Tôhoku Math. Journ.
25 (1973), 33-52.

EULER-POINCARÉ CHARACTERISTICS AND CURVATURE TENSORS

SHÛKICHI TANNO*)

(Received June 15, 1972)

1. Introduction. Let (M, g) be a Riemannian manifold with positive definite metric tensor $g = (g_{ij})$. By $R = (R^i{}_{jkl})$, $R_1 = (R_{jk})$ and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively. The dimension of M is denoted by m. We denote $(R, R) = R_{ijkl}R^{ijkl}$ and $(R_1, R_1) = R_{jk}R^{jk}$. Some significance of (R, R), (R_1, R_1) and S is explained in [6] by M. Berger or in [7] by M. Berger-P. Gauduchon-E. Mazet in connection with the Gauss-Bonnet theorem or the spectre of Riemannian manifolds.

We define $A(g)$ and $B(g)$ by

$$A(g) = (R, R) - \frac{2}{m-1}(R_1, R_1) \,, \tag{1.1}$$

$$B(g) = (R_1, R_1) - \frac{1}{m}S^2 \,. \tag{1.2}$$

Then we have $A(g) \geqq 0$, and the equality holds on M, $m \geqq 3$, if and only if (M, g) is of constant curvature. $B(g) \geqq 0$ holds, and the equality holds on M, if and only if (M, g) is an Einstein space.

For $m = 2$, $A(g) = B(g) = 0$. (cf. (2.10))

For $m = 3$, $A(g) = 3B(g)$. (cf. (8.3))

For $m \geqq 3$, the best inequality is

$$A(g) - \frac{2m\beta}{(m-1)(m-2)}B(g) \geqq 0 \,, \tag{1.3}$$

where β is a real number; $-\infty < \beta < 1$ (cf. Theorem 5.7). The equality holds (at x) if and only if (M, g) is of constant curvature (at x).

After some preliminaries in §2, we study relations among $A(g)$, $B(g)$, Euler-Poincaré characteristic $\chi(M)$, curvature and curvature tensors, Betti numbers, and real homology spheres.

THEOREM A. *Let (M, g) be a compact orientable Riemannian manifold, $m \geqq 3$. Assume one of the followings:*

(a) *(M, g) has positive scalar curvature S and satisfies*

*) The author is partially supported by the Matsunaga Foundation.

$$2S^2 > (m^2 - m - 2)(R, R)\ ,$$

(b) *(M, g) has Ricci curvature $\geqq \gamma > 0$ and satisfies*

$$4\gamma^2 > (m-1)^2(R, R) + (2-3m)(R_1, R_1) + S^2\ ,$$

(c) *(M, g) has Ricci curvature $\geqq \gamma > 0$ and satisfies*

$$\frac{4}{m-2+\varepsilon}\gamma - \frac{2(m-\varepsilon)}{m(m-1)(m-2+\varepsilon)}S > \left[A(g) - \frac{2m(m-2)+2\varepsilon(2m-\varepsilon)}{(m-1)(m-2+\varepsilon)^2}B(g)\right]^{1/2}$$

where ε is a real number such that $\varepsilon \neq 2-m$,

(d) *$m = 3$, S is positive, and $S^2 > 2(R_1, R_1)$,*

(e) *$m = 4$, S is positive, and $2S^2 > 9(R, R)$,*

(f) *(M, g) is of class 1 or 2 and satisfies*

$$S - m(m-1)[K^i{}_{jk}{}^l K^j{}_{rs}{}^k K^r{}_{uv}{}^s K^u{}_{il}{}^v]^{1/4} > 0\ ,$$

where $(K^i{}_{jkl})$ denotes the concircular curvature tensor.

Then (M, g) is a real homology sphere (cf. *Theorem* 4.2, *Corollary* 5.3, *Theorem* 5.5 *and* (5.12), *Theorem* 8.1, *Theorem* 9.4, *Theorem* 13.2).

By finding sufficient conditions (a) for Ricci curvature to be positive, and (b) for $F(\ ,\ )$ to be positive, we have

THEOREM B. *Let (M, g) be a compact orientable Riemannian manifold, $m \geqq 3$.*

(a) *If S is positive and if*

$$\frac{1}{m-1}S^2 > (R_1, R_1)\ ,$$

then the first Betti number $b_1(M) = 0$ (cf. *Theorem* 6.2).

(b) *If S is positive and if*

$$\frac{(m-p)^2}{m^2-m-2}S^2 > \frac{1}{2}[(p-1)^2(R, R) + (m-4p+2)(R_1, R_1) + S^2]$$

then the p-th Betti number $b_p(M) = 0$, where $2 \leqq p \leqq [m/2]$ (cf. *Theorem* 7.2).

By (1.3) and the Gauss-Bonnet theorem, we have

THEOREM C. *Let (M, g) be a compact orientable Riemannian manifold of 4 dimension. Then the followings hold:*

(a) $$(20 - 8\beta)\int B(g)dM + 192\pi^2\chi(M) \geqq \int S^2 dM\ ,$$

(b) $$\int [S^2 - (5-2\beta)(R, R)]dM \leqq 32(1+2\beta)\pi^2\chi(M)\ ,$$

where β is a real number < 1. The equality holds (in (a) *or in* (b)*), if* *and only if* (M, g) *is of constant curvature* (cf. *Theorem* 9.1, *Theorem* 9.3).

Theorem C (a) is a general form of a result by R. L. Bishop-S. I. Goldberg [8] and A. Avez [3] (which says that if (M, g) is a compact orientable Einstein space of 4 dimension then $S^2 \operatorname{Vol}(M) \leqq 192\pi^2\chi(M)$ holds, where the equality is equivalent to the fact that (M, g) is of constant curvature).

In §10, we give some results on non-existence of Killing vectors and Killing tensors.

The author is grateful to Professor S. Sasaki for the useful criticism.

2. **Preliminaries.** Riemannian manifolds are assumed to be connected and of class C^∞. By ∇ we denote the Riemannian connection defined by g. When we need local coordinate neighbourhood, we use $(U, x^i, i = 1, \cdots, m)$. The Riemannian curvature tensor $R = (R^i{}_{jkl})$ is defined by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ for vector fields X, Y, and Z, and $R^i{}_{jkl}\partial/\partial x^i = R(\partial/\partial x^k, \partial/\partial x^l)\partial/\partial x^j$. The Ricci curvature tensor and the scalar curvature are $R_1 = (R_{jk}) = (R^i{}_{jki})$ and $S = R_{jk}g^{jk}$.

$$\text{(i)} \qquad A(g) = R_{ijkl}R^{ijkl} - \frac{2}{m-1} R_{jk}R^{jk} \geqq 0 ,$$

where the equality holds on M (at x, resp.) if and only if (M, g) is of constant curvature (at x, resp.) for $m \geqq 3$.

$$\text{(ii)} \qquad B(g) = R_{jk}R^{jk} - \frac{1}{m}S^2 \geqq 0 ,$$

where the equality holds on M if and only if (M, g) is an Einstein space.

Proofs of (i) and (ii) are given in [6] by M. Berger (or in M. Berger-P. Gauduchon-E. Mazet [7], p. 74–p. 75). A simple proof of (i) is as follows: $P = (P^i{}_{jkl})$ defined by

$$P^i{}_{jkl} = R^i{}_{jkl} - \frac{1}{m-1}(R_{jk}\delta^i_l - R_{jl}\delta^i_k) \tag{2.1}$$

is called the Weyl's projective curvature tensor. The vanishing of P (at x) is equivalent to the fact that (M, g) is of constant curvature (at x). On the other hand, we have $(P, P) = A(g)$. This proves (i).

The concircular curvature tensor $K = (K^i{}_{jkl})$ is defined by

$$R^{ij}{}_{kl} = \frac{S}{m(m-1)}(\delta^i_l\delta^j_k - \delta^i_k\delta^j_l) + K^{ij}{}_{kl} , \tag{2.2}$$

$K^{ij}{}_{kl} = g^{jr}K^i{}_{rkl}$. K satisfies $K_{ijkl} = K_{klij}$. $K = 0$ holds (at x) if and only if (M, g) is of constant curvature (at x).

Since $P_{ijkl} = g_{ir}P^r{}_{jkl}$ does not necessarily satisfy $P_{ijkl} = P_{klij}$, we consider

$$(2.3)\qquad {}^*P^i{}_{jkl} = R^i{}_{jkl} - \frac{1}{2(m-1)}(R_{jk}\delta^i_l - R_{jl}\delta^i_k + R^i_l g_{jk} - R^i_k g_{jl}) \,.$$

Then we have ${}^*P_{ijkl} = {}^*P_{klij}$.

Let $w = (w^{ij})$ be a skew-symmetric tensor (field). We denote by $W = W_x$ the vector space of all such tensors at a point x. W is of $m(m-1)/2$-dimension. We define linear operators $[R]$, $[K]$, and $[{}^*P]$ of W as follows:

$$(2.4)\qquad [R]\colon w \longrightarrow [R]w = (-R^{ij}{}_{kl}w^{kl}) \,,$$

$$(2.5)\qquad [K]\colon w \longrightarrow [K]w = (K^{ij}{}_{kl}w^{kl}) \,,$$

$$(2.6)\qquad [{}^*P]\colon w \longrightarrow [{}^*P]w = ({}^*P^{ij}{}_{kl}w^{kl}) \,.$$

These operators are symmetric in the sense that

$$([R]w, v) = (-R_{ijkl}w^{kl}v^{ij}) = (w, [R]v) \,,$$

etc. $[R]$ is called the curvature operator (cf. M. Berger [4]). If w is non-zero and decomposable $w = X \wedge Y$, then $([R]w, w)/4|X \wedge Y|^2$ is the sectional curvature for the 2-plane (X, Y). If $[R]$ is positive, i.e., if $([R]w, w) > 0$ for any $w \neq 0$, then every sectional curvature is positive. The converse is not true in general [an example is as follows: Let $CP^n(k)$ be a complex projective space with constant holomorphic sectional curvature k and let $(e_\alpha, e_{\alpha^*} = Je_\alpha)$ be an adapted frame at x. Then we have $R_{ij12} = R_{ij1^*2^*}$, and hence $[R](e_1 \wedge e_2 - e_{1^*} \wedge e_{2^*}) = 0$. ⟨Note: Denote by Ω the fundamental 2-form of $CP^n(k)$; then $[R]\Omega \neq 0$, i.e., $R_{ijkl}J^{kl} = -(n+1)kJ_{ij}$⟩].

If (M, g) is complete and $[R]$ is positive ($\geqq \delta > 0$), then M is compact and the first Betti number $b_1(M) = 0$ (cf. S. B. Myers [17]), and $b_2(M) = 0$ too (M. Berger [4]). Recently, a beautiful result was proved by D. Meyer (for $F(\ ,\ )$, see §7):

LEMMA iii (D. Meyer [16]). *Let (M, g) be a Riemannian manifold. If $[R]$ is positive (negative, resp.), then $F(\ ,\ )$ is positive (negative, resp.). Hence, if (M, g) is compact orientable and if $[R]$ is positive, then the p-th Betti number $b_p(M) = 0$ for $p = 1, \cdots, m-1$. That is (M, g) is a real homology sphere.*

A sufficient condition for positiveness of $[R]$ was given by A. Weinsten [28] as we refer in §13. We give other sufficient conditions for positiveness of $[R]$ in terms of scalars defined by curvature tensors.

LEMMA iv (S. Tanno [22]). *Let H be a symmetric linear operator of a vector space W with inner product. Then, for every integer $k \geqq 1$ and every $w \in W$, we have*

$$(2.7)\qquad (Hw, w)^2 \leqq [\mathrm{trace}\, H^{2k}]^{1/k}(w, w)^2 \,.$$

If k and h are integers with $1 \leqq k < h$, we have

$$[\text{trace } H^{2k}]^{1/k} \geqq [\text{trace } H^{2h}]^{1/h} . \tag{2.8}$$

If rank $(H) \geqq 2$, *the strict inequality holds in* (2.8). *Further*

$$\lim_{k\to\infty}[\text{trace } H^{2k}]^{1/2k} = \|H\| \tag{2.9}$$

where $\|H\|$ denotes the operator norm of H.

For $m = 2$, we have $(R, R) = S^2$ and $(R_1, R_1) = S^2/2$. Hence

$$A(g) = B(g) = 0 . \tag{2.10}$$

3. Scalar inequalities and positive curvature operators. Let w be a skew-symmetric tensor field on (M, g), $w = (w^{ij})$. As an operator on $W = W_x$, we define $[I]$ by $[I] = (\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)/2$. Then (2.2) is written as

$$[R] = \frac{2S}{m(m-1)}[I] - [K] . \tag{3.1}$$

Applying Lemma iv, we have

$$\begin{aligned}([R]w, w) &= \frac{2S}{m(m-1)}(w, w) - ([K]w, w) \\ &\geqq \frac{2S}{m(m-1)}(w, w) - |([K]w, w)| \\ &\geqq \left[\frac{2S}{m(m-1)} - (\text{trace } [K]^{2k})^{1/2k}\right](w, w) .\end{aligned} \tag{3.2}$$

From this we have a sufficient condition for $[R]$ to be positive. Similarly we have a sufficient condition for $[R]$ to be negative. Thus,

THEOREM 3.1. *In a Riemannian manifold (M, g), if*

$$\frac{2|S|}{m(m-1)} > (\text{trace } [K]^{2k})^{1/2k} \tag{3.3}$$

holds at x for some integer $k \geqq 1$, then the curvature operator $[R]$ is positive at x for $S > 0$ and negative at x for $S < 0$.

