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ON A CONFORMAL C-KILLING VECTOR FIELD IN A COMPACT SASAKIAN MANIFOLD

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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0. Introduction. Y. Ogawa has defined the notion of a C-Killing vector field in a Sasakian manifold ([1]).

In this paper, we shall introduce the notion of a conformal C-Killing vector field in a Sasakian manifold and we shall show the following theorem:

THEOREM. *In a compact* $(2n+1)$*-dimensional Sasakian manifold, the following facts hold good:*

(i) *If* $n > 2$, *then every special conformal C-Killing vector field is special C-Killing.*

(ii) *If* $n = 2$, *for any conformal C-Killing vector field* u, $\bar{u}$ *is a closed vector field.* (*Cf. Theorem* 3.4.)

REMARK. (i) of the above theorem is analogous to the Theorem 29.2 in [2].

We suppose that our manifolds are connected and the differentiable structures are of class C^{∞}.

1. Preliminaries. Let M be a $(2n+1)$-dimensional Sasakian manifold with structure tensors $(\varphi_{\mu\lambda}, g_{\mu\lambda}, \eta_\lambda)$, where the Greek indices run from 1 to $2n+1$. It is well-known that M is orientable.

A vector field ξ on a Sasakian manifold is called to be C-Killing if it satisfies ([1])

$$\delta\xi = 0\ , \qquad \mathscr{L}(\xi)(g_{\mu\lambda} - \eta_\mu\eta_\lambda) = 0\ , \tag{1.1}$$

where the operator δ is the codifferential and $\mathscr{L}(\xi)$ denotes the Lie derivative with respect to a vector field ξ. Especially we call a C-Killing vector field ξ such that

$$i(\eta)\xi = \text{constant} \tag{1.2}$$

to be special C-Killing, where $i(\eta)\xi = \eta_\varepsilon\xi^\varepsilon$.

2. Conformal C-Killing vector fields on a Sasakian manifold. We call a vector field u on a Sasakian manifold M to be conformal C-Killing if it satisfies

$$\mathscr{L}(u)(g_{\mu\lambda} - \eta_\mu\eta_\lambda) = 2f(g_{\mu\lambda} - \eta_\mu\eta_\lambda) \,, \tag{2.1}$$

where f is a scalar field on M that is called the associated scalar field with respect to the conformal C-Killing vector field u. Especially we call a conformal C-Killing vector field u such that

$$u' = i(\eta)u = \text{constant} \tag{2.2}$$

to be special conformal C-Killing. Clearly an infinitesimal η-conformal transformation which is an infinitesimal contact transformation at the same time is conformal C-Killing ([2].) It is clear that the set of all conformal C-Killing vector fields constitutes a Lie algebra.

The equation (2.1) can be expressed as

$$\begin{aligned} \nabla_\mu u_\lambda + \nabla_\lambda u_\mu - 2u^\varepsilon(\varphi_{\varepsilon\mu}\eta_\lambda + \varphi_{\varepsilon\lambda}\eta_\mu) - \eta_\mu\nabla_\lambda u' - \eta_\lambda\nabla_\mu u' \\ = 2f(g_{\mu\lambda} - \eta_\mu\eta_\lambda) \,, \end{aligned} \tag{2.3}$$

where ∇ denotes the operator of the covariant differentiation. Transvecting (2.3) with η^λ, we have

$$\eta^\varepsilon\nabla_\varepsilon u_\lambda + \varphi_\lambda{}^\varepsilon u_\varepsilon - (i(\eta)\nabla_\eta u)\eta_\lambda = 0 \,, \tag{2.4}$$

where $\nabla_\eta u^\lambda = \eta^\varepsilon\nabla_\varepsilon u^\lambda$.

Thus we have

PROPOSITION 2.1. *In a Sasakian manifold, for a special conformal C-Killing vector field u, we have*

$$\mathscr{L}(\eta)u_\lambda = 0 \,. \tag{2.5}$$

Next, transvecting (2.3) with $g^{\mu\lambda}$, we have

$$\nabla^\varepsilon u_\varepsilon - \eta^\varepsilon\nabla_\varepsilon u' = 2nf \,. \tag{2.6}$$

Thus we have from (2.6) and the Green's Theorem

PROPOSITION 2.2. *In a compact Sasakian manifold, a special conformal C-Killing vector field with a constant associated scalar field is special C-Killing.*

Let u be a special conformal C-Killing vector field. Then making use of (2.2) and (2.3), we have

$$\nabla_\mu u_\lambda + \nabla_\lambda u_\mu = 2u^\varepsilon(\varphi_{\varepsilon\mu}\eta_\lambda + \varphi_{\varepsilon\lambda}\eta_\mu) + 2f(g_{\mu\lambda} - \eta_\mu\eta_\lambda) \,. \tag{2.7}$$

Differentiating (2.7) with ∇^μ, we get

$$\nabla^\rho\nabla_\rho u_\lambda = -R_{\lambda\varepsilon}u^\varepsilon - 4u_\lambda + 4(n+1)u'\eta_\lambda + 2(1-n)f_\lambda - 2(f_\varepsilon\eta^\varepsilon)\eta_\lambda - 2Du\eta_\lambda\,, \tag{2.8}$$

where $f_\lambda = \nabla_\lambda f$ and $Du = \varphi^{\rho\sigma}\nabla_\rho u_\sigma$.

Transvecting (2.8) with η^λ, we have

$$\eta^\sigma\nabla^\rho\nabla_\rho u_\sigma = 2nu' - 2nf_\varepsilon\eta^\varepsilon - 2Du\,.$$

On the other hand, we have from (2.2)

$$\eta^\sigma\nabla^\rho\nabla_\rho u_\sigma = 2nu' - 2Du\,.$$

Thus we have from the above two equations

PROPOSITION 2.3. *In a Sasakian manifold, for an associated scalar field f of a special conformal C-Killing vector field we have*

$$f_\varepsilon\eta^\varepsilon = 0\,. \tag{2.9}$$

From the above Proposition, for a special conformal C-Killing vector field u, (2.8) can be written as

$$\nabla^\rho\nabla_\rho u_\lambda = -R_{\lambda\varepsilon}u^\varepsilon - 2Du\eta_\lambda - 4u_\lambda + 4(n+1)u'\eta_\lambda - 2(n-1)f_\lambda\,. \tag{2.10}$$

LEMMA 2.4. *In a compact Sasakian manifold, for a vector field u satisfying* (2.7) *and* (2.9), *we have $u' =$ constant.*

PROOF. Differentiating (2.7) with ∇^μ, we have

$$\nabla^\rho\nabla_\rho u_\lambda = -R_{\lambda\varepsilon}u^\varepsilon - \nabla_\lambda(\nabla_\varepsilon u^\varepsilon) - 2Du\eta_\lambda - 2(\nabla_\rho u_\varepsilon)\varphi_\lambda{}^\varepsilon\eta^\rho + (4n+2)u'\eta_\lambda - 2u_\lambda + 2f_\lambda\,. \tag{2.11}$$

On the other hand, transvecting (2.7) with $g^{\mu\lambda}$, we get

$$\nabla_\varepsilon u^\varepsilon = 2nf\,. \tag{2.12}$$

Substituting (2.12) in (2.11), we have

$$\nabla^\rho\nabla_\rho u_\lambda = -R_{\lambda\varepsilon}u^\varepsilon - 2Du\eta_\lambda - 2(\nabla_\rho u_\varepsilon)\varphi_\lambda{}^\varepsilon\eta^\rho + (4n+2)u'\eta_\lambda - 2u_\lambda + 2(1-n)f_\lambda\,.$$

Calculating the Laplacian of u', we have from the above equation $\nabla^\rho\nabla_\rho u' = 0$. Since the manifold is compact, u' must be constant ([3]).

q.e.d.

From this lemma, we obtain

THEOREM 2.5. *A vector field u on a compact Sasakian manifold is special conformal C-Killing if and only if it satisfies the relations* (2.7) *and* (2.9).

Next, we consider the relation between a conformal C-Killing vector field and a special conformal C-Killing vector field. Then we have

THEOREM 2.6. *For a conformal C-Killing vector field u on a Sasakian manifold, a vector field v defined by*

$$v_\lambda = u_\lambda - u'\eta_\lambda \tag{2.13}$$

is special conformal C-Killing. Conversely for a special conformal C-Killing vector field v and any scalar field h, a vector field u defined by

$$u_\lambda = v_\lambda + h\eta_\lambda \tag{2.14}$$

is conformal C-Killing.

Since the proof of the above theorem is completely same as Theorem 3.4 in [1], we omit it.

3. Special conformal C-Killing vector fields on a compact Sasakian manifold. In this section, we show that a special conformal C-Killing vector field on a compact $(2n+1)$-dimensional Sasakian manifold $(n>2)$ is special C-Killing.

