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THE FAILURE OF INTERIOR-EXTERIOR FACTORIZATION IN THE POLYDISC AND THE BALL

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Let G be a connected open set in complex n-space C^n. Let $H^\infty(G)$ denote the algebra of bounded analytic functions in G, with the supremum norm. It is known that $H^\infty(G)$ is a conjugate Banach space (see [7], Theorem 4.5, or [3]) and so has a weak-star (w^*) topology. In fact, the pre-dual of H^∞ can be taken to be a quotient space of $L^1(G)$ and hence is separable. Thus by a theorem of Banach ([1], Chapitre VIII, Théorème 5) a subspace of H^∞ is w^* closed if and only if it is sequentially w^* closed (that is, contains all limits of sequences).

There is also the strict topology on $H^\infty(G)$ defined by Buck [2]. It is known that the w^* and the strict topologies have the same convergent sequences: a sequence $\{f_n\}$ converges to f if and only if

$$\sup \|f_n\| < \infty \quad \text{and} \quad \lim f_n(z) = f(z), \quad \text{all} \quad z \in G$$

(bounded pointwise convergence). In fact, the strict topology is the strongest topology with precisely these convergent sequences. (See [7], §3, and [6] for these results. The discussion there is for $n = 1$, but the proofs carry over to several variables. See also [5] for a recent survey.)

If $f \in H^\infty$, then $(f) = fH^\infty$ denotes the principal ideal generated by f, and $(f)^-$ denotes its w^* closure.

DEFINITION. A function $f \in H^\infty$ is exterior if (f) is w^* dense in H^∞, and is interior if (f) is w^* closed.

Also, f is a unit if $1/f \in H^\infty$. Thus f is both interior and exterior if and only if f is a unit. Note that an exterior function in a region cannot have any zeros there.

It was shown in [7] that in the unit disc Δ, the exterior functions are precisely the outer functions, while the interior functions are precisely the inner functions multiplied by units. Hence, every bounded analytic

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function in Δ has a factorization, unique up to units, as the product of an interior function and an exterior function.

In this note, we show that the analogous factorization fails, when $n > 1$, in the unit polydisc

$$\Delta^n = \{z_1, \cdots, z_n : |z_1| < 1, \cdots, |z_n| < 1\}$$

and in the unit ball

$$B^n = \{z_1, \cdots, z_n : |z_1|^2 + \cdots + |z_n|^2 < 1\} .$$

We shall show first that no function in the principal ideal (z_1) in the unit ball has an interior-exterior factorization. These results should be compared with [8], Theorem 5.4.8, which is concerned with inner-outer factorizations.

THEOREM 1. *Let $g(z_1, \cdots, z_n) = z_1 f(z_1, \cdots, z_n)$, where $f \in H^\infty(B^n)$, $n > 1$ $f \not\equiv 0$. Then g has no interior-exterior factorization.*

PROOF. Let

$$h_\beta(z_1, \cdots, z_n) = (1 - \beta z_n)^{-1/2},\ |\beta| = 1 .$$

We first show that $||gh_\beta|| \leqq \sqrt{2}\,||f||$. Indeed,

$$|g(z_1, \cdots, z_n)h_\beta(z_1, \cdots, z_n)|^2 \leqq ||f||^2 \frac{|z_1|^2}{|1 - \beta z_n|} \leqq ||f||^2 \frac{1 - |z_n|^2}{1 - |z_n|} \leqq 2||f||^2 .$$

Next, we show that $gh_\beta \in (g)^-$. For let us choose a sequence $r_k \uparrow 1$, $0 < r_k < 1$, and let

$$h_{\beta,k}(z_1, \cdots, z_n) = (1 - r_k \beta z_n)^{-1/2} ,$$

so that $h_{\beta,k} \in H^\infty$. Just as above, $||gh_{\beta,k}|| \leqq \sqrt{2}\,||f||$. Thus $gh_{\beta,k}$ converges pointwise boundedly to gh_β and so $gh_\beta \in (g)^-$.

Now suppose, by way of contradiction, that $g = IE$, where I is interior and E is exterior. Then

$$gH^\infty = I(EH^\infty) \subseteqq IH^\infty ,$$

which is w^* closed, and so $(g)^- \subseteqq IH^\infty$ (actually, we have equality here). Hence, for each β with $|\beta| = 1$, there exists a function $\varphi_\beta \in H^\infty$ such that

$$gh_\beta = I\varphi_\beta .$$

Since $g \not\equiv 0$, we have $I \not\equiv 0$, and thus

$$E(z_1, \cdots, z_n) = (\varphi_\beta / h_\beta)(z_1, \cdots, z_n) = \varphi_\beta(z_1, \cdots, z_n)(1 - \beta z_n)^{1/2} .$$

Hence

$$\lim_{z_n \to \beta} E(0, \cdots, 0, z_n) = 0$$

and so $E(0, \cdots, 0, z_n) \equiv 0$. This is a contradiction, since an exterior function has no zeros. Q.E.D.

We shall next show that in the polydisc Δ^n, $n > 1$, no function in the ideal $(z_1 - z_2)$ has an interior-exterior factorization.

THEOREM 2. *Let $g(z_1, \cdots, z_n) = (z_1 - z_2)f(z_1, \cdots, z_n)$, where $f \in H^\infty(\Delta^n)$, $n > 1$, $f \not\equiv 0$. Then g has no interior-exterior factorization.*

PROOF. if $F \in H^\infty(\Delta^1)$, we let

$$g_F(z_1, \cdots, z_n) = (F(z_1) - F(z_2))f(z_1, \cdots, z_n) .$$

We shall show that $g_F \in (g)^-$. First, assume that F is a polynomial. Then $g_F = pg$, where $p(z_1, \cdots, z_n) = (F(z_1) - F(z_2))/(z_1 - z_2)$ is a polynomial.

Next, for general $F \in H^\infty(\Delta^1)$, let σ_k denote the k-th Cesàro mean of the Taylor series of F. It is well-known that $\|\sigma_k\| \leqq \|F\|$. Thus $(\sigma_k(z_1) - \sigma_k(z_2))f(z_1, \cdots, z_n)$ is uniformly bounded and converges pointwise to g_F and hence $g_F \in (g)^-$.

Now suppose, by way of contradiction, that $g = IE$ is an interior-exterior factorization. Then, as in the proof of Theorem 1, $(g)^- \subseteqq IH^\infty$, and consequently $g_F \in IH^\infty$, say $g_F = I\varphi_F$. From this, we obtain

$$\varphi_F(z_1, \cdots, z_n) = \frac{F(z_1) - F(z_2)}{z_1 - z_2} E(z_1, \cdots, z_n) .$$

Therefore,

$$\varphi_F(\lambda, \lambda, z_3, \cdots, z_n) = F'(\lambda)E(\lambda, \lambda, z_3, \cdots, z_n) \tag{1}$$

for all $\lambda \in \Delta^1$. Fix $z_3, \cdots, z_n$, and also choose a fixed ζ, $|\zeta| = 1$, and then choose F so that $F'(r\zeta) \to \infty$ as $r \to 1-$. From (1) we see that $E(r\zeta, r\zeta, z_3, \cdots, z_n) \to 0$ since the left side is bounded. Since this holds for all ζ with $|\zeta| = 1$, we have

$$E(\lambda, \lambda, z_3, \cdots, z_n) = 0 , \qquad \lambda \in \Delta^1 ,$$

which is impossible since an exterior function can have no zeros. Q.E.D.

In defining the notion of interior function, we used the w^* topology. We now show that we could have used the norm topology instead; that is, a principal ideal in H^∞ is norm closed if and only if it is w^* closed.

THEOREM 3. *Let G be a region in C^n and let $f \in H^\infty(G)$. Then (f) is w^* closed if and only if it is norm closed.*

PROOF. If (f) is w^* closed, then it is norm closed, since the norm topology is stronger.

Conversely, suppose that (f) is norm closed. Then the operator M_f of multiplication by f on $H^\infty(G)$ has a closed range. Since M_f is one: one, it must be bounded below, by the closed graph theorem. To show w^* closure, it is enough to show sequential closure. Suppose that $fg_n \to h$ pointwise boundedly. Since the sequence $\{M_f(g_n)\}$ is bounded, the sequence $\{g_n\}$ must also be bounded. Furthermore, $g_n \to \varphi = h/f$ pointwise in G, off the zeros of f. Thus $\varphi \in H^\infty(G)$, and $h = f\varphi$, which completes the proof.

In conclusion, we list some open problems.

1) Which regions have interior functions other than units? In particular, what about the unit ball B^n, $n > 1$? (P. Malliavin is reported to have shown that there are no non-constant inner functions in B^n for $n > 1$.)

2) Is it true in B^n and $\varDelta^n$ that a function is interior if and only if its radial boundary values are bounded away from zero almost everywhere (that is, the boundary function is invertible in L^∞ of the distinguished boundary)? This is true for $n = 1$.

3) Is every interior function in B^n or $\varDelta^n$ an inner function multiplied by a unit?

4) In which regions do there exist bounded analytic functions with no interior-exterior factorization? In particular, does every bounded region in C^n, for $n > 1$, have this property? Even for $n = 1$, there are such regions. In fact, C. W. Neville, in his forthcoming thesis (University of Illinois) has shown that there are regions $G \subsetneqq C^1$ in which the function $f(z) = z$ has no interior-exterior factorization. In [4], based partly on Neville's work, there is a characterization, in terms of analytic capacity, of those regions for which this happens.

5) Does there exist a region $G \subsetneqq C^1$ and an $f \in H^\infty(G)$ such that no function in the principal ideal (f) admits an interior-exterior factorization?

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