Consequently, we have by Lemma iii,

THEOREM 3.2. *Let (M, g) be a compact orientable Riemannian manifold with positive scalar curvature S. If* (3.3) *holds on M for some integer $k \geqq 1$, then (M, g) is a real homology sphere.*

For $k = 1$, we have

COROLLARY 3.3. *Let (M, g) be a compact orientable Riemannian manifold. If S is positive and if*

$$(3.4)\qquad 4 > \left[\frac{(R, R)}{S^2} - \frac{2}{m(m-1)}\right] m^2(m-1)^2$$

holds on M, then (M, g) is a real homology sphere.

If we use $A(g)$ and $B(g)$ and if we write the right hand side of (3.4) by $C_2(g)$, we have

$$(3.5)\qquad C_2(g) = \frac{m^2(m-1)^2}{S^2}\left[A(g) + \frac{2}{m-1}B(g)\right].$$

We have $C_2(g) \geqq 0$. The equality holds (at x) if and only if (M, g) is of constant curvature (at x). $C_2(g)$ is invariant by every homothetic deformation $g \rightarrow \sigma g$ (σ: constant), i.e., $C_2(g) = C_2(\sigma g)$. If we define $C_{2k}(g)$ for $S \neq 0$ by

$$(3.6)\qquad C_{2k}(g) = \frac{m^{2k}(m-1)^{2k}}{S^{2k}}(\text{trace } [K]^{2k}),$$

then (3.3) is written as $2^{2k} > C_{2k}(g)$. $C_{2k}(g)$ is also a homothetic invariant.

The case $k = 2$ is also of some interest.

COROLLARY 3.4. *Let (M, g) be a compact orientable Riemannian manifold. If S is positive and if $16 > C_4(g)$, where*

$$(3.7)\qquad \begin{aligned} C_4(g) = m^4(m-1)^4\Big[&\frac{1}{S^4}R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{tu}R^{tu}{}_{ij} \\ &- \frac{8}{m(m-1)S^3}R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{ij} + \frac{24}{m^2(m-1)^2S^2}(R, R) \\ &- \frac{24}{m^3(m-1)^3}\Big], \end{aligned}$$

then (M, g) is a real homology sphere.

PROOF. Put $2S/m(m-1) = L$. Then $[K] = L[I] - [R]$, and hence

$$[K]^4 = L^4[I] - C_1^4L^3[R] + C_2^4L^2[R]^2 - C_3^4L[R]^3 + [R]^4.$$

Using trace $[I] = m(m-1)/2$ and trace $[R] = S$, etc., trace $[K]^4$ is calculated easily.

REMARK. Although (3.7) in Corollary 3.4 is complicated more than (3.4) in Corollary 3.3, Corollary 3.4 is better than Corollary 3.3 as is seen from (2.8) and (3.2).

REMARK. If the maximum of absolute eigenvalues of $[K]$ is given by a positive eigenvalue, then $[R]$ is positive for $S > 0$ at x, if and only if there is an integer $k \geqq 1$ such that (3.3) holds at x, as is seen from (2.9) and (3.2).

4. Scalar inequalities and curvature operators. Theorem 3.1 works for every integer $k \geqq 1$; and the criterion is getting better as $k \to \infty$. In this section we use a method which can be applied only for $k = 1$. Let T be a positive number. For the case $S > 0$, we put

$$[R] = T[I] - (T[I] - [R]) . \tag{4.1}$$

Using Lemma iv, we have

$$\begin{aligned} ([R]w, w) &= T(w, w) - ((T[I] - [R])w, w) \\ &\geqq [T - [\operatorname{trace}(T[I] - [R])^2]^{1/2}](w, w) . \end{aligned} \tag{4.2}$$

Since $\operatorname{trace}(T[I] - [R])^2 = \operatorname{trace}(T^2[I] - 2T[R] + [R]^2)$, if

$$T > \left[\frac{m(m-1)}{2}T^2 - 2ST + (R, R)\right]^{1/2} \tag{4.3}$$

holds, then $[R]$ is positive for $S > 0$. For this, it suffices that

$$(m^2 - m - 2)T^2 - 4ST + 2(R, R) < 0$$

has a positive solution T. Consequently, for $m \geqq 3$,

$$D = 4[4S^2 - 2(m^2 - m - 2)(R, R)] > 0$$

is sufficient. The case $S < 0$ is also studied by putting

$$[R] = -T[I] + (T[I] + [R]) ,$$

and we have

THEOREM 4.1. *In a Riemannian manifold (M, g), $m \geqq 3$, if*

$$\frac{2}{m^2 - m - 2}S^2 > (R, R) \tag{4.4}$$

holds at x, then $[R]$ is positive at x for $S > 0$ and negative at x for $S < 0$.

THEOREM 4.2. *If a compact orientable Riemannian manifold (M, g), $m \geqq 3$, has positive scalar curvature S and satisfies (4.4) on M, then (M, g) is a real homology sphere.*

REMARK. Theorem 4.2 is better than Corollary 3.3. Because (3.4) is equivalent to $2(m^2 - m + 2)S^2/m^2(m-1)^2 > (R, R)$, and we have an inequality $2/(m^2 - m - 2) > 2(m^2 - m + 2)/m^2(m-1)^2$.

5. Ricci curvatures and positive curvature operators. Assuming that (M, g) is of positive Ricci curvature, as usual we define a scalar field γ on M by

$$\gamma(x) = \text{the minimum of Ricci curvatures at } x . \tag{5.1}$$

Then we have $R_{rs}w^{ri}w^{s}{}_{i} \geqq \gamma(w, w)$. (2.3) gives

$$(5.2)\qquad ([{}^*P]w,\ w) \geqq -\ ([R]w,\ w) + \frac{2}{m-1}\gamma(w,\ w)\ .$$

Similarly as in §3, we have

$$(5.3)\qquad \begin{aligned} ([R]w,\ w) &\geqq \frac{2}{m-1}\gamma(w,\ w) - |([{}^*P]w,\ w)| \\ &\geqq \left[\frac{2}{m-1}\gamma - (\text{trace}\ [{}^*P]^{2k})^{1/2k}\right](w,\ w)\ . \end{aligned}$$

Therefore, we get

THEOREM 5.1. *Let (M, g) be a Riemannian manifold with Ricci curvature $\geqq \gamma > 0$. If*

$$(5.4)\qquad \frac{2}{m-1}\gamma > (\text{trace}\ [{}^*P]^{2k})^{1/2k}$$

holds at x for some integer $k \geqq 1$, then $[R]$ is positive at x.

By Theorem 5.1 and Lemma iii, we have

THEOREM 5.2. *Let (M, g) be a compact orientable Riemannian manifold with Ricci curvature $\geqq \gamma > 0$. If (5.4) holds on M for some integer $k \geqq 1$, then (M, g) is a real homology sphere.*

For $k = 1$, we have

$$(5.5)\qquad \begin{aligned} \text{trace}\ [{}^*P]^2 &= R_{ijkl}R^{ijkl} + \frac{2-3m}{(m-1)^2}R_{jk}R^{jk} + \frac{1}{(m-1)^2}S^2 \\ &= A(g) - \frac{m}{(m-1)^2}B(g) \geqq 0\ . \end{aligned}$$

COROLLARY 5.3. *Let (M, g) be a compact orientable Riemannian manifold with Ricci curvature $\geqq \gamma > 0$. If*

$$(5.6)\qquad \frac{4}{(m-1)^2}\gamma^2 > (R,\ R) + \frac{2-3m}{(m-1)^2}(R_1,\ R_1) + \frac{1}{(m-1)^2}S^2$$

holds on M, then (M, g) is a real homology sphere.

COROLLARY 5.4. *Let (M, g) be a compact orientable Riemannian manifold with sectional curvature $\geqq \delta > 0$. If*

$$(5.7)\qquad 4(m-1)^2\delta^2 > (m-1)^2(R,\ R) + (2-3m)(R_1,\ R_1) + S^2$$

holds on M, then (M, g) is a real homology sphere.

PROOF. This follows from Corollary 5.3 and $\gamma \geqq (m-1)\delta$.

Next we consider ${}^{\varepsilon}C$ for $\varepsilon \neq 2-m$ defined by

$$(5.8)\qquad {}^{\varepsilon}C_{ijkl} = R_{ijkl} - \frac{1}{m-2+\varepsilon}(R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik}) + \frac{(m-\varepsilon)S}{m(m-1)(m-2+\varepsilon)}(g_{jk}g_{il} - g_{jl}g_{ik}) .$$

If $\varepsilon = 0$, $({}^{0}C^{i}{}_{jkl}) = (C^{i}{}_{jkl})$ is the Weyl's conformal curvature tensor. We have ${}^{\varepsilon}C_{ijkl} = {}^{\varepsilon}C_{klij}$. We define $[{}^{\varepsilon}C]w = ({}^{\varepsilon}C^{ij}{}_{kl}w^{kl})$. Then (5.8) gives

$$(5.9)\qquad [{}^{\varepsilon}C]w = -[R]w - \frac{2}{m-2+\varepsilon}(R^{i}{}_{r}w^{jr} - R^{j}{}_{r}w^{ir}) - \frac{2(m-\varepsilon)S}{m(m-1)(m-2+\varepsilon)}w .$$

By an inequality immediately after (5.1), we have

$$(5.10)\qquad ([R]w, w) \geqq \frac{4}{m-2+\varepsilon}\gamma(w, w) - \frac{2(m-\varepsilon)S}{m(m-1)(m-2+\varepsilon)}(w, w) - |([{}^{\varepsilon}C]w, w)| ,$$

from which we have

THEOREM 5.5. *Let (M, g) be a Riemannian manifold with Ricci curvature $\geqq \gamma > 0$. If*

$$(5.11)\qquad \frac{4}{m-2+\varepsilon}\gamma - \frac{2(m-\varepsilon)}{m(m-1)(m-2+\varepsilon)}S > (\mathrm{trace}\,[{}^{\varepsilon}C]^{2k})^{1/2k}$$

holds at x for some integer $k \geqq 1$ and for some $\varepsilon \neq 2-m$, then $[R]$ is positive at x.

THEOREM 5.6. *Let (M, g) be a compact orientable Riemannian manifold with Ricci curvature $\geqq \gamma > 0$. If* (5.11) *holds on M for some $k \geqq 1$ and for some $\varepsilon \neq 2-m$, then (M, g) is a real homology sphere.*

If ${}^{\varepsilon}C_{ijkl} = 0$ holds at x for $\varepsilon \neq 0$ and $\varepsilon \neq 2-m$, then (M, g) is of constant curvature at x. The converse is also true.

By a calculation we have

$$(5.12)\qquad \mathrm{trace}\,[{}^{\varepsilon}C]^2 = A(g) - \frac{2m(m-2) + 2\varepsilon(2m-\varepsilon)}{(m-1)(m-2+\varepsilon)^2}B(g) \geqq 0 .$$

Since we have for $m \geqq 3$

$$(5.13)\qquad \frac{2m(m-2) + 2\varepsilon(2m-\varepsilon)}{(m-1)(m-2+\varepsilon)^2} < \frac{2m}{(m-1)(m-2)}$$

and since the left hand side of (5.13) converges to the right hand side as $\varepsilon \to 0$, we have

THEOREM 5.7. *In a Riemannian manifold (M, g) the following inequality*

$$A(g) - \frac{2m\beta}{(m-1)(m-2)}B(g) \geqq 0\,, \qquad \text{i.e.,} \tag{5.14}$$

$$(R, R) - \frac{2m + 2m\beta - 4}{(m-1)(m-2)}(R_1, R_1) + \frac{2\beta}{(m-1)(m-2)}S^2 \geqq 0 \tag{5.14$'$}$$

holds, where β is a real number; $-\infty < \beta < 1$. The equality holds at x, if and only if (M, g) is of constant curvature at x.

PROOF. By $\beta < 1$ and by (5.13), we have some ε ($\neq 0$) such that is (5.14) written as

$$\text{trace}\,[{}^{\varepsilon}C]^2 + (\text{positive number})B(g) \geqq 0\,.$$

The equality implies that $B(g) = 0$ and trace $[{}^{\varepsilon}C]^2 = 0$.

REMARK. In Theorem 5.7, the number $2m/(m-1)(m-2)$ and the relation $\beta < 1$ are the best possible. In fact, (M, g) is conformally flat if and only if $A(g) - 2mB(g)/(m-1)(m-2) = 0$ for $m \geqq 4$, and conformally flat Riemannian manifolds are not necessarily of constant curvature (with respect to (5.14), cf. S. Tanno [24]).