Let u be a special conformal C-Killing vector field. Then differentiating (2.7) with ∇_ν and adding them cyclically with respect to the indices ν, μ and λ, we have

$$\begin{aligned}\nabla_\nu\nabla_\mu u_\lambda - R_{\mu\lambda\nu\varepsilon}u^\varepsilon &= (\nabla_\nu u^\varepsilon)(\varphi_{\varepsilon\mu}\eta_\lambda + \varphi_{\varepsilon\lambda}\eta_\mu) + (\nabla_\mu u^\varepsilon)(\varphi_{\varepsilon\nu}\eta_\lambda + \varphi_{\varepsilon\lambda}\eta_\nu)\\ &\quad - (\nabla_\lambda u^\varepsilon)(\varphi_{\varepsilon\mu}\eta_\nu + \varphi_{\varepsilon\nu}\eta_\mu) + 2u'\eta_\lambda g_{\nu\mu} + 2u^\varepsilon(\eta_\mu\eta_\nu g_{\lambda\varepsilon}\\ &\quad - \eta_\nu\eta_\lambda g_{\mu\varepsilon} - \eta_\lambda\eta_\mu g_{\nu\varepsilon}) - 2u^\varepsilon(\varphi_{\varepsilon\lambda}\varphi_{\lambda\mu} + \varphi_{\varepsilon\mu}\varphi_{\nu\lambda})\\ &\quad + f_\nu(g_{\mu\lambda} - \eta_\mu\eta_\lambda) + f_\mu(g_{\nu\lambda} - \eta_\nu\eta_\lambda)\\ &\quad - f_\lambda(g_{\nu\mu} - \eta_\nu\eta_\mu) + 2f(\varphi_{\nu\lambda}\eta_\mu + \varphi_{\mu\lambda}\eta_\nu)\,.\end{aligned} \tag{3.1}$$

Let $\bar{u}_\lambda = \varphi_\lambda{}^\varepsilon u_\varepsilon$, then we can easily obtain

$$d\bar{u} = e(\eta)u + \Gamma u - 4f\varphi\,, \tag{3.2}$$

where the operators d, $e(\eta)$ and Γ are same as the operators defined in [1].

The following lemmas have proved by Y. Ogawa ([1]):

LEMMA 3.1. *In a compact Sasakian manifold, for any vector field u satisfying $u' =$ constant we have*

$$(\Gamma u, e(\eta)u) = -(e(\eta)u, e(\eta)u)\,, \tag{3.3}$$

where (u, v) denotes the global inner product of any p-forms u and v.

LEMMA 3.2. *In a compact $(2n+1)$-dimensional Sasakian manifold, we have for any p-form u and $(p+1)$-form v*

$$(\Gamma u, v) = (u, Dv) - 2n(e(\eta)u, v)\ , \tag{3.4}$$

where $(Dv)_{\lambda_1\cdots\lambda_p} = \varphi^{\rho\sigma}\nabla_\rho v_{\sigma\lambda_1\cdots\lambda_p}$.

PROPOSITION 3.3. *In a compact* $(2n+1)$*-dimensional Sasakian manifold, we have for a special conformal C-Killing vector field* u

$$(\Gamma u, \Gamma u) = (e(\eta)u, e(\eta)u) - 4n(n-2)(f, f)\ . \tag{3.5}$$

PROOF. Calculating $D\Gamma u$ for a special conformal C-Killing vector field u, we have from (2.8) and (3.1)

$$(D\Gamma u)_\lambda = (Du - u')\eta_\lambda - (2n-1)u_\lambda + 2(n-2)f_\lambda\ . \tag{3.6}$$

Thus we have

$$(u, D\Gamma u) = (u', Du) - (2n-1)(u, u) - (u', u') - 4n(n-2)(f, f)\ . \tag{3.7}$$

Since $\Gamma u' = 0$ for a special conformal C-Killing vector field u, we have $(u', Du) = 2n(u', u')$. Substituting the last equation in (3.7), we have

$$(u, D\Gamma u) = -(2n-1)\{(u, u) - (u', u')\} - 4n(n-2)(f, f)\ . \tag{3.8}$$

Taking account of Lemmas 3.1, 3.2, and (3.8) we obtain (3.5). q.e.d.

Calculating $(d\bar{u}, d\bar{u})$ for a special conformal C-Killing vector field u and taking account of (3.2), (3.3), and (3.5), we have

$$(d\bar{u}, d\bar{u}) = -4n(n-2)(f, f) + 16(f\varphi, f\varphi) - 8(\Gamma u, f\varphi)\ .$$

On the other hand, since $(f\varphi, f\varphi) = n(f, f)$ and $(\Gamma u, f\varphi) = 2n(f, f)$, we have

$$(d\bar{u}, d\bar{u}) = -4n(n-2)(f, f)\ . \tag{3.9}$$

Thus we have

THEOREM 3.4. *In a compact* $(2n+1)$*-dimensional Sasakian manifold, the following facts hold good:*

(i) *If* $n > 2$, *every special conformal C-Killing vector field is special C-Killing.*

(ii) *If* $n = 2$, *for any conformal C-Killing vector field* u, $\bar{u}$ *is a closed vector field.*

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