that $m(Q^*)\leqq\dfrac{c}{\lambda}\displaystyle\int_{-\infty}^{\infty}$ $|f(t)|\,dm(t)$, it is sufficient to show $m(I_n^*)\leqq cm(I_n)$ due to (2.6). But this is clear from the following lemma.

LEMMA 3. *If we put* $I=(a-k, a+k)$, $a>k>0$, $I^*=(a-2k, a+2k)$, *then* $m(I^*)\leqq cm(I)$.

PROOF. $m(a, a+2k)=m(a, a+k)+m(a+k, a+2k)\leqq m(a-k, a+k)$, so that we need only to show that $m(a-2k, a)\leqq cm(a-k, a)$. But this comes from an elementary calculation; for example, consider the two cases $0<k<a/4$ and $a/4<k<a$.

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