6. Ricci curvature tensors and Ricci curvatures. We give sufficient conditions for Ricci curvature to be positive or negative in terms of (R_1, R_1) and S.

THEOREM 6.1. *Let (M, g) be a Riemannian manifold.*

(a) *$(R_1, R_1) \geqq (1/m)S^2$ holds always.*

(b) *If the inequality*

$$\frac{1}{m-1}S^2 > (R_1, R_1) \geqq \frac{1}{m}S^2 \tag{6.1}$$

holds at x, then Ricci curvature is positive at x for $S > 0$ and negative at x for $S < 0$. Furthermore:

(c) *The positive minimum value γ or the negative maximum value γ satisfies the following relation*

$$|\gamma| \geqq \frac{1}{m}[|S - [m(m-1)(R_1, R_1) - (m-1)S^2]^{1/2}]\,. \tag{6.2}$$

PROOF. We write $R_1(X, Y) = R_{jk}X^jY^k$. (a) is (ii). We prove (b). First we assume $S > 0$. Let T be a positive number, and let X be an arbitrary non-zero vector. By Lemma iv we have

$$\begin{aligned} R_1(X, X) &= T(X, X) - (T(X, X) - R_1(X, X)) \\ &\geqq T(X, X) - [(Tg_{ij} - R_{ij})(Tg^{ij} - R^{ij})]^{1/2}(X, X) \\ &= [T - (mT^2 - 2ST + (R_1R_1))^{1/2}](X, X)\,. \end{aligned} \tag{6.3}$$

If $T > [mT^2 - 2ST + (R_1, R_1)]^{1/2}$, i.e.,

$$(m-1)T^2 - 2ST + (R_1, R_1) < 0\,, \tag{6.4}$$

we have $R_1(X, X) > 0$. (6.4) has a solution T, if and only if

$$D = 4S^2 - 4(m-1)(R_1, R_1) > 0\,.$$

This solution T is positive by (6.4) and $S > 0$. This proves (b) for $S > 0$. The case $S < 0$ is similarly proved by considering

$$R_1(X, X) = -T(X, X) + (T(X, X) + R_1(X, X))\,.$$

Next we show (c). Assume $S > 0$. We put

$$T - [mT^2 - 2ST + (R_1, R_1)]^{1/2} \geqq Q\,. \tag{6.5}$$

We solve T in the positive range of Q. (6.5) is written as

$$(m-1)T^2 - 2(S-Q)T + (R_1, R_1) - Q^2 \leqq 0\,. \tag{6.6}$$

(6.6) has a solution T if and only if

$$D = 4(S-Q)^2 - 4(m-1)[(R_1, R_1) - Q^2] \geqq 0\,. \tag{6.7}$$

We put $4f(Q) = D$, i.e.,

$$f(Q) = mQ^2 - 2SQ + S^2 - (m-1)(R_1, R_1)\,. \tag{6.8}$$

Since $(R_1, R_1) < S^2/(m-1)$, we have $f(0) > 0$ and $df/dQ_{|Q=0} = -2S < 0$. Let Q_0 be the smaller solution of $f(Q) = 0$. Then

$$mQ_0 = S - [m(m-1)(R_1, R_1) - (m-1)S^2]^{1/2}\,. \tag{6.9}$$

Since $Q_0 = \mathrm{Max}\{Q\colon \text{satisfying (6.5)},\ T > 0\}$, we have (6.2) for the case $S > 0$. The case $S < 0$ is similar.

THEOREM 6.2. *Let (M, g) be a compact orientable Riemannian manifold with non-negative scalar curvature S. If*

$$\frac{1}{m-1}S^2 \geqq (R_1, R_1) \geqq \frac{1}{m}S^2 \tag{6.10}$$

holds on M, then $b_1(M) \leqq m$. If the strict inequality holds somewhere, then $b_1(M) = 0$.

REMARK. Theorem 6.2 is better than a result by Y. Tomonaga [26] (or Corollary 3.2 in S. Tanno [22]). In fact,

$$\frac{S}{m} - \left[(R_1, R_1) - \frac{S^2}{m}\right]^{1/2} \geqq 0 \tag{6.11}$$

is equivalent to $(m+1)S^2/m^2 \geqq (R_1, R_1)$. Hence, $1/(m-1) > (m+1)/m^2$ implies that Theorem 6.2 is better.

7. Positivity and negativity of $F(w, w)$. Let $w = (w_{ijr\dots s})$ be a skew symmetric tensor. The classical $F(w, w) = {}^pF(w, w)$ is defined to be

$$F(w, w) = \left(\frac{p-1}{2} R_{ijkl} + R_{ik}g_{jl}\right) w^{ijr\dots s} w^{kl}{}_{r\dots s} \tag{7.1}$$

(cf. for example, K. Yano–S. Bochner [29]). We define $F = {}^pF$ by

$$F_{ijkl} = \frac{p-1}{2} R_{ijkl} + \frac{1}{4}(R_{ik}g_{jl} - R_{jk}g_{il} + R_{jl}g_{ik} - R_{il}g_{jk}) . \tag{7.2}$$

Then $F_{ijkl} = F_{klij}$ holds, and $[F](= [{}^pF])$ is defined by

$$[F]w = (F^{ij}{}_{kl} w^{klr\dots s})$$

for $w = (w^{ijr\dots s})$. Lemma iv is valid also for $[F]$ and w under the following notations:

$$([F]w, w) = F^{ij}{}_{kl} w^{klr\dots s} w_{ijr\dots s} , \qquad \text{etc.} \tag{7.3}$$

By (7.1), (7.2) and (7.3) we have $([F]w, w) = F(w, w)$. In [22] and [23] we studied relations between scalar inequalities and sign of $F(w, w)$. Here we apply another method. For a positive number T, we put $[F] = T[I] - (T[I] - [F])$. Then we have

$$\begin{aligned} ([F]w, w) &= T(w, w) - ((T[I] - [F])w, w) \\ &\geqq [T - (\operatorname{trace}(T[I] - [F])^2)^{1/2}](w, w) . \end{aligned} \tag{7.4}$$

Since $\operatorname{trace}(T[I] - [F])^2 = T^2 m(m-1)/2 - 2T \operatorname{trace}[F] + (F, F)$, and

$$\operatorname{trace}[F] = F^{ij}{}_{ij} = \frac{m-p}{2} S ,$$

$$(F, F) = \frac{(p-1)^2}{4}(R, R) + \frac{m-2-4(p-1)}{4}(R_1, R_1) + \frac{1}{4} S^2 ,$$

if inequality

$$4T^2 > 2m(m-1)T^2 - 4(m-p)ST + (p-1)^2(R, R) + (m-4p+2)(R_1, R_1) + S^2$$

has a positive solution T, then $F(w, w) > 0$. Similarly as in §4, the condition for the existence of such T is calculated and we have

THEOREM 7.1. *In a Riemannian manifold (M, g), if*

$$\frac{(m-p)^2}{m^2-m-2} S^2 > \frac{1}{2}[(p-1)^2(R, R) + (m-4p+2)(R_1, R_1) + S^2] \tag{7.5}$$

holds at x, then $F(,)$ is positive definite at x for $S > 0$, and negative definite at x for $S < 0$, where $2 \leqq p \leqq m-1$.

