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ON THE UNIQUENESS OF SOLUTIONS IN THE HULL

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We shall consider an almost periodic system, or more generally, a system with the compact hull, and assume that in either case the system has a bounded solution. The purpose of this note is to prove a uniqueness theorem for every solution in the hull of the bounded solution. Kato and Yoshizawa have assumed in [1] the condition

(c) *solutions of every system in the hull are unique for initial conditions*

in order to prove that a bounded solution of a system with the compact hull is totally stable if it is uniformly asymptotically stable. Kato has weakened the condition (c) in [2]. Moreover, he has constructed a system with a uniformly asymptotically stable but not totally stable solution, which lacks the uniqueness property of a solution of a system in the hull.

Concerning the uniqueness of a given solution, Okamura has given a necessary and sufficient condition in [3]. His condition is to require the existence of a kind of Liapunov function. Yoshizawa has improved the method to construct the Liapunov function (see p. 5-8 in [5]). Using his method, we shall show a necessary and sufficient condition for the uniqueness of every solution in the hull.

We shall use the following notations throughout this note. We set $I = [0, \infty)$, $R = (-\infty, +\infty)$, $R^n =$ a real Euclidean n-space, $S_{B^*} = \{x \in R^n; |x| < B^*\}$, where $|\cdot|$ is a norm, and $C(I \times S_{B^*}, R^n) =$ the family of R^n-valued continuous functions defined on $I \times S_{B^*}$. For any $f \in C(I \times S_{B^*}, R^n)$ and $\tau \in I$, we set $f_\tau(t, x) = f(t + \tau, x)$ for $(t, x) \in I \times S_{B^*}$. The hull of f, denoted by $H(f)$, is the closure of the set $\{f_\tau; \tau \in I\}$ in the sense of the uniform convergence on any compact subset of $I \times S_{B^*}$.

1. We shall consider a system of differential equations

$$\frac{dx}{dt} = f(t, x) , \tag{1}$$

and assume that $f \in C(I \times S_{B^*}, R^n)$ and $H(f)$ is compact. Let $u(t)$ be a solution of the system (1) such that for a constant B, $0 < B < B^*$,

$$|u(t)| \leqq B \quad \text{for all} \quad t \in I\,.$$

Then, u_τ is obviously a bounded solution remaining in $\overline{S_B}$ on I of the system

$$(1, \tau) \qquad \frac{dx}{dt} = f_\tau(t, x)$$

for any $\tau \in I$. Since $|f_\tau(t, x)| \leqq L$ for some $L = L(B)$ and all $(t, x) \in I \times \overline{S_B}$, u_τ satisfies the Lipschitz condition

$$|u_\tau(t) - u_\tau(s)| \leqq L|t - s| \quad \text{for all} \quad t, s \in I\,.$$

Therefore, $H(u)$ and $H(u, f)$ are compact. Here, for $(v, g) \in H(u, f)$ there exists a sequence $\{\tau_k\}$, $\tau_k \in I$, such that

(*) $u_{\tau_k} \to v$ and $f_{\tau_k} \to g$ as $k \to \infty$ uniformly on any compact subset of $I \times S_{B^*}$.

v is a bounded solution remaining in $\overline{S_B}$ on I of the system

$$(2) \qquad \frac{dx}{dt} = g(t, x)\,.$$

We shall denote the tubular neighborhoods of $u(t)$, $\tau \leqq t \leqq \tau + T$, and $u_\tau(t)$, $0 \leqq t \leqq T$, by the following;

$$N(\tau, T, \varepsilon) = \{(t, x);\, \tau \leqq t \leqq \tau + T \quad \text{and} \quad |x - u(t)| < \varepsilon\}\,,$$
$$M(\tau, T, \varepsilon) = \{(t, x);\, 0 \leqq t \leqq T \quad \text{and} \quad |x - u_\tau(t)| < \varepsilon\}\,.$$

Clearly, $(t, x) \in N(\tau, T, \varepsilon)$ if and only if $(t - \tau, x) \in M(\tau, T, \varepsilon)$.

We have obtained the following theorem concerning the uniqueness of the solution v of the system (2).

THEOREM. *Let T and ε be given, where $0 < T$ and $0 < \varepsilon < B^* - B$. Then, for any $(v, g) \in H(u, f)$ v is a unique solution to the right of the system* (2) *if and only if there exist continuous functions $V(t, x, \tau)$ defined on $N(\tau, T, \varepsilon)$ for all $\tau \in I$, which satisfy the following conditions*:

(i) $V(t, u(t), \tau) \equiv 0$ *for all* $t \in [\tau, \tau + T]$.

(ii) $a(|x - u(t)|) \leqq V(t, x, \tau) \leqq |x - u(t)|$ *for all* $(t, x) \in N(\tau, T, \varepsilon)$, *where $a(r)$ is a positive definite continuous function of $r \in [0, \varepsilon)$, which may depend on T and ε but not on τ.*

(iii) $|V(t, x, \tau) - V(t, y, \tau)| \leqq |x - y|$ *for all* (t, x), $(t, y) \in N(\tau, T, \varepsilon)$.

(iv) $V'_{(1)}(t, x, \tau) \leqq 0$ *for all* $(t, x) \in N(\tau, T, \varepsilon)$.

PROOF. Sufficiency. If $(v, g) \in H(u, f)$, there exists a sequence $\{\tau_k\}$, $\tau_k \in I$, such that the condition (*) holds. Let $y(t)$ be a solution of (2) defined on $[t_0, t_1)$, for some t_0 and $t_1 \in I$, $t_0 < t_1$, such that $y(t_0) = v(t_0)$.

We shall show that $y(t) = v(t)$ for all $t \in [t_0, t_1)$ sufficiently close to t_0. Considering $\{\tau_k + t_0\}$ instead of $\{\tau_k\}$, we can assume that $t_0 = 0$. Since $\{u_{\tau_k}\}$ converges v uniformly on $[0, T]$, there exists a small $t_2 > 0$ such that $(t, v(t))$ and $(t, y(t)) \in M(\tau_k, T, \varepsilon)$ for all $t \in [0, t_2]$ and sufficiently large k. Set

$$W(t, x, \tau_k) = V(t + \tau_k, x, \tau_k) , \tag{3}$$

which is defined on $M(\tau_k, T, \varepsilon)$. