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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. An interesting class of phase spaces for autonomous functional differential equations with infinite retardation has been discussed by Coleman and Mizel [1]. A special case of their results concerns the Banach space V of functions mapping the interval $(-\infty, 0]$ into the n-dimensional Euclidean space R^n with norm $\|\cdot\|_V$ defined by

$$\|\varphi\|_V = |\varphi(0)| + \int_{-\infty}^{0} |\varphi(s)| e^s \, ds$$

for $\varphi \in V$. But, as is pointed out by Hale [3] we can easily see that the right-hand sides of even simple differential difference equations will not be continuous in the above norm of Coleman and Mizel. Taking a view of this point, the author considered a Banach space B of measurable functions mapping $(-\infty, 0]$ into R^n with the following properties:

(i) All bounded continuous functions mapping $(-\infty, 0]$ into R^n are in B.

(ii) If φ is in B, then $T^\sigma\varphi$, $T_\sigma\varphi$ and $\chi^\sigma\varphi$ are in B for each $\sigma \geqq 0$, where

$$T^\sigma\varphi(s) = \begin{cases} \varphi(0) & \text{for } s \in [-\sigma, 0], \\ \varphi(s+\sigma) & \text{for } s \in (-\infty, -\sigma), \end{cases}$$

$$T_\sigma\varphi(s) = \varphi(s-\sigma) \qquad \text{for } s \in (-\infty, 0]$$

and

$$\chi^\sigma\varphi(s) = \begin{cases} \varphi(s) - \varphi(-\sigma) & \text{for } s \in [-\sigma, 0], \\ 0 & \text{for } s \in (-\infty, -\sigma). \end{cases}$$

(iii) The norm $\|\cdot\|_B$ has the following properties: If φ, ψ are in B and $|\varphi| \leqq |\psi|$ holds almost everywhere, then $\|\varphi\|_B \leqq \|\psi\|_B$. For any sequence $\{\varphi_n\} \in B$ which converges pointwise to some function $\varphi \in B$, $\|\varphi_n - \varphi\|_B \to 0$ as $n \to \infty$. Furthermore, $\|T^\sigma\varphi - <\varphi(0)>\|_B \to 0$ as $\sigma \to \infty$, where $<\varphi(0)>$ is the constant function β such that $\beta(s)=\varphi(0)$ for all $s\in(-\infty, 0]$. $\|\varphi\|_B=0$ if and only if φ is zero almost everywhere.

The class of phase spaces considered by Coleman and Mizel is contained in our case. Furthermore, let C be a space of continuous functions φ mapping $(-\infty, 0]$ into R^n such that $\varphi(s)e^s \to 0$ as $s\to-\infty$ with norm $\|\varphi\|_C=\sup|\varphi(s)|e^s$, $\varphi\in C$. Then this space C also is contained in our case, but not in Coleman and Mizel's.

When I was discussing the asymptotic behavior of solutions of a nonautonomous system on the phase space mentioned above, Professor Yoshizawa informed me that Hale considered a more general class of Banach space and discussed the asymptotic behavior of the solutions of autonomous functional differential equations with infinite retardation. Therefore, in this paper we shall extend Hale's results to a nonautonomous system.

2. The phase space considered by Hale. Let $|y|$ be any vector norm of R^n. Let $B=B((-\infty, 0], R^n)$ be a Banach space of functions mapping $(-\infty, 0]$ into R^n with norm $\|\cdot\|$. For any φ in B and any σ in $[0,\infty)$, let φ^σ be the restriction of φ to the interval $(-\infty, -\sigma]$. This is a function mapping $(-\infty, -\sigma]$ into R^n. We shall denote by B^σ the space of such functions φ^σ. For any $\eta\in B^\sigma$, we define the norm $\|\eta\|_{B^\sigma}$ of η by

$$\|\eta\|_{B^\sigma} = \inf_{\varphi}\{\|\varphi\|, \ \varphi^\sigma=\eta\}\ .$$

Then the space B^σ is a Banach space with norm $\|\cdot\|_{B^\sigma}$. If x is any function defined on $(-\infty, a)$, $a>0$, then for each t in $[0, a)$ define the function x_t by the relation $x_t(s)=x(t+s)$, $-\infty<s\leqq 0$. Let A^a, $a>0$, be the class of functions mapping $(-\infty, a)$ into R^n such that for each x in A^a, x is a continuous function on $[0, a)$ and $x_0\in B$. The space B is assumed to have the following properties:

(I) If x is in A^∞, then x_t is in B for all t in $[0,\infty)$ and x_t is a continuous function of t.

(II) All bounded continuous functions mapping $(-\infty, 0]$ into R^n are in B.

(III) There is a $\nu\geqq 0$ such that if $\{\varphi_k\}$ is any uniformly bounded sequence in B converging to φ, uniformly on every compact subset of $(-\infty, 0]$ then φ^ν is in B and $\|\varphi_k^\nu-\varphi^\nu\|_{B^\nu}\to 0$ as $k\to\infty$.*)

(IV) There are continuous, nondecreasing, nonnegative functions $b(r)$, $c(r)$, $r\geqq 0$, $b(0)=c(0)=0$, and a constant $K\geqq 0$ such that for any φ in B,

$$\|\varphi\| \leqq Kb[\sup_{-\sigma\leqq s\leqq 0}|\varphi(s)|] + c(\|\varphi^\sigma\|_{B^\sigma})$$

for any $\sigma\geqq 0$.

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*) In the assumption (III), Hale has assumed that $\{\varphi_k\}$ is pointwise converging sequence. However, if so, the space C mentioned in Section 1 is not contained in Hale's space.

(V) If $\sigma \geqq 0$, $\varphi \in B$, and $\bar{\varphi}^{\sigma} \in B^{\sigma}$ is defined by $\bar{\varphi}^{\sigma}(s) = \varphi(\sigma + s)$, then $\|\bar{\varphi}^{\sigma}\|_{B^{\sigma}} \to 0$ as $\sigma \to \infty$.

It is verified without difficulty that the space mentioned in Section 1 is a special case of this space considered by Hale.

3. Almost periodic systems. Let B be the Banach space constructed by Hale. We shall discuss asymptotic behaviors of solutions of a nonautonomous functional differential equation. We shall use the following notations. Let $d(\varphi, D)$ be the distance of the point φ from the set D in B, and B_H be the subset of B such that for any $\varphi \in B_H$ $\|\varphi\| < H$ and $\bar{B}_H$ be its closure. If $x(t)$ is a continuous function on an interval $[t_0, \infty)$ into R^n and $x_{t_0} \in B$, the positive limit set of $x(t)$ denoted by $\Gamma^+(x(t))$ consists of all points z for which there is a sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ with $x(t_m) \to z$, and the positive limit set $\Lambda^+(x_t)$ consists of all functions $\varphi \in B$ such that there is a sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ with $\|x_{t_m} - \varphi\| \to 0$.

We shall consider a system

$$\dot{x}(t) = P(t, x_t) \tag{1}$$

and its perturbed system

$$\dot{x}(t) = P(t, x_t) + R(t, x_t) + G(t, x_t), \tag{2}$$

where the following assumptions will be made:

(H1) R and G are continuous on $[0, \infty) \times B$ with values in R^n.

(H2) P is continuous on $R^1 \times B$ and almost periodic in t uniformly for $\varphi \in B$.

(H3) For any $H > 0$, there exists a positive constant $L(H)$ such that $|P(t, \varphi)| \leqq L(H)$ on $R^1 \times \bar{B}_H$.

(H4) Let $y(t)$ be any function defined on R^1, $y_0 \in B$, which is continuous for $t \geqq 0$ and $\|y_t\| < H^*, t \geqq 0$, for some $H^* > 0$. For any sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ and for $t \in R^1$,

$$\lim_{m \to \infty} \int_{t_m}^{t_m + t} |G(s, y_s)| ds = 0.$$

(H5) Let Q be a fixed subset. For any $H > 0$ and each $\varepsilon > 0$ there are fixed numbers $\delta > 0$ and $b > 0$ depending on ε and H such that $|R(t, \varphi)| < \varepsilon$ holds whenever $t \geqq b$, $\varphi \in \bar{B}_H$ and $d(\varphi, Q) < \delta$.

(H6) B is separable.

These hypotheses are sufficient to guarantee a local existence theorem for solutions of (1).

The following lemma is well known in the case where B is separable.

LEMMA 1. *If $P(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in B$ and if $\{\tau'_k\}$ is any real sequence, then there exists a subsequence $\{\tau_k\}$ and a $P^*(t, \varphi)$, which is almost periodic in t uniformly for $\varphi \in B$, such that*

$$(3) \qquad P(t+\tau_k, \varphi) \to P^*(t, \varphi) \quad \text{as } k \to \infty$$

uniformly for all $t \in R^1$ and φ on compact subsets of B.