By a well known theorem on harmonic forms (cf. S. Bochner [10], K. Yano and S. Bochner [29]), we have

THEOREM 7.2. *If a compact orientable Riemannian manifold* (M, g) *has positive scalar curvature and if* (7.5) *holds on* M, *then the p-th Betti number* $b_p(M) = 0$, *where* $2 \leqq p \leqq [m/2]$.

THEOREM 7.3. *In a Riemannian manifold* (M, g), *if* $m - 4p + 2 < 0$ *and if*

$$\left[\frac{2(m-p)^2}{m^2 - m - 2} - \frac{m - 4p + 2}{m} - 1\right]S^2 > (p-1)^2(R, R) \tag{7.6}$$

holds at x, *then* $F(\ ,\ )$ *is positive definite at* x *for* $S > 0$ *and negative definite at* x *for* $S < 0$, *where* $2 \leqq p \leqq m - 1$.

PROOF. This follows from (7.5) and (ii).

8. 3-dimensional Riemannian manifold. Since the Riemannian curvature tensor R of a 3-dimensional Riemannian manifold is given by

$$R^i{}_{jkl} = \delta^i_l R_{jk} - \delta^i_k R_{jl} + R^i_l g_{jk} - R^i_k g_{jl} - \frac{S}{2}(\delta^i_l g_{jk} - \delta^i_k g_{jl}) \,, \tag{8.1}$$

we have

$$(R, R) - 4(R_1, R_1) + S^2 = 0 \,. \tag{8.2}$$

Therefore $A(g)$ and $B(g)$ are dependent, and we have

$$A(g) = 3B(g) \,. \tag{8.3}$$

By (8.2), (4.4) is written as

$$2S^2 > 4(R, R) = 16(R_1, R_1) - 4S^2 \,.$$

Hence, a compact orientable Riemannian manifold, $m = 3$, satisfying $S > 0$ and $3S^2 > 8(R_1, R_1)$ on M is a real homology sphere by Theorem 4.2. However, applying Theorem 6.2, we have

THEOREM 8.1. *Let* (M, g) *be a compact orientable Riemannian manifold of* 3-*dimension. If the scalar curvature* S *is positive and* $S^2 > 2(R_1, R_1)$ *holds on* M, *then* (M, g) *is a real homology sphere.*

PROOF. By $b_1(M) = 0$ and by the duality, we have $b_2(M) = 0$.

9. 4-dimensional Riemannian manifolds. Let (M, g) be a 4-dimensional compact orientable Riemannian manifold. Denote by $\chi(M)$ the Euler-Poincaré characteristic of M. Then the Gauss-Bonnet formula is given by

$$\int [(R, R) - 4(R_1, R_1) + S^2] dM = 32\pi^2 \chi(M) \,, \tag{9.1}$$

where dM denotes the volume element of (M, g) (cf. for example, M.

Berger [6]). By (i) and (ii), the followings are known:

(v) If (M, g) is an Einstein space, $\int(R, R)dM = 32\pi^2\chi(M) \geqq 0$.

(vi) $\int(R, R)dM \geqq 32\pi^2\chi(M)$ holds. The equality holds if and only if (M, g) is an Einstein space (A. Avez [2]).

(vii) If (M, g) is an Einstein space, we have

$$S^2 \operatorname{Vol}(M) \leqq 192\,\pi^2\chi(M)\,.$$

The equality holds if and only if (M, g) is of constant curvature (R. L. Bishop and S. I. Goldberg [8], A. Avez [3]).

The best result including (vii) is as follows:

THEOREM 9.1. *In a compact orientable Riemannian manifold (M, g) of 4-dimension, we have*

$$(9.2) \qquad (20 - 8\beta)\int B(g)dM + 192\pi^2\chi(M) \geqq \int S^2 dM$$

for every constant β; $-\infty < \beta < 1$. The equality holds if and only if (M, g) is of constant curvature.

PROOF. The integration (5.14)$'$ gives

$$(9.3) \qquad \int[6(R, R) - (4 + 8\beta)(R_1, R_1) + 2\beta S^2]dM \geqq 0\,.$$

Eliminating (R, R) from (9.1) and (9.3), we have

$$(9.4) \qquad \int[(6 - 2\beta)S^2 - (20 - 8\beta)(R_1, R_1)]dM \leqq 192\pi^2\chi(M)\,.$$

By $B(g) = (R_1, R_1) - (1/4)S^2$, we have (9.2). q.e.d.

By (9.4) with $\beta \doteqdot 1$, if $S^2 > 3(R_1, R_1)$, we have $\chi(M) > 0$. Generally we have

THEOREM 9.2. *In a 4-dimensional compact orientable Riemannian manifold (M, g), if $S > 0$ and $S^2 > 3(R_1, R_1)$ hold on M, then $\chi(M) \geqq 2$.*

PROOF. By Theorem 6.2 and $S^2 > 3(R_1, R_1)$, we have $b_1(M) = 0$. By the duality we have $b_3(M) = 0$. Hence, $\chi(M) = b_0(M) + b_2(M) + b_4(M) \geqq 2$.

THEOREM 9.3. *In a compact orientable Riemannian manifold (M, g) of 4-dimension, we have*

$$(9.5) \qquad \int[S^2 - (5 - 2\beta)(R, R)]dM \leqq 32(1 + 2\beta)\pi^2\chi(M)$$

for every constant β; $-\infty < \beta < 1$. The equality holds if and only if (M, g) is of constant curvature.

PROOF. Eliminating (R_1, R_1) from (9.1) and (9.3), we have (9.5).

By Theorem 9.3 with $\beta \doteqdot 1$, if $(R, R) < (1/3)S^2$, then we have $\chi(M) > 0$. Generally we have

THEOREM 9.4. *In a 4-dimensional compact orientable Riemannian manifold (M, g), if S is positive and if*

$$9(R, R) < 2S^2 \tag{9.6}$$

holds on M, then (M, g) is a real homology sphere.

PROOF. By $A(g) = (R, R) - (2/3)(R_1, R_1) \geqq 0$, $(2/9)S^2 > (R, R)$ implies $(1/3)S^2 > (R_1, R_1)$. Then, Theorem 6.2 shows that $b_1(M) = 0$. By Theorem 7.3 for $m = 4$ and $p = 2$, $(3/10)S^2 > (R, R)$ implies that $b_2(M) = 0$. Next, by the duality, we have Theorem 9.4.

REMARK. Theorem 9.4 is better than Theorem 4.2 for $m = 4$.

10. Killing vectors and Killing tensors. By Theorem 6.1 (and its proof), we have

THEOREM 10.1. *Let (M, g) be a compact orientable Riemannian manifold. Assume that S is non-positive and*

$$S^2 \geqq (m-1)(R_1, R_1)\ . \tag{10.1}$$

Then, Killing vectors are parallel. If the strict inequality holds somewhere, there is no non-zero Killing vector.

REMARK. This result is better than Corollary 2.2 in [23]. Cf. Remark in §6.

By Theorem 7.1, we have

THEOREM 10.2. *Let (M, g) be a compact orientable Riemannian manifold with negative scalar curvature S. If, for p $(2 \leqq p \leqq m-1)$, (7.5) holds on M, there is no non-zero Killing tensor of order p.*

By Theorem 4.1 and Lemma iii, we have

THEOREM 10.3. *If a compact orientable Riemannian manifold (M, g) has negative scalar curvature S and if (4.4) holds on M, then there is no non-zero Killing tensor of order p, $1 \leqq p \leqq m-1$.*

REMARK. We have also results corresponding to Theorems 3.1, 5.1.

11. Positive sectional curvature, I. Let (X, Y) be an arbitrary orthonormal pair. By (2.2) and Lemma iv, we have

$$R_{ijkl}X^jX^kY^iY^l = \frac{S}{m(m-1)} + K_{ijkl}X^jX^kY^iY^l \tag{11.1}$$

$$\geqq \frac{S}{m(m-1)} - (K_{ijkl}X^jX^kK^{irsl}X_rX_s)^{1/2} ,$$

since $K_{ijkl}X^jX^k$ is symmetric in i and l. Next, if we put $N_{jkrs} = K^i{}_{jk}{}^l K_{irsl}$, then we have $N_{jkrs} = N_{rsjk}$. By Lemma iv, we have

$$N_{jkrs}(X^jX^k)(X^rX^s) \leqq (N_{jkrs}N^{jkrs})^{1/2}(X^rX_r)(X^sX_s) , \tag{11.2}$$

where the vector space W is of (2, 0)-tensors. Hence, (11.1) and (11.2) show that if

$$\frac{S}{m(m-1)} - (K^i{}_{jk}{}^l K_{irsl}K^{ujkv}K_u{}^{rs}{}_v)^{1/4} > 0 , \tag{11.3}$$

then the sectional curvature is positive. Thus,

THEOREM 11.1. *Let (M, g) be a Riemannian manifold. If* (11.3) *holds at x, then sectional curvatures are positive at x.*

REMARK. The explicit form of (11.3) is as follows:

$$\begin{aligned} &\Big[R_{ijkl}R^i{}_{rs}{}^l R^{ujkv}R_u{}^{rs}{}_v + \frac{4S}{m(m-1)}R_{ijkl}R^i{}_{rs}{}^l R^{sjkr} - \frac{4S}{m(m-1)}R_{ijkl}R^{il}R^{jk} \\ &\quad + \frac{6S^2}{m^2(m-1)^2}R_{ijkl}R^{ijkl} + \frac{4(m-3)S^2}{m^2(m-1)^2}R_{jk}R^{jk} - \frac{m^2-7m+12}{m^3(m-1)^3}S^4\Big]^{1/4} \\ &\quad < \frac{S}{m(m-1)} . \end{aligned} \tag{11.3}'$$

The upper bound of $R_{ijkl}X^jX^kY^iY^l$ is similarly calculated:

$$R_{ijkl}X^jX^kY^iY^l \leqq \frac{S}{m(m-1)} + (K^i{}_{jkl}K_{irs}{}^l K^{ujkv}K_u{}^{rs}{}_v)^{1/4} . \tag{11.4}$$

By the well known sphere theorem (cf. W. Klingenberg [14], D. Gromoll-W. Klingenberg-W. Meyer [12]), we have

THEOREM 11.2. *Let (M, g) be a complete and simply connected Riemannian manifold. If*

$$\frac{3S}{m(m-1)} - 5(K^i{}_{jk}{}^l K^j{}_{rs}{}^k K^r{}_{uv}{}^s K^u{}_{il}{}^v)^{1/4} > 0 \tag{11.5}$$

holds on M, then (M, g) is homeomorphic to a sphere.

PROOF. The condition $\Delta \leqq R_{ijkl}X^jX^kY^iY^l \leqq 4\Delta$ for some positive Δ is calculated by (11.1), (11.2) and (11.4), getting (11.5).

REMARK. We can also apply differentiable pinching theorems (for example, see M. Sugimoto-K. Shiohama [21]).

12. Positive sectional curvature, II. Let (e_i) be an orthonormal basis at x. We put $g(R(e_m, e_b)e_m, e_a) = R_{ammb}$, where $a, b = 1, \cdots, m-1$. By V we denote the $(m-1)$-dimensional subspace of the tangent space at x defined by $(e_1, \cdots, e_{m-1})$. Let Y be a unit vector in V, $Y = (Y^a)$. Let T be a positive number.

$$\begin{aligned}(12.1)\qquad R_{ammb}Y^aY^b &= T(Y, Y) - (T\delta_{ab} - R_{ammb})Y^aY^b \\ &\geqq T - [(T\delta_{ab} - R_{ammb})(T\delta^{ab} - R^{ammb})]^{1/2} .\end{aligned}$$

Since $\delta_{ab}\delta^{ab} = m-1$, $\delta_{ab}R^{ammb} = \delta_{ij}R^{immj} = R_{mm}$ and $R_{ammb}R^{ammb} = R_{immj}R^{immj}$, we see that if

$$(12.2)\qquad T > [(m-1)T^2 - 2R_{mm}T + R_{immj}R^{immj}]^{1/2} ,$$

the sectional curvature for each 2-plane which contains e_m is positive. A sufficient condition for the existence of T satisfying (12.2) is

$$(R_{mm})^2 - (m-2)\sum_{i,j} R_{immj}R^{immj} > 0 .$$

For $(R_{mm})^2$ we use (6.2), and for $R_{immj}R^{immj}$ we use

$$R_{ijkl}R^i{}_{rs}{}^lX^jX^kX^rX^s \leqq [R^i{}_{jk}{}^lR^j{}_{uv}{}^kR^u{}_{rs}{}^vR^r{}_{il}{}^s]^{1/2}X^jX_jX^rX_r$$

(cf. (11.2)). Then we have

THEOREM 12.1. *In a Riemannian manifold (M, g), if S is positive and if*

$$(12.3)\quad [S-[m(m-1)(R_1, R_1)-(m-1)S^2]^{1/2}]^2 > m^2(m-2)[R^i{}_{jk}{}^lR^j{}_{uv}{}^kR^u{}_{rs}{}^vR^r{}_{il}{}^s]^{1/2}$$

holds at x, then sectional curvatures are positive at x.

13. Riemannian manifolds of class one or two. Every C^ω-Riemannian manifold (M, g) can be imbedded locally and isometrically in an $[(m(m+1)/2]$-dimensional Euclidean space (L. Schlaefli [19], M. Janet [13], E. Cartan [11]). For C^∞-case, this problem is open (cf. S. Kobayashi-K. Nomizu [15], Vol. 2, p. 354). If a Riemannian manifold (M, g) can be C^∞-imbedded locally and isometrically in an $(m+p)$-dimensional Euclidean space, and if for some point x any neighborhood of x can not be C^∞-imbedded isometrically in an $(m+p-1)$-dimensional Euclidean space, we say that (M, g) is of class p (cf. G. Ricci [18], T. Y. Thomas [25], C. B. Allendoerfer [1], etc.).

LEMMA viii (A. Weinstein [28]). *If a Riemannian manifold (M, g) is isometrically immersed in E^{m+2} and if (M, g) is of positive curvature, then the curvature operator $[R]$ is positive.*

Lemma iii and Lemma viii give

PROPOSITION 13.1. *If a compact orientable Riemannian manifold (M, g) is of class* 1 *or* 2 *and if (M, g) is of positive curvature, then (M, g) is a real homology sphere.*

REMARK. Proposition 13.1 is a generalization of a result of Y. Tomonaga [29] (his assumption is "m = even and class 1" and the conclusion is "$\chi(M) > 0$").

THEOREM 13.2. *If a compact orientable Riemannian manifold (M, g) is of class* 1 *or* 2 *and satisfies* (11.3), *then (M, g) is a real homology sphere.*

REFERENCES

[1] C. B. ALLENDOERFER, Rigidity for spaces of class greater than one, Amer. Journ. Math., 61 (1939), 633-644.

[2] A. AVEZ, Applications de la formule de Gauss-Bonnet-Chern aux variétés à quatre dimensions, C. R. Acad. Sci. Paris, 256 (1963), 5488-5490.

[3] A. AVEZ, Characteristic classes and Weyl tensor: Applications to general relativity, Proc. Nat. Acad. Sci., 66 (1970), 265-268.

[4] M. BERGER, Sur les variétés à opérateur de courbure positif, C. R. Acad. Sci. Paris, 253 (1961), 2832-2834.

[5] M. BERGER, Sur les variétés d'Einstein compactes, C. R. II^e Réunion de Gr. Math. d'Exp. Latine, Louvain-Belgique (1966), 35-55.

[6] M. BERGER, Le spectre des variétés riemanniennes. Rev. Roum. Math. Pures et Appl., 13 (1968), 915-931.

[7] M. BERGER, P. Gauduchon and E. Mazet, Le Spectre d'une variété riemannienne, Lec. Notes in Math. No. 194, Springer-Verlag.

[8] R. L. BISHOP AND S. I. GOLDBERG, Some implications of the generalized Gauss-Bonnet theorem, Trans. Amer. Math. Soc., 112 (1964), 508-535.

[9] S. BOCHNER, Vector fields and Ricci curvature, Bull. Amer. Math. Soc., 52 (1946), 776-797.

[10] S. BOCHNER, Curvature and Betti numbers, Ann. of Math., 49 (1948), 379-390.

[11] E. CARTAN, Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien, Ann. de Soc. Polon. Math., 6 (1927), 1-7.

[12] D. GROMOLL, W. Klingenberg, and W. Meyer, Riemannsche Geometrie im Großen, Lec. Notes in Math. No. 55, Springer-Verlag.

[13] M. JANET, Sur la possibilité de plonger un espace riemannien donné dans espace euclidien, Ann. de Soc. Polon. Math., 5 (1926), 38-43.

[14] W. KLINGENBERG, Über Riemannsche Mannigfaltigkeiten mit nach oben beschränkter Krümmung, Ann. Mat. Pura Appl., 60 (1962), 49-59.

[15] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol. I, II, Intersci. Publ., 1963, 1969.

[16] D. MEYER, Sur les variétés riemanniennes à opérateur de courbure positif, C. R. Acad. Sci. Paris, 272 (1971), 482-485.

[17] S. B. MYERS, Riemannian manifolds with positive mean curvature, Duke Math. Journ., 8 (1941), 401-404.

[18] G. RICCI, Principii di una teoria delle forme differenziale quadratiche, Ann. di Mat., 12 (1883), 135-168.

[19] L. SCHLAEFLI, Nota alla memoria der Sig. Beltrami, Ann. di Mat., 5 (1871-1873), 170-193.
[20] I. M. SINGER AND J. A. THORPE, The curvature of 4-dimensional Einstein spaces, Global Analysis, in h. of K. Kodaira, Princeton Univ. Press, (1966), 355-365.
[21] M. SUGIMOTO AND K. SHIOHAMA, On the differentiable pinching problems, Math. Ann., 195 (1971), 1-16.
[22] S. TANNO, Betti numbers and scalar inequalities, Math. Ann., 190 (1970), 135-148.
[23] S. TANNO, Curvature tensors and non-existence of Killing vectors, Tensor, N. S., 22 (1971), 387-394.
[24] S. TANNO, An inequality for 4-dimensional Kählerian manifolds, to appear.
[25] T. Y. THOMAS, Imbedding theorems in differential geometry, Bull. Amer. Math. Soc., 45 (1939), 841-850.
[26] Y. TOMONAGA, Note on Betti numbers of Riemannian manifolds, I, Journ. Math. Soc. Japan, 5 (1953), 59-64.
[27] Y. TOMONAGA, Euler-Poincaré characteristic and sectional curvature, Diff. Geometry in h. of K. Yano, Kinokuniya, Tokyo, (1972), 501-502.
[28] A. WEINSTEIN, Positively curved n-manifolds in R^{n+2}, Journ. Diff. Geometry, 4 (1970), 1-4.
[29] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. Math. Studies, No. 32, Princeton, 1953.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN