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APPROXIMATION OF SEMI-GROUPS OF NONLINEAR OPERATORS

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1. Introduction. Let us consider an evolution equation

$$(d/dt)u(t) = Au(t),\ u(0) = x \tag{1.1}$$

in a Banach space X. Here A is an operator, not necessarily linear, in X and is assumed to be time independent. And we introduce an approximating scheme to the evolution equation as follows. Take a sequence $\{h_n\}$ of positive numbers going to 0 as $n\to\infty$. The solution $u_n(t)$ to the n-th approximating equation is calculated inductively, for t integral multiples of h_n, by the following system of equations:

$$u_n((k+1)h_n) = C_n u_n(kh_n),\ u_n(0) = x \tag{1.2}$$

for $k = 0, 1, 2, \cdots$ and n, where each C_n is an operator from X into itself. In case when A is a linear operator whose domain $D(A)$ is dense in X, Trotter [13] proved the following results which show the existence of solution $u(t) = u(t;x)$ of (1.1) and the convergence of approximating solutions, that is,

$$u_n([t/h_n]h_n) = C_n^{[t/h_n]}x \to u(t\,;\,x),$$

where $[\cdot]$ denotes the Gaussian blacket.

THEOREM A. *Let $\{C_n\}$ be a sequence of bounded linear operators satisfying the consistency condition and the stability condition*:

$$\lim_{n\to\infty} h_n^{-1}(C_n - I)x = Ax \qquad \textit{for } x \in D(A) \tag{C1}$$

and the domain $D(A)$ is dense linear in X,

$$\|C_n^{\,k}\| \leqq Ke^{Mkh_n} \qquad \textit{for } k \textit{ and } n, \tag{S1}$$

where K and M are some constants independent of k and n. Suppose that

(T) *for some $\lambda_0 > M$, $Cl[R(\lambda_0 I - A)] = X$(or $R(\lambda_0 I - A) = X$), where $R(\lambda_0 I - A)$ is the range of $\lambda_0 I - A$ and $Cl[R(\lambda_0 I - A)]$ is the closure of $R(\lambda_0 I - A)$.*

Then the closure $\bar{A}$ of A (or A) generates a linear semi-group $\{T(t)\}$ of class (C_0) and for each $x \in X$

$$T(t)x = \lim_{n\to\infty} C_n^{[t/h_n]}x \qquad \text{for } t \geqq 0. \tag{1.3}$$

Now, for the approximating scheme (1.2) to the nonlinear evolution equation we set the following two basic requirements, instead of the conditions (C1) and (S1);

$$\lim_{n\to\infty} h_n^{-1}(C_n - I)x = Ax \qquad \text{for } x \in D(A), \tag{C}$$

$$\|C_n x - C_n y\| \leqq e^{Mh_n}\|x-y\| \qquad \text{for } x, y \in X \text{ and } n, \tag{S}$$

where M is a constant independent of x, y and n.

The main purpose of this paper is to extend the Trotter theorem to the case of nonlinear operators for which there exist approximating schemes satisfying the above conditions (C) and (S). In §2 we shall state the main results, and in §3 we shall prepare some lemmas. The proofs of the theorems mentioned in §§2.1 and 2.3 are given in §§4 and 5, respectively.

2. Theorems. In this section we shall state the main results of the present paper.

2.1. We first introduce some notions of nonlinear semi-groups. Let $\{T(t)\,;\, t \geqq 0\}$ be a family of operators, not necessarily linear, from X into itself satisfying the following conditions:

(2.1) $T(0) = I$ (the identity mapping), $T(t)T(s) = T(t+s)$ for s, $t \geqq 0$;

(2.2) for each $x \in X$, $T(t)x$ is strongly continuous in $t \geqq 0$;

(2.3) there is a constant $\omega \geqq 0$ such that

$$\|T(t)x - T(t)y\| \leqq e^{\omega t}\|x-y\| \quad \text{for } x, y \in X \text{ and } t \geqq 0.$$

Then we call such a family $\{T(t)\}$ simply a *nonlinear semi-group of local type*. And we define the *infinitesimal generator* A_0 of a nonlinear semi-group $\{T(t)\}$ by

(2.4) $$A_0x = \lim_{h\to 0+} h^{-1}(T(h)-I)x$$

and the *weak infinitesimal generator* A' by

(2.5) $$A'x = \text{w-}\lim_{h\to 0+} h^{-1}(T(h)-I)x,$$

where "w-lim" means the weak limit in X.

In view of these notions, the main theorem is stated in the following form.

THEOREM 1. *Let X^* be uniformly convex. Suppose that*

(i) (C) *and* (S) *are satisfied, and $D(A)$ is dense in X,*

(ii) $Cl[R(I-h_0A)]=X$ *for some* $h_0 \in (0, 1/M)$.

Then we have the following:

(a) *The closure* $\bar{A}$[1] *of A, which is not necessarily single-valued, generates a nonlinear semi-group $\{T(t)\}$ of local type such that for each $x \in D(\bar{A})$, $T(t)x$ is strongly absolutely continuous on every finite interval, $T(t)x \in D(\bar{A})$ for all $t \geqq 0$ and*

$$(d/dt)T(t)x \in \bar{A}T(t)x \qquad \text{for a. a. } t \geqq 0.$$

(b) *For each $x \in X$, the convergence*

(2.6) $$T(t)x = \lim_{n\to\infty} C_n^{[t/h_n]}x$$

holds true, uniformly with respect to t in every finite interval.

(c) *In particular, if we assume $R(I-h_0A)=X$ instead of* (ii) *then $\bar{A}=A$, A is the weak infinitesimal generator of $\{T(t)\}$ and for each $x \in D(A)$, $T(t)x$ has the weak derivative $AT(t)x$ which is weakly continuous in $t \geqq 0$.*

(b) of the above theorem states the convergence of the approximating scheme (1.2). We shall say that *the approximating scheme is convergent to the semi-group $\{T(t)\}$* if (b) holds true, in the following.

Next, let us assume that the evolution equation (1.1) is well posed in the following sense:

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1) An operator T, not necessarily single-valued, is said to be the closure of A if $G(T)=Cl[G(A)]$, where $G(\cdot)$ denotes the graph of operator; we write $T=\bar{A}$.

(E) There is a nonlinear semi-group $\{T(t)\}$ of local type such that for each $x \in D(A)$, $(\text{w-}d/dt)T(t)x = AT(t)x$ for a. a. $t \geqq 0$, where $(\text{w-}d/dt)u(t)$ is the weak derivative of $u(t)$.

Then (b) of Theorem 1 holds true without the uniform convexity of X^* and the assumption (ii), that is, we have the following

THEOREM 2. *Suppose that* (C), (S) *and* (E) *are satisfied, and that* $D(A)$ *is dense in* X. *Then the approximating scheme is convergent to the semi-group* $\{T(t)\}$.

The proofs of the above theorems are given in §4.

2. 2. In this paragraph we shall consider some relations between abstract Cauchy problems and nonlinear semi-groups.

Let A be a not necessarily single-valued operator in a Banach space X. For such an operator A, we introduce the following abstract Cauchy problem:

(CP) Given an element $x \in X$, find a function $y(t\,;\,x)$ such that

(α) $y(t\,;\,x)$ is strongly absolutely continuous on any finite subinterval of $[0, \infty)$;

(β) $y(0\,;\,x) = x$ and

$$(d/dt)y(t\,;\,x) \in Ay(t\,;\,x) \qquad \text{for a. a. } t. \tag{2.7}$$

Here, if A is single-valued then "$\in$" in the above problem is replaced by "$=$".

In the above (CP) we may consider the following equation

$$(\text{w-}d/dt)y(t\,;\,x) \in Ay(t\,;\,x) \qquad \text{for a. a. } t, \tag{2.8}$$

instead of (2. 7). We write (w-CP) for the (CP) in which (2. 7) is replaced by (2. 8). But in view of (α), any solution of (w-CP) is necessarily a solution of (CP) (see [4; Theorem 3. 8. 6]).

Before stating the theorem, we introduce a notion of the duality mapping J of X. The *duality mapping* of X is a multi-valued mapping from X into X^* defined by $x \to \{x^* \in X^*;\ \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ (see, for example, Browder [1]). By virtue of Kato's results [5; Lemma 1. 2], the assumption that X^* is uniformly convex implies that the duality mapping J is single-valued and uniformly continuous on every bounded set of X.

THEOREM 3. (a) *Assume that*

(D) $A - \omega I$ *is dissipative for some* $\omega \geqq 0$, *that is, for every* $x, y \in D(A)$, $x' \in (A - \omega I)x$ *and* $y' \in (A - \omega I)y$ *there is an* $f \in J(x - y)$ *such that*

$$\mathrm{re}\langle x'-y', f\rangle \leqq 0.$$

And suppose that for each $x \in D(A)$ there is a solution $y(t\,;\,x)$ of the (w-CP).

Then there is a unique nonlinear semi-group $\{T(t)\}$ of local type defined on $Cl[D(A)]$ such that $T(t)x = y(t\,;\,x)$ for all $t \geqq 0$ and $x \in D(A)$. In particular, if A is single-valued then A coincides with the weak infinitesimal generator of $\{T(t)\}$ on a dense subset of $Cl[D(A)]$.

(b) *Conversely, if X is reflexive and if A is the weak infinitesimal generator of a nonlinear semi-group $\{T(t)\}$ of local type, then the property* (D) *is satisfied and for each $x \in D(A)$, $T(t)x$ is a unique solution of the* (w-CP).

PROOF. (a) We first note that if A satisfies the condition (D) then the (w-CP) has at most one solution for each initial value. For, let $y(t\,;\,x)$ and $z(t\,;\,x)$ be the solutions of the (w-CP) for an initial value x, and set $u(t) = y(t\,;\,x) - z(t\,;\,x)$ for $t \geqq 0$. Then $u(s)$ has the weak derivative $(\text{w-}d/ds)u(s)$ $(\equiv u_1(s)) \in Ay(s\,;\,x) - Az(s\,;\,x)$ at a. a. s and $\|u(s)\|$ is absolutely continuous in any finite interval. Then by the Kato lemma [5; Lemma 1. 3], for a. a. s

$$\|u(s)\|(d/ds)\|u(s)\| = \mathrm{re}\langle u_1(s), f\rangle$$

for every $f \in J(u(s))$. Combining (D) with this, we have

$$\|u(s)\|(d/ds)\|u(s)\| \leqq \omega\|u(s)\|^2 \qquad \text{for a. a. } s.$$

Hence

$$\|u(t)\|^2 = 2\int_0^t \|u(s)\|(d/ds)\|u(s)\|\,ds \leqq 2\omega\int_0^t \|u(s)\|^2\,ds$$

for all $t \geqq 0$. This inequality implies that $u(t) = 0$ for $t \geqq 0$.

Next, for any pair x, $z \in D(A)$ and $t \geqq 0$ we put $v(t) = y(t\,;\,x) - y(t\,;\,z)$. Since $(\text{w-}d/ds)v(s) \in Ay(s\,;\,x) - Ay(s\,;\,z)$ for a. a. s and $\|v(s)\|$ is absolutely continuous in any finite subinterval of $[0, \infty)$, similarly as in the above, we get

$$\|v(t)\|^2 \leqq \|x-z\|^2 + 2\omega\int_0^t \|v(s)\|^2\,ds \qquad \textit{for } t \geqq 0.$$

This integral inequality implies that $\|v(t)\|^2 \leqq e^{2\omega t}\|x-z\|^2$, that is,

$$\|y(t\,;\,x) - y(t\,;\,z)\| \leqq e^{\omega t}\|x-z\|$$

for all x, $z \in D(A)$ and $t \geqq 0$.

If we define $T_0(t)$ by

$$T_0(t)x = y(t\,;\,x) \qquad \text{for } x \in D(A) \text{ and } t \geqq 0,$$

then $T_0(t)x$ is strongly continuous in $t \geqq 0$, $T_0(t)x \in Cl[D(A)]$ and $\|T_0(t)x - T_0(t)y\| \leqq e^{\omega t}\|x-y\|$ for $x, y \in D(A)$ and $t \geqq 0$. So that each $T_0(t)$ has a unique extension $T(t)$ which maps $Cl[D(A)]$ into itself and satisfies the same Lipschitz condition. Moreover, for each $x \in Cl[D(A)]$, $T(t)x$ is strongly continuous in $t \geqq 0$. Finally, the semi-group property follows from the unicity of solution of (w-CP). In fact, let $x \in D(A)$ and set $E_x = \{s \geqq 0\,;\, y(s\,;\,x) \in D(A)\}$. Then from the unicity of solution of (w-CP) we see that $y(t+s\,;\,x) = y(t\,;\,y(s\,;\,x))$ for $t \geqq 0$ and $s \in E_x$, that is, $T(t+s)x = T(t)T(s)x$ for $t \geqq 0$ and $s \in E_x$. Thus we have

$$T(t+s)x = T(t)T(s)x \qquad \text{for } t, s \geqq 0,$$

because $s \in E_x$ for a. a. $s \geqq 0$ and $T(s)x$ is strongly continuous in $s \geqq 0$.

Now, suppose that A is single-valued, and let A' be the weak infinitesimal generator of $\{T(t)\}$. Then from the semi-group property, for each $x \in D(A)$, we have $A'T(t)x = (\text{w-}d/dt)T(t)x = AT(t)x$ for a. a. $t \geqq 0$. Hence we see that for each $x \in D(A)$, there is a sequence $\{t_n\}$, going to 0, such that $AT(t_n)x = A'T(t_n)x$ for all n. Since $T(t_n)x \to x$, if we put $D = \bigcup_{x \in D(A)} \{T(t)x\,;\, AT(t)x = A'T(t)x\}$, then D is dense in $D(A)$. Consequently $Cl[D] = Cl[D(A)]$.

(b) Let $\{T(t)\}$ be a nonlinear semi-group of local type, and let A be its weak infinitesimal generator. Then there is an $\omega \geqq 0$ such that $\|T(t)x - T(t)y\| \leqq e^{\omega t}\|x-y\|$ for $x, y \in X$ and $t \geqq 0$. For each pair $x, y \in D(A)$ we have

$$\mathrm{re} < A_h x - A_h y, f > \leqq h^{-1}(e^{\omega h} - 1)\|x-y\|^2$$

for every $f \in J(x-y)$, where $A_h = h^{-1}(T(h) - I)$; and hence

$$\mathrm{re} < Ax - Ay, f > \leqq \omega \|x-y\|^2$$

for every $f \in J(x-y)$. Thus A satisfies the condition (D).

Let x be an element of $D(A)$. Then $\sup\{\|A_h x\|\,;\, 0 < h \leqq 1\} = K < \infty$ and $\|T(t+h)x - T(t)x\| \leqq Ke^{\omega t}h$ for $t \geqq 0$ and $h \in (0, 1]$. This shows that $T(t)x$ is strongly absolutely continuous on every finite interval. Since X is reflexive, the strong absolute continuity of $T(t)x$ implies that it is strongly differentiable at a. a. t and $(d/dt)T(t)x = A_0 T(t)x$ for a. a. t, where A_0 is the infinitesimal generator of $\{T(t)\}$ (see Kōmura [7; Appendix]). Since $A \supset A_0$, $T(t)x$ is a solution of the (w-CP) for A. The uniqueness follows from the fact that A has the property (D). Q. E. D.

REMARK. 1) Since any solution of (w-CP) is necessarily a solution of (CP), in (a) of the above theorem we may also conclude that A coincides with the infinitesimal generator of $\{T(t)\}$ on a dense subset of $Cl[D(A)]$ if A is single-valued.

2) The above theorem is an extension of the Dorroh theorem [3; Theorem 2.5] which is stated as follows:

If a single-valued operator A satisfies

$$\|(\lambda I-A)x-(\lambda I-A)y\| \geqq \lambda\|x-y\| \text{ for } x, y \in D(A) \text{ and } \lambda>0$$

(this is equivalent to the condition that A is dissipative, by the Kato lemma [5; Lemma 1.1]), and if for each $x \in D(A)$ there is a continuously differentiable funciton $y(t; x)$ from $[0, \infty)$ into X such that $y(0; x)=x$ and $(d/dt)y(t; x)=Ay(t; x)$ for all $t \geqq 0$, then A has an extension which is the infinitesimal generator of a (nonlinear contraction) semi-group of class $(C, Cl[D(A)])'$.

Now, we consider the (CP) for an operator A with dense domain and suppose that there is an approximating scheme $\{C_n\}$ satisfying the following condition

(C′) there is a subset D of $D(A)$ such that

$$\lim_{n\to\infty} h_n^{-1}(C_n-I)x = A_1 x$$

exists for each $x \in D$ and the closure $\overline{A}_1$ of A_1, not necessarily single-valued, coincides with A.

Then in terms of the (CP), we may restate Theorem 1 in the following form.

THEOREM 4. *Let X^* be uniformly convex. Suppose that*

(i′) (C′) *and* (S) *are satisfied,*

(ii′) $R(I-h_0A)=X$ *for some* $h_0 \in (0, 1/M)$.

Then for each $x \in D(A)$ there is a unique solution $y(t; x)$ of the (CP) *and*

$$y(t; x) = \lim_{n\to\infty} C_n^{[t/h_n]}x \tag{2.9}$$

uniformly with respect to t in every finite interval.

PROOF. It follows from $\bar{A}_1 = A$ and $Cl[D(A)] = X$ that D is dense in X. Thus (i′) states that (i) of Theorem 1 is satisfied for the operator A_1. Take any $y \in X$. Then there is an $x \in D(A)$ with $y \in (I - h_0 A)x$, i. e., $(x-y)/h_0 \in Ax$. Since $A = \bar{A}_1$, we may take a sequence $\{x_n\}$ such that $x_n \to x$ and $A_1 x_n \to (x-y)/h_0$, i. e., $(I - h_0 A_1)x_n \to y$. This means that $Cl[R(I - h_0 A_1)] = X$. So that (ii) of Theorem 1 holds true for A_1. Thus by Theorem 1, A generates a nonlinear semi-group $\{T(t)\}$ of local type such that for each $x \in D(A)$, $T(t)x$ is strongly absolutely continuous on every finite interval, $T(t)x \in D(A)$ for all $t \geqq 0$ and $(d/dt)T(t)x \in AT(t)x$ for a. a. $t \geqq 0$. Moreover the approximating scheme is convergent to the semi-group $\{T(t)\}$. Thus if we put $y(t\,;\,x) = T(t)x$ for each $x \in D(A)$, then $y(t\,;\,x)$ is a solution of (CP) and the convergence (2. 9) holds true. Now we note that the operator A satisfies the condition (D). For, by (S), $\mathrm{re} < h_n^{-1}(C_n - I)x - h_n^{-1}(C_n - I)y, J(x-y) > \leqq h_n^{-1}(e^{Mh_n} - 1)\|x-y\|^2$ for $x, y \in D$. In view of (C′), passing to the limit as $n \to \infty$, we have $\mathrm{re} < A_1 x - A_1 y, J(x-y) > \leqq M\|x-y\|^2$ for $x, y \in D$. Now, take any pair $x, y \in D(A)$ and any $x' \in Ax$ and $y' \in Ay$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in D such that $x_n \to x$, $A_1 x_n \to x'$ and $y_n \to y$, $A_1 y_n \to y'$. Since $\mathrm{re} < A_1 x_n - A_1 y_n, J(x_n - y_n) > \leqq M\|x_n - y_n\|^2$ for each n, and since J is strongly continuous, we have

$$\mathrm{re} < x' - y', J(x-y) > \leqq M\|x-y\|^2 .$$

This is nothing else but the condition (D). Consequently, $y(t;\,x) = T(t)x$ ($x \in D(A)$) is the unique solution of (CP) (see the proof of Theorem 3 (a)). Q. E. D.

Finally, by combining Theorem 3 with Theorem 2, we may obtain an analogy of the Lax theorem on the difference approximation of initial value problems [12; p. 45] in the following form.

THEOREM 5. *Let A be a single-valued, densely defined operator, and suppose that for each $x \in D(A)$ there is a solution $u(t\,;\,x)$ of the* (w-CP). *And assume that the approximating scheme* (1. 2) *satisfies* (C) *and* (S). *Then there is a unique nonlinear semi-group $\{T(t)\}$ of local type such that $T(t)x = u(t\,;\,x)$ for each $x \in D(A)$ and $t \geqq 0$, and the approximating scheme is convergent to the semi-group $\{T(t)\}$. In particular, for each $x \in D(A)$,*

$$u(t\,;\,x) = \lim_{n\to\infty} C_n^{[t/h_n]}x$$

uniformly with respect to t in every finite interval.

PROOF. As we see from the proof of Theorem 4, (C) and (S) imply that A satisfies the condition (D) with $\omega = M$. Thus the conclusions follow from Theorems 3 and 2. Q. E. D.

2. 3. The proofs of Theorems 1 and 2 are based on the convergence theorems of nonlinear semi-groups and Lemma 4 in §3. By using the same lemma, we may also deal with the representation of nonlinear semi-groups. For linear semi-group $\{T(t)\}$ of class (C_0), the following exponential formulas are well known (see, for example, Hille and Phillips [4; p. 354]):

For each $x \in X$,

$$T(t)x = \lim_{h \to 0+} \exp(tA_h)x, \tag{2.10}$$

$$T(t)x = \lim_{n \to \infty} \exp[tA(I-n^{-1}A)^{-1}] \cdot x \tag{2.11}$$

$$T(t)x = \lim_{n \to \infty} [I-(t/n)A]^{-n}x \tag{2.12}$$

uniformly with respect to t in any finite interval, where $A_h = h^{-1}(T(h)-I)$ and A is the infinitesimal generator of $\{T(t)\}$.

We shall deal with an analogy of the above formulas for nonlinear semi-groups of local type. In the following, we assume that $\{T(t)\}$ is a nonlinear semi-group of local type with

$$\|T(t)x-T(t)y\| \leqq e^{\omega t}\|x-y\| \qquad \text{for } x, y \in X \text{ and } t \geqq 0 \tag{2.13}$$

where $\omega \geqq 0$ is a constant, and that A' is the weak infinitesimal generator of $\{T(t)\}$. And for shorter statements, we write $C^1([0, \infty); X)$ for the set of all strongly continuously differentiable X-valued functions defined on $[0, \infty)$. First, corresponding to (2.10), we obtain the following

THEOREM 6. *If we put $A_h = h^{-1}(T(h)-I)$ for $h > 0$, then each A_h is the infinitesimal generator of a nonlinear semi-group $\{T(t; A_h)\}$ of local type such that $T(t; A_h)x \in C^1([0, \infty); X)$ for each $x \in X$; and for each $x \in Cl[D(A')]$,*

$$T(t)x = \lim_{h \to 0+} T(t; A_h)x \tag{2.14}$$

uniformly with respect to t in any finite interval.

Next, corresponding to (2.11) and (2.12), we obtain the analogy in the following forms.

THEOREM 7. *Let X^* be uniformly convex, and suppose that $Cl[R(I-h_0A')]=X$ for some $h_0 \in (0,1/\omega)$. Then for each $h \in (0,1/\omega)$, $(I-hA')^{-1}$ exists and it has a unique extension $J(h)$ defined on X such that $\|J(h)x-J(h)y\| \leqq (1-h\omega)^{-1}\|x-y\|$ for $x, y \in X$; and then we have the following*:

(a) *For each $n>\omega$, $A_n=n[J(1/n)-I]$ is the infinitesimal generator of a nonlinear semi-group $\{T(t;A_n)\}$ of local type such that $T(t;A_n)x \in C^1([0,\infty);X)$ for $x \in X$; and for each $x \in Cl[D(A')]$,*

$$T(t)x = \lim_{n\to\infty} T(t;A_n)x \tag{2.15}$$

uniformly with respect to t in any finite interval.

(b) *For each $x \in Cl[D(A')]$,*

$$T(t)x = \lim_{n\to\infty} J(t/n)^n x \tag{2.16}$$

uniformly with respect to t in any finite interval, where we define $J(0)$ by I.

The proofs of Theorems 6 and 7 are given in §5.

REMARK. 1) In the previous paper [8], Miyadera proved that the convergence (2.14) holds true for $x \in Cl[D_0]$ under an additional assumption that "there exists a set D_0 such that $D_0 \subset D(A_0)$ and for any $x \in D_0$, $T(t)x \in D(A_0)$ for a.a. $t \geqq 0$, where A_0 is the infinitesimal generator of $\{T(t)\}$." Also, see Dorroh [3; Theorem 2.9].

2) If, in Theorem 7, we assume "$R(I-h_0A')=X$ for some $h_0 \in (0,1/\omega)$" instead of "$Cl[R(I-h_0A')]=X$", then we have that $J(h)=(I-hA')^{-1}$ for $h \in (0,1/\omega)$ and $A_n = A'(I-n^{-1}A')^{-1}$ for $n>\omega$.

3) Oharu [10] showed that the convergence (2.16) holds true for $x \in Cl[D_0]$, in an arbitrary Banach space, if there exists a set D_0 such that $D_0 \subset D(A_0)$ and for any $x \in D_0$, the strong right-hand derivative $D^+T(t)x$ exists and it is continuous for $t \geqq 0$ and if $Cl[R(I-h_0A_0)]=X$ for some $h_0 \in (0,1/\omega)$, where A_0 is the infinitesimal generator of $\{T(t)\}$. Under the similar conditions, Dorroh [3; Theorems 4.5 and 4.8] has also treated the convergences (2.15) and (2.16).

3. Lemmas. In this section we prove some basic lemmas for the proofs of the theorems stated in the preceding section.

Let C be an operator (not necessarily linear) from a Banach space X into itself satisfying

$$\|Cx-Cy\| \leqq \alpha\|x-y\| \qquad \text{for } x,y \in X, \tag{3.1}$$

where $\alpha \geqq 1$ is a constant independent of x and y.

We start from the following

LEMMA 1. *$C-I$ is the infinitesimal generator of a unique nonlinear semi-group $\{T(t; C-I)\}$ of local type satisfying the following conditions;*

$$\|T(t; C-I)x - T(t; C-I)y\| \leqq e^{(\alpha-1)t} \|x-y\| \tag{3.2}$$

for $x, y \in X$ and $t \geqq 0$,

(3.3) *for each $x \in X$, $T(t; C-I)x \in C^1([0, \infty); X)$ and*

$$(d/dt)T(t; C-I)x = (C-I)T(t; C-I)x \qquad \text{for } t \geqq 0,$$

and

$$g_x(t) = x + \int_0^t e^s C(e^{-s} g_x(s)) ds \qquad \text{for } x \in X \text{ and } t \geqq 0, \tag{3.4}$$

where $g_x(t) = e^t T(t; C-I)x$.

PROOF. It follows from (3.1) that the integral equation (3.4) has a unique solution $g_x(t) \in C^1([0, \infty); X)$ for any $x \in X$, and that

$$\|g_x(t) - g_y(t)\| \leqq \|x-y\| + \alpha \int_0^t \|g_x(s) - g_y(s)\| \, ds$$

for $x, y \in X$ and $t \geqq 0$. This inequality implies that

$$\|g_x(t) - g_y(t)\| \leqq e^{\alpha t} \|x-y\| \qquad \text{for } x, y \in X \text{ and } t \geqq 0.$$

If we define $T(t; C-I)$ by

$$T(t; C-I)x = e^{-t} g_x(t) \qquad \text{for } t \geqq 0 \text{ and } x \in X,$$

then $\{T(t; C-I)\}$ is the desired nonlinear semi-group of operators. Q. E. D.

Setting $T(t; U) = T(h^{-1}t; C-I)$ $(h > 0)$, we have the following

COROLLARY 1. *Let $h > 0$. Then $U = h^{-1}(C-I)$ generates a unique nonlinear semi-group $\{T(t; U)\}$ of local type such that*

(a) *for every $x, y \in X$ and $t \geqq 0$,*

$$\|T(t\,;\,U)x-T(t\,;\,U)y\|\leqq e^{(\alpha-1)t/h}\|x-y\|,$$

(b) *for each* $x\in X$, $T(t\,;\,U)x\in C^1([0,\infty)\,;\,X)$ *and*

$$(d/dt)T(t\,;\,U)x=UT(t\,;\,U)x \qquad \textit{for } t\geqq 0.$$

LEMMA 2. *For each* $x\in X$ *and* $t\geqq 0$,

$$\|g_x(t)-e^t x\|\leqq e^{\alpha t}t\|(C-I)x\|.$$

PROOF. By (3.4),

$$g_x(t)-e^t x=\int_0^t e^s[C(e^{-s}g_x(s))-x]\,ds$$

$$=\int_0^t e^s(C-I)e^{-s}g_x(s)ds+\int_0^t (g_x(s)-e^s x)\,ds.$$

Since $\|(C-I)e^{-s}g_x(s)\| = \|(C-I)T(s\,;\,C-I)x\|\leqq e^{(\alpha-1)s}\|(C-I)x\|$ for all $s\geqq 0$, we have

$$\|g_x(t)-e^t x\|\leqq\left(\int_0^t e^{\alpha s}\,ds\right)\|(C-I)x\|+\int_0^t \|g_x(s)-e^s x\|ds. \tag{3.5}$$

Then it follows from the induction that

$$\|g_x(t)-e^t x\|\leqq\left(\int_0^t \sum_{k=0}^n \frac{(t-s)^k}{k!}e^{\alpha s}\,ds\right)\|(C-I)x\| \tag{3.6}$$

$$+\frac{1}{n!}\int_0^t (t-s)^n\|g_x(s)-e^s x\|\,ds$$

for all $t\geqq 0$ and non-negative intergers n. Passing to the limit as $n\to\infty$ in (3.6), we get

$$\|g_x(t)-e^t x\|\leqq e^t\int_0^t e^{(\alpha-1)s}ds\,\,\|(C-I)x\|$$

$$\leqq e^{\alpha t}t\|(C-I)x\|. \qquad \text{Q. E. D.}$$

LEMMA 3. *Let $x \in X$. For any $t \geqq 0$ and positive integer m,*

$$\|T(t;\, C-I)x - C^m x\| \leqq e^{-t}\alpha^{m-1} \sum_{j=0}^{\infty} \frac{|j-m|\, t^j \alpha^j}{j!} \|(C-I)x\|.$$

PROOF. By (3.4),

$$g_x(t) - e^t C^m x = (x - C^m x) + \int_0^t e^s [C(e^{-s} g_x(s)) - C^m x]\, ds;$$

and hence

$$\|g_x(t) - e^t C^m x\| \leqq \|x - C^m x\| + \alpha \int_0^t \|g_x(s) - e^s C^{m-1} x\|\, ds.$$

Repeating this argument, we arrive at

$$\|g_x(t) - e^t C^m x\| \leqq \sum_{j=0}^{m-1} \frac{t^j \alpha^j}{j!} \|x - C^{m-j} x\| \tag{3.7}$$

$$+ \frac{\alpha^m}{(m-1)!} \int_0^t (t-s)^{m-1} \|g_x(s) - e^s x\|\, ds$$

for $t \geqq 0$. Since $\|x - C^k x\| \leqq \sum_{l=1}^{k} \|C^{l-1} x - C^l x\| \leqq k\alpha^{k-1} \|x - Cx\|$,

we have

$$\sum_{j=0}^{m-1} \frac{t^j \alpha^j}{j!} \|x - C^{m-j} x\| \leqq \alpha^{m-1} \sum_{j=0}^{m-1} \frac{(m-j)\, t^j}{j!} \|x - Cx\| \tag{3.8}$$

$$\leqq \alpha^{m-1} \sum_{j=0}^{m-1} \frac{(m-j)\, t^j \alpha^j}{j!} \|(C-I)x\|.$$

By Lemma 2,

$$\int_0^t (t-s)^{m-1} \|g_x(s) - e^s x\|\, ds \leqq \int_0^t (t-s)^{m-1} s e^{\alpha s} ds\, \|(C-I)x\|$$

$$= \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \int_0^t (t-s)^{m-1} s^{j+1}\, ds\, \|(C-I)x\|$$

$$=(m-1)!\sum_{j=0}^{\infty}\frac{(j+1)\alpha^j t^{m+j+1}}{(m+j+1)!}\|(C-I)x\|,$$

because

$$\int_0^t (t-s)^{m-1}s^{j+1}\,ds = t^{m+j+1}\int_0^1 (1-s)^{m-1}s^{j+1}\,ds$$

$$=t^{m+j+1}\frac{(j+1)!(m-1)!}{(m+j+1)!}.$$

Consequently

$$\frac{\alpha^m}{(m-1)!}\int_0^t (t-s)^{m-1}\|g_x(s)-e^s x\|\,ds \tag{3.9}$$

$$\leqq \alpha^m\sum_{j=0}^{\infty}\frac{(j+1)t^{m+j+1}\alpha^j}{(m+j+1)!}\|C-I)x\|$$

$$\leqq \alpha^{m-1}\sum_{j=m+1}^{\infty}\frac{(j-m)t^j\alpha^j}{j!}\|(C-I)x\|.$$

By (3.7), (3.8) and (3.9), we have

$$\|T(t\,;\,C-I)x-C^m x\| = e^{-t}\|g_x(t)-e^t C^m x\|$$

$$\leqq e^{-t}\alpha^{m-1}\sum_{j=0}^{\infty}\frac{|j-m|t^j\alpha^j}{j!}\|(C-I)x\|. \qquad \text{Q.E.D.}$$

LEMMA 4. *For any $x\in X$ and positive integer m,*

$$\|T(m\,;\,C-I)x-C^m x\|$$

$$\leqq \alpha^m e^{m(\alpha-1)}\{m^2(\alpha-1)^2+m(\alpha-1)+m\}^{1/2}\|(C-I)x\|.$$

PROOF. Putting $t=m$ in Lemma 3,

$$\|T(m\,;\,C-I)x-C^m x\| \leqq \alpha^m e^{-m}\sum_{j=0}^{\infty}\frac{|j-m|m^j\alpha^j}{j!}\|(C-I)x\|.$$

It follows from the Schwarz inequality that

$$\sum_{j=0}^{\infty} \frac{|j-m| m^j \alpha^j}{j!} \leqq e^{m\alpha/2} \left(\sum_{j=0}^{\infty} (j-m)^2 \frac{m^j \alpha^j}{j!} \right)^{1/2}$$

$$= e^{m\alpha} (m^2(\alpha-1)^2 + m(\alpha-1) + m)^{1/2}.$$

This completes the proof. Q. E. D.

In particular, if $\alpha = 1$ in the above lemma, we get the following

COROLLARY 2. *If C is a contraction operator from X into itself (i.e., $\|Cx-Cy\| \leqq \|x-y\|$ for $x, y \in X$), then*

$$\|T(m\,;\, C-I)x - C^m x\| \leqq m^{1/2} \|(C-I)x\|$$

for every $x \in X$ and positive integer m.

In case of linear contraction operators, this corollary has been proved by Chernoff [2]. (Note that $T(m\,;\, C-I) = e^{m(C-I)}$ if C is linear.)

4. Proofs of Theorems 1 and 2.

4.1. Let $\{T^{(n)}(t)\}_{n=1,2,3,\dots}$ be a sequence of nonlinear semi-groups of local type satisfying the stability condition

$$\|T^{(n)}(t)x - T^{(n)}(t)y\| \leqq e^{\omega t} \|x-y\|$$

for $t \geqq 0$, n and x, $y \in X$, where ω is a non-negative constant independent of x, y, t and n. Let $A^{(n)}$ be the infinitesimal generator of $\{T^{(n)}(t)\}$, and suppose that $\lim_{n\to\infty} A^{(n)}x = Ax$ is defined on a set D. The following theorems have been proved by Miyadera ([8; Theorem 2.1] and [9; Theorem 1]).

THEOREM B. *Suppose that*

(i) *A is a restriction of the weak infinitesimal generator of a nonlinear semi-group $\{T(t)\}$ of local type,*

(ii) *there exists a set $D_0 \subset D$ such that for each $x \in D_0$,*

(ii$_1$) *for each n, $T^{(n)}(t)x \in D(A^{(n)})$ for a. a. $t \geqq 0$,*

(ii$_2$) *$T(t)x \in D$ for a. a. $t \geqq 0$.*

Then for each $x \in Cl[D_0]$ we have

$$T(t)x = \lim_{n\to\infty} T^{(n)}(t)x \qquad \textit{for each } t \geqq 0,$$

and the convergence is uniform with respect to t in any finite interval.

THEOREM C. *Let X^* be uniformly convex and assume that for each n, $R(I-\alpha_n A^{(n)}) = X$ for some $\alpha_n \in (0, 1/\omega)$.*
Suppose that

(i) *D is dense in X,*

(ii) *$Cl[R(I-h_0A)] = X$ for some $h_0 \in (0, 1/\omega)$.*

Then we have

(a) *the closure $\bar{A}$, not necessarily single-valued, generates a nonlinear semi-group $\{T(t)\}$ of local type such that for each $x \in D(\bar{A})$, $T(t)x$ is strongly absolutely continuous on every finite interval, $T(t)x \in D(\bar{A})$ for all $t \geqq 0$ and $(d/dt)T(t)x \in \bar{A}T(t)x$ for a. a. $t \geqq 0$;*

(b) *for each $x \in X$,*

$$T(t)x = \lim_{n\to\infty} T^{(n)}(t)x \qquad \textit{for each } t \geqq 0,$$

and the convergence is uniform with respect to t in every finite interval;

(c) *in particular if we assume that $R(I-h_0A) = X$, instead of* (ii), *then $\bar{A} = A$ and A is the weak infinitesimal generator of $\{T(t)\}$ and for each $x \in D$, $T(t)x$ has the weak derivative $AT(t)x$ which is weakly continuous in $t \geqq 0$.*

REMARK. Theorem C remains true even if $A^{(n)}$ is the weak infinitesimal generator of $\{T^{(n)}(t)\}$.

4. 2. We shall now prove the theorems. In both Theorems 1 and 2 we assumed that the approximating scheme satisfies the following conditions;

$$\text{(C)} \qquad \lim_{n\to\infty} h_n^{-1}(C_n - I)x = Ax \qquad \text{for } x \in D(A),$$

$$\text{(S)} \qquad \|C_n x - C_n y\| \leqq e^{Mh_n}\|x-y\| \qquad \text{for } x, y \in X \text{ and } n.$$

Throughout this paragraph we set

$$\cdot\ A_n = h_n^{-1}(C_n - I) \text{ and } M_n = h_n^{-1}(e^{Mh_n}-1),$$

and let M_0 be a constant such that $M_n \leqq M_0$ for n. Then it follows from

Corollary 1 (§3) that each A_n is the infinitesimal generator of a nonlinear semi-group $\{T(t; A_n)\}$ of local type satisfying

$$\|T(t; A_n)x - T(t; A_n)y\| \leqq e^{M_n t}\|x-y\| \leqq e^{M_0 t}\|x-y\| \tag{4.1}$$

for $t \geqq 0$ and $x, y \in X$, and that for each $x \in X$,

$$T(t; A_n)x \in C^1([0, \infty); X) \text{ and } (d/dt)T(t; A_n)x = A_n T(t; A_n)x \tag{4.2}$$

for $t \geqq 0$. For the semi-groups $\{T(t; A_n)\}$ we have the following

LEMMA 5. *Let $t \geqq 0$ and set $k_n = [t/h_n]$. Then for each $x \in X$ and n,*

$$\|T(t; A_n)x - T(k_n h_n; A_n)x\| \leqq e^{M_0 t} h_n \|A_n x\|, \tag{4.3}$$

$$\|T(k_n h_n; A_n)x - C_n^{k_n} x\| \leqq M(t) h_n^{1/2} \|A_n x\|, \tag{4.4}$$

where $M(t) = e^{(M+M_0)t}\{t^2 M_0^2 c_0 + tM_0 c_0 + t\}^{1/2}$ and c_0 is a constant such that $h_n \leqq c_0$ for all n.

PROOF. It follows from (4.1) and (4.2) that

$$T(t; A_n)x - T(k_n h_n; A_n)x = \int_{k_n h_n}^{t} A_n T(s; A_n)x \, ds$$

and $\|A_n T(s; A_n)x\| \leqq e^{M_0 s}\|A_n x\|$ for $s \geqq 0$. Hence we get (4.3). Applying Lemma 4 with $C = C_n$, $\alpha = e^{M h_n}$ and $m = k_n$, we have

$$\begin{aligned}
&\|T(k_n; C_n - I)x - C_n^{k_n} x\| \\
&\qquad \leqq e^{(M+M_n)t}\{t^2 M_n^2 + tM_n + t/h_n\}^{1/2} \|(C_n - I)x\| \\
&\qquad \leqq e^{(M+M_0)t}\{t^2 M_0^2 h_n + tM_0 h_n + t\}^{1/2} h_n^{1/2} \|A_n x\| \\
&\qquad \leqq M(t) h_n^{1/2} \|A_n x\|.
\end{aligned}$$

Since $T(k_n; C_n - I) = T(k_n h_n; A_n)$, we get the desired estimation (4.4).

Q. E. D.

PROOF OF THEOREM 1. Since each A_n is Lipschitz continuous uniformly in $x \in X$, we have that

$$R(I-\alpha_n A_n) = X \qquad \text{for sufficiently small } \alpha_n > 0.$$

Thus if we set $T^{(n)}(t) = T(t\,;A_n)$, $A^{(n)} = A_n$ and $D=D(A)$, then the assumptions of Theorem C are satisfied. Consequently, $\bar{A}$ generates a unique nonlinear semi-group $\{T(t)\}$ of local type with the properties in Theorem C; and for each $x \in X$,

$$T(t)x = \lim_{n\to\infty} T(t\,;A_n)x \tag{4.5}$$

uniformly with respect to t in every finite interval.

Let $x \in D(A)$, $t \geqq 0$ and put $k_n = [t/h_n]$. It follows from Lemma 5 that

$$\begin{aligned}\|T(t)x - C_n{}^{k_n}x\| &\leqq \|T(t)x - T(t\,;A_n)x\| \\ &\quad + \|T(t\,;A_n)x - T(k_nh_n\,;A_n)x\| + \|T(k_nh_n\,;A_n)x - C_n{}^{k_n}x\| \\ &\leqq \|T(t)x - T(t\,;A_n)x\| + (e^{M_0t}h_n + M(t)h_n{}^{1/2})\|A_nx\|\end{aligned}$$

for all n. Thus the condition (C) and the convergence (4.5) yield that

$$T(t)x = \lim_{n\to\infty} C_n{}^{k_n}x \tag{4.6}$$

uniformly with respect to t in any finite interval.

Since $\|C_n{}^{k_n}x - C_n{}^{k_n}y\| \leqq e^{Mt}\|x-y\|$ and $\|T(t)x - T(t)y\| \leqq e^{Mt}\|x-y\|$ for all $x, y \in X$ and $t \geqq 0$, the assumption $Cl[D(A)] = X$ implies that the convergence (4.6) holds true for every $x \in X$. Q. E. D.

PROOF OF THEOREM 2. Let $\{T(t)\}$ be the semi-group in the assumption (E), and set $E_x = \{T(t)x\,; (\text{w-}d/dt)T(t)x = AT(t)x\}$ for $x \in D(A)$ and $D = \bigcup_{x\in D(A)} E_x$. Obviously, $D \subset D(A)$, $\lim_{n\to\infty} A_nx = Ax$ for $x \in D$ (by(C)) and the restriction of A to D is a restriction of the weak infinitesimal generator of $\{T(t)\}$. Thus the condition (i) of Theorem B is satisfied. And the condition (ii) is also satisfied by taking $D_0 = D$. For, (ii$_1$) is obvious from "$D(A_n) = X$", and (E) shows that for each $x \in D(\subset D(A))$, $T(t)x \in D$ for a. a. $t \geqq 0$. Moreover, it follows from (E), $Cl[D(A)] = X$ and the strong continuity of $T(t)$ that D is dense in X. Therefore, by Theorem B, we have that for each $x \in X$,

$$T(t)x = \lim_{n\to\infty} T(t\,;A_n)x$$

uniformly with respect to t in every finite interval. Combining this with Lemma 5, as in the proof of Theorem 1, we obtain that for each $x \in X$,

$$T(t)x = \lim_{n\to\infty} C_n^{[t/h_n]}x$$

uniformly with respect to t in any finite interval. Q. E. D.

5. **Proofs of Theorems 6 and 7.** Let $\{T(t)\}$ be a nonlinear semi-group of local type satisfying

$$\|T(t)x-T(t)y\| \leqq e^{\omega t}\|x-y\| \qquad \text{for } x, y \in X \text{ and } t \geqq 0, \tag{5.1}$$

where $\omega \geqq 0$ is a constant, and let A' be the weak infinitesimal generator of $\{T(t)\}$.

PROOF OF THEOREM 6. Fix an $h>0$. Since $\|T(h)x-T(h)y\| \leqq e^{\omega h}\|x-y\|$ for $x, y \in X$, Corollary 1 (§3) yields that $A_h = h^{-1}(T(h)-I)$ is the infinitesimal generator of a nonlinear semi-group $\{T(t; A_h)\}$ of local type such that

$$\|T(t; A_h)x-T(t; A_h)y\| \leqq e^{\omega(h)t}\|x-y\| \tag{5.2}$$

for $x, y \in X$ and $t \geqq 0$, where $\omega(h) = h^{-1}(e^{\omega h}-1)$, and for each $x \in X$

$$T(t; A_h)x \in C^1([0, \infty); X) \text{ and } (d/dt)T(t; A_h)x = A_h T(t; A_h)x \text{ for } t \geqq 0. \tag{5.3}$$

Let $t \geqq 0$ and put $n_h = [t/h]$. Then by Lemma 4 (with $C = T(h)$, $\alpha = e^{\omega h}$ and $m = n_h$) and $T(hn_h; A_h) = T(n_h; T(h)-I)$, we have

$$\|T(hn_h; A_h)x - T(hn_h)x\| \leqq M(t; h)h^{1/2}\|A_h x\|,$$

where $M(t; h) = e^{(\omega+\omega(h))t}\{t^2\omega(h)^2 h + t\omega(h)h + t\}^{1/2}$. Next, by (5.2) and (5.3), we have

$$\|T(hn_h; A_h)x - T(t; A_h)x\| = \|\int_{hn_h}^{t} A_h T(s; A_h)x\, ds\| \leqq e^{\omega(h)t}h\|A_h x\|.$$

Hence for each $x \in X$,

$$\begin{aligned}\|T(t)x - T(t; A_h)x\| &\leqq \|T(t)x - T(hn_h)x\| + \|T(hn_h)x - T(hn_h; A_h)x\| \\ &\quad + \|T(hn_h; A_h)x - T(t; A_h)x\| \\ &\leqq e^{\omega t}\|T(t-hn_h)x - x\| + M(t; h)h^{1/2}\|A_h x\| + e^{\omega(h)t}h\|A_h x\|.\end{aligned}$$

This shows that for each $x \in D(A')$ the convergence

$$T(t)x = \lim_{h\to 0+} T(t; A_h)x$$

holds true, uniformly with respect to t in every finite interval. Then it follows from (5.1) and (5.2) that the above convergence holds true for $x \in Cl[D(A')]$.

Q. E. D.

PROOF OF THEOREM 7. Since $\mathrm{re}<h^{-1}(T(h)-I)x - h^{-1}(T(h)-I)y, J(x-y)> \leqq h^{-1}(e^{\omega h}-1)\|x-y\|^2$, we have

$$\mathrm{re}<A'x-A'y, J(x-y)> \leqq \omega\|x-y\|^2 \qquad \text{for } x, y \in D(A'),$$

that is, $A'-\omega I$ is dissipative. From this and $Cl[R(I-h_0A')] = X$ we have the following properties (see [9; Lemma 2]):

For each $h \in (0, 1/\omega)$,

(a′) $Cl[R(I-hA')] = X$,

(b′) $(I-hA')^{-1}$ exists and it has a unique extension $J(h)$ defined on X such that

$$\|J(h)x - J(h)y\| \leqq (1-h\omega)^{-1}\|x-y\| \qquad \text{for } x, y \in X, \tag{5.4}$$

(c′) $\bar{A}'-hI$ is dissipative and $J(h) = (I-h\bar{A}')^{-1}$, where $\bar{A}'$ is the closure of A'. (If we assume "$R(I-h_0A') = X$" here, instead of "$Cl[R(I-h_0A')] = X$," then $\bar{A}'$ coincides with A'.)

If we set

$$A_{1/h} = h^{-1}(J(h)-I) \qquad \text{for } h \in (0, 1/\omega),$$

then each $A_{1/h}$ generates a nonlinear semi-group $\{T(t; A_{1/h})\}$ of local type satisfying the following conditions (see Corollary 1);

$$\|T(t; A_{1/h})x - T(t; A_{1/h})y\| \leqq \exp(\omega t/(1-h\omega))\|x-y\| \tag{5.5}$$

for $t \geqq 0$ and $x, y \in X$, and for each $x \in X$

$$T(t; A_{1/h})x \in C^1([0, \infty); X),\ (d/dt)T(t; A_{1/h})x = A_{1/h}T(t; A_{1/h})x \tag{5.6}$$

for $t \geqq 0$.

Since A' has the property (c'), by Kato's generation theorem [6] we see that A' generates a nonlinear semi-group $\{\widehat{T}(t)\}$ of local type defined on $Cl[D(\overline{A}')]$ $(=Cl[D(A')])$ such that

(a'') for each $x \in Cl[D(\overline{A}')]$,

$$\widehat{T}(t)x = \lim_{h\to 0+} T(t; A_{1/h})x$$

uniformly with respect to t in any finite interval,

(b'') for each $x \in D(\overline{A}')$, $\widehat{T}(t)x$ is strongly absolutely continuous on any finite interval and $(d/dt)\widehat{T}(t)x \in \overline{A}'\widehat{T}(t)x$ for a. a. $t \geqq 0$.

Now it is easy to see the convergence (2.15). In fact, the above (b'') shows that for each $x \in D(\overline{A}')$, $\widehat{T}(t)x$ is a solution of (CP) for $\overline{A}'$. On the other hand, for each $x \in D(A')$, $T(t)x$ is a solution of (w-CP) for A' (see Theorem 3(b)) and a fortiori it is a solution of (w-CP) for $\overline{A}'$ because $A' \subset \overline{A}'$. Since $\overline{A}'-hI$ is dissipative, the (w-CP) for $\overline{A}'$ has at most one solution for each initial value (see the proof of Theorem 3(a)). Consequently, $\widehat{T}(t)x = T(t)x$ for $x \in D(A')$ and $t \geqq 0$, and hence $\widehat{T}(t)x = T(t)x$ for $x \in Cl[D(A')]$ and $t \geqq 0$. Thus for each $x \in Cl[D(A')]$,

$$T(t)x = \lim_{h\to 0+} T(t;A_{1/h})x \tag{5.7}$$

uniformly with respect to t in any finite interval.

Next we shall prove the convergence (2.16). We first note that

$$\|A_{1/h}x\| \leqq (1-h\omega)^{-1}\|A'x\| \text{ for } x \in D(A') \text{ and } h \in (0, 1/\omega). \tag{5.8}$$

For, by the property (b'), $\|J(h)x-x\| = \|J(h)x-J(h)(I-hA')x\| \leqq (1-h\omega)^{-1}h\|A'x\|$ for $x \in D(A')$ and $h \in (0, 1/\omega)$. Fix a $t>0$ and let n be a positive integer such that $n \geqq 2\omega t$. We now use Lemma 4 with $C = J(t/n)$, $\alpha = (1-\omega t/n)^{-1}$ and $m=n$. Then, by nothing that $(1-\omega t/n)^{-n} \leqq e^{2\omega t}$ and $n[(1-\omega t/n)^{-1}-1] \leqq 2\omega t$, we have

$$\begin{aligned} &\|T(n;J(t/n)-I)x - J(t/n)^n x\| \\ &\leqq e^{4\omega t}(4\omega^2 t^2+2\omega t+n)^{1/2}\|[J(t/n)-I]x\| \\ &\leqq e^{4\omega t}(4\omega^2 t^2+2\omega t+1)^{1/2} t n^{-1/2}\|A_{t/n}x\| . \end{aligned}$$

Since $T(n; J(t/n)-I) = T(t; A_{t/n})$ and $\|A_{t/n}x\| \leqq (1-\omega t/n)^{-1}\|A'x\| \leqq 2\|A'x\|$ for $x \in D(A')$ (by (5.8)), we have the following estimation

$$\|T(t; A_{t/n})x - J(t/n)^n x\| \leqq 2te^{4\omega t}(4\omega^2 t^2+2\omega t+1)^{1/2} n^{-1/2}\|A'x\| \tag{5.9}$$

for $x \in D(A')$. Now, it follows from (5.7) and (5.9) that for each $x \in D(A')$,

$$T(t)x = \lim_{n\to\infty} J(t/n)^n x \tag{5.10}$$

uniformly with respect to t in any finite interval, where we define $J(0)$ by I. And the convergence (5.10) remains true for each $x \in Cl[D(A')]$, since $\|J(t/n)^n x - J(t/n)^n y\| \leqq (1-\omega t/n)^{-n}\|x-y\|$. This completes the proof. Q. E. D.

APPENDIX

After this paper was submitted for publication we obtained the following which is a generalization of Lemma 4.

LEMMA 4'. *Let X_0 be a closed subset of a Banach space X, and let C be an operator from X_0 into itself satisfying condition* (3.1) *on X_0. If*

$$R(I-\lambda(C-I)) \supset X_0 \qquad \textit{for every } \lambda \in (0, 1/(\alpha-1)), \tag{$*$}$$

then (i) *$C-I$ is the infinitesimal generator of a unique nonlinear semi-group $\{T(t; C-I)\}$ of local type defined on X_0, satisfying conditions* (3.2) *and* (3.3) *on X_0, and* (ii)

$$\|T(m; C-I)x - C^m x\| \leqq \alpha^m e^{m(\alpha-1)}\{m^2(\alpha-1)^2 + m(\alpha-1)+m\}^{1/2}\|(C-I)x\|$$

for any $x \in X_0$ and positive integer m.

Indeed, one can directly prove (i). Setting $g_x(t) = e^t T(t; C-I)x$ for $x \in X_0$ and $t \geqq 0$ we obtain $g_x(t) = x + \int_0^t e^s C(e^{-s} g_x(s))ds$. We then have the conclusion (ii) in the similar way to the proof of Lemma 4.

In particular, if X_0 is a closed convex subset of X, then using the fixed point theorem we see that $(*)$ holds true. Hence we have the following

COROLLARY. *Let X_0 be a closed convex subset of a Banach space X. If C is an operator from X_0 into itself satisfying condition* (3.1) *on X_0, then the conclusions in Lemma 4' hold true.*

This corollary was first obtained by Brezis and Pazy in a Hilbert space. (See Brezis and Pazy "Semigroups of nonlinear contractions on convex sets", to appear). Recently Professor Pazy informed us that they also obtained the same results as ours.

We have discussed in this paper the case of approximating scheme $\{C_n\}$ such that each C_n is defined on X. But in view of Lemma 4′ we may proceed with similar arguments for the case of approximating scheme $\{C_n\}$ such that C_n is defined as an operator from a closed subset of X into itself. Also, as for the operator A which is to be approximated, we may replace conditions for $\overline{A}$ to be an m-dissipative operator by other weaker conditions. For instance our main theorem can be extended as follows:

THEOREM 1′. *Let X^* be uniformly convex. Let $\{C_n\}$ be a sequence of operators such that $D(C_n)$ is closed convex, C_n maps $D(C_n)$ into itself, and the following conditions are satisfied*:

$$\text{(S)} \qquad \|C_n x - C_n y\| \leqq e^{Mh_n}\|x-y\| \qquad \textit{for } x, y \in D(C_n);$$

$$\text{(C)} \qquad \begin{cases} D(C_n) \supset D(A) & \textit{and} \\ \lim_{n\to\infty} h_n^{-1}(C_n - I)x = Ax & \textit{for } x \in D(A); \end{cases}$$

$$(**) \qquad \textit{for } \lambda \in (0, 1/M),\ R(I-\lambda A) \cap \overline{\mathrm{co}(D(A))} \textit{ is dense in } \overline{\mathrm{co}(D(A))}.$$

Then there is a semi-group $\{T(t)\}$ of local type defined on $\overline{D(A)}$ such that

$$T(t)x = \lim_{n\to\infty} C_n^{[t/h_n]}x \qquad \textit{for } x \in \overline{D(A)}.$$

Furthermore, Lemma 4′ turns out to extend our results on the representation to the case of semigroups of local type defined on closed subsets of X.

It is sometimes convenient for applications to extend our results in these forms. For details, we shall publish elsewhere.

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