The following lemma is due to Hale (p. 43 in [3]).

LEMMA 2. *For any $\delta \geqq 0$ and any φ in B, there is a continuous function $e(\delta, \varphi)$, $e(0, 0) = 0$ such that for any x in A^∞ with $x_0 = \varphi$ and $|x(s)| \leqq \delta$, $s \geqq 0$, and for any t in $[0, \infty)$,*

$$\|x_t\| \leqq e(\delta, \varphi).$$

THEOREM 1. *Suppose that hypotheses* (H1) *through* (H6) *hold for some fixed set $Q \subset B$ and that a bounded solution $x(t)$ for $t \geqq t_0$ of* (2) *tends to Q as $t \to \infty$. Then for any sequence $\{t'_m\}$ such that $t'_m \to \infty$ as $m \to \infty$, there exists a subsequence $\{t_m\}$ of $\{t'_m\}$, a function $y(t)$ and a function $P^*(t, \varphi)$ such that*

$$(4) \qquad \|x_{t_m+t} - y_t\| \to 0, \text{ as } m \to \infty, \text{ uniformly on compact subsets of } R^1,$$

$$(5) \qquad P(t_m + t, \varphi) \to P^*(t, \varphi) \text{ as } m \to \infty,$$

uniformly for all $t \in R^1$ and φ on compact subsets of B, and

$$(6) \qquad \dot{y}(t) = P^*(t, y_t) \quad \text{for } t \in R^1.$$

PROOF. Let $x(t) = x(t, t_0, \psi)$, $x_{t_0} = \psi$, be a solution of (2) which satisfies the hypotheses of the theorem and $|x(t)| < \zeta$ for some constant ζ and for every $t \geqq t_0$. By Lemma 2, there exists a constant $\eta(\zeta, \psi)$ such that $\|x_t\| \leqq \eta$ for all $t \geqq t_0$. By boundedness of $x(t)$, there is a z and a subsequence $\{t_m\}$ of $\{t'_m\}$ such that $x(t_m) \to z$ as $m \to \infty$. Take any compact interval $I_N = [-N, N]$ and let $\{\varepsilon_m\}$ be a decreasing positive sequence such that $\varepsilon_m \to 0$ as $m \to \infty$. We can assume that

$$(7) \qquad t_0 + N < t_1, \qquad t_m < t_{m+1}, \qquad \text{for } m = 1, 2, 3, \cdots,$$

and that

$$(8) \qquad \int_{t_m - N}^{t_m + N} |G(s, x_s)|\, ds < \varepsilon_m \qquad m = 1, 2, 3, \cdots,$$

by (H4), taking a subsequence again, if necessary. From (H5) it follows that there are numbers $s_m > 0$ and $\delta_m > 0$ such that we have $|R(t, \varphi)| < \varepsilon_m$ whenever $t \geqq s_m$, $\varphi \in \bar{B}_\eta$ and $d(\varphi, Q) < \delta_m$. Since $x_t \to Q$ as $t \to \infty$, the numbers s_m can be taken so large that $d(x_t, Q) < \delta_m$ for all $t \geqq s_m$. We assume that $t_m \geqq s_m + N$ and consequently that

$$(9) \qquad |R(t, x_t)| < \varepsilon_m \qquad \text{for all } t \geqq t_m - N.$$

For any $t > t^1$, $t^1 > t_0$, we have

$$(10) \qquad x(t) = x(t^1) + \int_{t^1}^{t} \{P(s, x_s) + R(s, x_s) + G(s, x_s)\}\, ds\,.$$

Set $x_m(t) = x(t_m + t)$ for $-\infty < t \leqq N$ and $m = 1, 2, 3, \cdots$. By (H3), let M be the number defined by

$$(11) \qquad M = \sup\{|P(t, \varphi)|;\ t \in R^1,\ \varphi \in \bar{B}_\eta\}\,.$$

Using (7) and (10), we see that

$$(12) \qquad x_m(t) = x_m(0) + \int_0^t P(t_m + s, (x_m)_s)\, ds + \int_{t_m}^{t_m + t} \{R(s, x_s) + G(s, x_s)\}\, ds\,.$$

The sequence $\{x_m(t);\ t \in I_N\}$ is uniformly bounded and equicontinuous, since for any m and for $-N \leqq s_1 < s_2 \leqq N$, it follows from (8), (9) and (11) that

$$\begin{aligned} |x_m(s_1) - x_m(s_2)| &\leqq \int_{s_1}^{s_2} M\, ds + \int_{s_1}^{s_2} \varepsilon_m\, ds + \int_{t_m - N}^{t_m + N} |G(s, x_s)|\, ds \\ &\leqq (M + \varepsilon_m)(s_2 - s_1) + \varepsilon_m\,. \end{aligned}$$

By Ascoli's theorem, we can choose a subsequence which converges uniformly on I_N. Letting $N = 1, 2, 3, \cdots$ and using the familiar diagonalization procedure, we can get a subsequence that will be uniformly convergent to a function y on all compact subsets of R^1. The limit function y is continuous and bounded, and hence y belongs to B by (II). Define $\hat{x}_t$ and $\tilde{\psi}^t$ by

$$\hat{x}_t(\theta) = \begin{cases} x(t+\theta) & \text{for } -(t-t_0) \leqq \theta \leqq 0, \\ \psi(0) & \text{for } -\infty < \theta < -(t-t_0). \end{cases}$$

and

$$\tilde{\psi}^t(\theta) = \begin{cases} 0 & \text{for } -(t-t_0) \leqq \theta < \infty \\ \bar{\psi}^{t-t_0}(\theta) - \psi(0) & \text{for } -\infty < \theta < -(t-t_0). \end{cases}$$

For the ν given in (III) and t_m such that $t_m + s \geqq \nu$ and $t_m + s \geqq t_0$, hypothesis (IV) implies

$$\begin{aligned} \|(x_m)_s - y_s\| &= \|\hat{x}_{t_m+s} + \tilde{\psi}^{t_m+s} - y_s\| \\ &\leqq \|\hat{x}_{t_m+s} - y_s\| + \|\tilde{\psi}^{t_m+s}\| \\ &\leqq Kb[\sup_{-\nu \leqq \theta \leqq 0} |x(t_m+s+\theta) - y(s+\theta)|] + c(\|\hat{x}_{t_m+s}{}^{\nu} - y_s{}^{\nu}\|_{B\nu}) \\ &\quad + Kb[\sup_{-(t_m+s \leqq \theta \leqq 0} |\tilde{\psi}^{t_m+s}(\theta)|] + c(\|\tilde{\psi}^{t_m+s}\|_{B^{t_m+s}}). \end{aligned}$$

Since $x(t_m+\theta) \to y(\theta)$ uniformly on compact sets and hypotheses (III) and (V) are satisfied, $\|(x_m)_s - y_s\| \to 0$ as $m \to \infty$ uniformly on compact sets. This proves (4).

Now let $H(P)$ be the set of functions $H(P) = \{P(t+h, \varphi), t \in R^1\}$ and $\bar{H}(P)$ be the uniform closure in the sense of (3) of $H(P)$. Using Lemma 1, we can assume that for the same subsequence $\{t_m\}$ and for some $P^*(t, \varphi) \in \bar{H}(P)$,

$$|P(t_m + t, \varphi) - P^*(t, \varphi)| \to 0 \tag{13}$$

uniformly for all $t \in R^1$ and φ on compact subsets of B, as $m \to \infty$. For any $t \in R^1$, we have

$$\begin{aligned} &\left| \int_0^t \{P(t_m + s, (x_m)_s) - P^*(s, y_s)\} ds \right| \\ &\quad \leqq \left| \int_0^t |P(t_m + s, (x_m)_s) - P^*(s, (x_m)_s)| ds \right| \\ &\quad + \left| \int_0^t |P^*(s, (x_m)_s) - P^*(s, y_s)| ds \right|. \end{aligned} \tag{14}$$

By (4), we can see that the set $\{(x_m)_s ; s \in R^1\}$ has the compact closure in B, and therefore it follows from (13) that the first term on the right-hand side of (14) tends to zero as $m \to \infty$. Since $P^*(t, \varphi)$ is uniformly continuous on compact sets of (t, φ), the second term also tends to zero. Using (8), (9), (12), (13) and (14), we can see that

$$y(t) = z + \int_0^t P^*(s, y_s)ds \qquad \text{for all } t \in R^1 .$$

This completes the proof.

REMARK 1. The above result (4) shows that the set $\{x_t \mid t \geqq t_0\}$ is relatively compact in B. The above result has been shown by Hale (Lemma 2 in [4], p. 44) for an autonomous system.

Considering a sequence $\{t_m\}$ such that $x_{t_m} \to \psi$ as $m \to \infty$, the following corollary follows immediately from Theorem 1.

COROLLARY 1. *Let the hypotheses of Theorem* 1 *hold. Then the positive limit set* $\Lambda^+(x_t)$ *of the path* x_t *is not empty, and for each element* $\psi \in \Lambda^+(x_t)$ *there is a sequence* $\{t_m\}$, *a function* $y(t)$ *and a function* $P^*(t, \varphi)$ *such that* $t_m \to \infty$ *as* $m \to \infty$,

$$\|x_{t_m+t} - y_t\| \to 0, \text{ as } m \to \infty, \text{ uniformly on compact subsets of } R^1,$$

$$P(t_m + t, \varphi) \to P^*(t, \varphi) \qquad \text{as } m \to \infty ,$$

uniformly for all t and φ *on compact subsets of B, and*

$$\dot{y}(t) = P^*(t, y_t) \qquad \text{on } -\infty < t < \infty$$

with $y_0 = \psi$.

REMARK 2. The above result has been obtained by Miller (Corollary 1 in [6]) in the case where the phase space is $C([-r, 0], R^n)$.

COROLLARY 2. *Let the hypotheses of Theorem* 1 *hold. Then for each point* $z \in \Gamma^+(x(t))$ *there is a sequence* $\{t_m\}$, *a function* $y(t)$ *and a function* $P^*(t, \varphi)$ *such that* $t_m \to \infty$ *as* $m \to \infty$,

$$x(t_m + t) \to y(t), \text{ as } m \to \infty, \text{ uniformly on compact subsets of } R^1,$$

$$P(t_m + t, \varphi) \to P^*(t, \varphi) \qquad \text{as } m \to \infty ,$$

uniformly for all $t \in R^1$ *and* φ *on compact subsets of B, and*

$$\dot{y}(t) = P^*(t, y_t) \qquad on \ -\infty < t < \infty$$

with $y(0) = z$.

This corollary also follows immediately from Theorem 1 by considering a sequence $\{t_m\}$ such that $x(t_m) \to z$ as $m \to \infty$. Furthermore, this corollary shows the relation $\{\psi(0) \mid \psi \in \Lambda^+(x_t)\} \supset \{z \mid z \in \Gamma^+(x(t))\}$.

REMARK 3. For the case where $P(t, \varphi)$ in (2) is autonomous, the above results correspond to Yoshizawa's [7]. Miller [5] has extended Yoshizawa's result to almost periodic systems, and furthermore, Miller [6] has extended his results to almost periodic systems of functional differential equations with finite retaradation.

DEFINITION. *A* subset U of B is said to be semi-invariant (see p. 54 in [8]) with respect to the almost periodic system (1), if for each element $\psi \in U$ there is an almost periodic function $P^*(t, \varphi) \in \bar{H}(P)$ and a solution $y(t)$ of $\dot{y}(t) = P^*(t, y_t)$ such that $y_0 = \psi$ and $y_t \in U$ for $-\infty < t < \infty$.

COROLLARY 3. *We assume the hypotheses* (H1) *through* (H6). *Let* Q_0 *be the largest semi-invariant subset of* Q *with respect to* (1). *If* $x(t)$ *is a bounded solution of* (2) *such that* $x(t) \to Q$ *as* $t \to \infty$, *then* $x_t \to Q_0$ *as* $t \to \infty$.

PROOF. The conclusion of Corollary 1 implies the semi-invariance of $\Lambda^+(x_t)$ with respect to (1). It is easily proved by (4) that $x_t \to \Lambda^+(x_t)$ as $t \to \infty$, since $\Lambda^+(x_t)$ is nonempty and compact by Lemma 3 in [3]. From Corollary 1 it follows that $\Lambda^+(x_t) \subset Q_0$. Therefore $x_t \to Q_0$ as $t \to \infty$.

4. Liapunov functions and asymptotic behaviors. In this section we shall apply the results of Section 3 to the system (1). If $W(t)$ is a function, then let

$$\dot{W}(t) = \limsup_{h \to 0^+} \frac{1}{h} \left(W(t+h) - W(t) \right).$$

THEOREM 2. *Let* $V(t, \varphi)$ *be a continuous function on* $R^1 \times B$ *such that* $\dot{V}(t, x_t) \leqq 0$ *for each solution* $x(t)$ *of* (1). *Suppose* $P(t, \varphi)$ *and* $V(t, \varphi)$ *are almost periodic in t uniformly for* $\varphi \in B$ *and* $P(t, \varphi)$ *satisfies* (H3). *Let* U *be the subset of* B *consisting of all trajectories* y_t *such that corresponding to* y_t *there exists a sequence* $\{t_m\}$, *a function* $P^* \in \bar{H}(P)$ *and a function*

$V^* \in \bar{H}(V)$ *such that* $t_m \to \infty$, *as* $m \to \infty$,

$$P(t_m + t, \varphi) \to P^*(t, \varphi), \quad V(t_m + t, \varphi) \to V^*(t, \varphi)$$

uniformly for all on compact subsets of B,

$$\dot{y}(t) = P^*(t, y_t) \qquad \textit{for } t \in R^1,$$

$$y(t) \textit{ is bounded for } t \in R^1,$$

and

$$\dot{V}^*(t, y_t) = 0 \qquad \textit{for } t \in R^1 .$$

If $x(t)$ *is a bounded solution for* $t \geqq t_0 \geqq 0$ *of* (1), *then* U *is nonempty and* $x_t \to U$ *as* $t \to \infty$.

Since the proof of this theorem is similiar to that of Miller (Theorem 3 in [6]), we omit it.

REMARK 4. The essential part in the proof of this theorem is that $V(t, x_t) \to V_0$ for a constant V_0 as $t \to \infty$ for a bounded solution $x(t)$, and $|V(t, x_t)|$ is bounded since we assume that $V(t, \varphi)$ is almost periodic. Therefore it is clear that we can replace the condition $\dot{V}(t, x_t) \leqq 0$ by $\dot{V}(t, x_t) \geqq 0$.

If P and V in Theorem 2 are periodic, the above theorem has the following form.

COROLLARY 4. *Let* $P(t, \varphi)$ *and* $V(t, \varphi)$ *be periodic in* t *of the same period* θ. *Suppose* $P(t, \varphi)$ *satisfies* (H3) *and* $\dot{V}(t, y_t) \leqq 0$ *for each solution* $y(t)$ *of* (1). *Let* U *be the set of solutions* y_t *such that* $y(t)$ *is bounded on* $(-\infty, \infty)$ *and* $\dot{V}(t, y_t) = 0$ *for* $-\infty < t < \infty$. *If* $x(t)$ *is a bounded solution for* $t \geqq t_0 \geqq 0$ *of* (1), *then* $x_t \to U$ *as* $t \to \infty$.

The following lemma is easily proved by modifing Hale's result (Lemma 1 in [2]).

LEMMA 3. *Let the function* $P(t, \varphi)$ *of* (1) *satisfy* (H3) *and* $V(t, \varphi)$ *be continuous on* $[0, \infty) \times B$. *Define*

$$U(C) = \{(t, \varphi) \,|\, V(t, \varphi) < C\} .$$

Suppose there is a constant K *such that for all* $(t, \varphi) \in U(C)$ *we have* $|\varphi(0)| \leqq K$ *and* $V(t, \varphi) \geqq 0$. *If* $\dot{V}(t, x_t) \leqq 0$ *for all solutions* $x(t)$ *of* (1), *then all solutions of* (1) *which start in the set* $U(C)$ *are bounded for all future time.*

PROOF. Let $x(t)$ be a solution of (1) with initial condition $(t_0, x_{t_0}) \in U(C)$. Since $\dot{V}(t, x_t) \leqq 0$, $V(t, x_t) < C$ for all $t \geqq t_0$ and $|x(t)| \leqq K$ as long as $x(t)$ exists. By (H3), we can see that $x(t)$ exists for all $t \geqq t_0$ and $|x(t)| \leqq K$ for all $t \geqq t_0$.

By combining Theorem 2 and Lemma 3 we can generalize certain result of Hale (Theorem 1 in [2]), Krasovskiĭ (p. 153 in [3]) and Miller (Corollary 4 in [6]).

COROLLARY 5. *Let* $V(t, \varphi)$ *be a continuous function on* $R^1 \times B$ *such that* $\dot{V}(t, x_t) \leqq 0$ *for each solution of* (1). *Let* $P(t, \varphi)$ *and* $V(t, \varphi)$ *be almost periodic in* t *uniformly for* $\varphi \in B$ *and* $P(t, \varphi)$ *satisfy* (H3). *Suppose that there are constants* C *and* K *such that* $V(t, \varphi) \geqq 0$, $|\varphi(0)| \leqq K$ *for each* $(t, \varphi) \in U(C)$. *If* U_0 *is the set of solutions* y_t *which satisfy the following conditions; corresponding to* y_t *there exists a sequence* $\{t_m\}$, *a function* $P^* \in \bar{H}(P)$ *and a function* $V^* \in \bar{H}(V)$ *such that* $t_m \to \infty$ *as* $m \to \infty$, $P(t_m + t, \varphi) \to P^*(t, \varphi)$, $V(t_m + t, \varphi) \to V^*(t, \varphi)$ *as* $m \to \infty$, *uniformly for all* $t \in R^1$ *and* φ *on compact subsets of* B,

$$\dot{y}(t) = P^*(t, y_t) \qquad \text{for } t \in R^1,$$

$$|y(t)| \leqq K \qquad \text{for } t \in R^1$$

and

$$\dot{V}^*(t, y_t) = 0 \qquad \text{for } t \in R^1.$$

Then all solutions $x(t)$ *of* (1) *with initial value in* $U(C)$ *remain in* $L(K)$, *where* $L(K) = \{x \mid |x| \leqq K\}$, *and* $x_t \to U_0$ *as* $t \to \infty$.

The following corollary corresponds to Corollary 5 in [6].

COROLLARY 6. *Let* $P(t, \varphi)$ *and* $V(t, \varphi)$ *are almost periodic in* t *uniformly for* $\varphi \in B$ *and* $P(t, \varphi)$ *satisfy* (H3). *Suppose that*

(i) *there exists a sequence* $\{t_m\}$, *a function* $P^* \in \bar{H}(P)$ *and a function* $V^* \in \bar{H}(V)$ *such that* $t_m \to \infty$, *as* $m \to \infty$, $P(t_m + t, \varphi) \to P^*(t, \varphi)$,

$V(t_m+t, \varphi) \to V^*(t, \varphi)$ *as* $m \to \infty$, *uniformly for all* $t \in R^1$ *and* φ *on compact subsets of* B,

(ii) $\dot{V}(t, x_t) \geqq 0$ *for each solution* $x(t)$ *of* (1),

(iii) *for each* $\varepsilon > 0$ *there is a* $\delta(\varepsilon) > 0$ *such that if* $\|\varphi\| < \delta$, *then* $|V(t, \varphi)| < \varepsilon$ *uniformly for* $t \in R^1$,

and

(iv) *there is no solution* $y(t)$ *of* $\dot{y} = P^*(t, y_t)$ *such that* $y(t) \not\equiv 0$, $y(t)$ *is bounded and* $V^*(t, y_t) = 0$ *on* $t \in R^1$.

If $x(t)$ *is a solution of* (1) *with* $V(T, x_T) > 0$ *for some* $T > 0$, *then for each* $H_1 > 0$ *there is a number* $t_1 = t_1(H_1, x(t)) > T$ *such that* $|x(t_1)| > H_1$.

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REFERENCES

[1] B. D. COLEMAN AND V. J. MIZEL, On the stability of solutions of functional differential equations, Arch. Rational Mech. Anal., 30(1968), 174-196.
[2] J. K. HALE, A stability theorem for functional differential equations, Proc. Nat. Acad. Sci., 50(1963), 942-946.
[3] J. K. HALE, Dynamical systems and stability, J. Math. Anal. Appl., 26(1969), 39-59.
[4] N. N. KRASOVSKIǏ, Stability of motion, Stanford Univ. Press, Stanford, Calif., 1963.
[5] R. K. MILLER, Asymptotic behavior of solutions of nonlinear differential equations, Trans. Amer. Math. Soc., 115(1965), 400-416.
[6] R. K. MILLER, Asymptotic behavior of nonlinear delay-differential equations, J. Differential Equations, 1(1965), 293-305.
[7] T. YOSHIZAWA, Asymptotic behavior of solutions of a system of differential equations, Contrib. Diff. Eqs., 1(1963), 371-387.
[8] T. YOSHIZAWA, Stability theory by Liapunov's second method, The Mathematical Society of Japan, 1966.

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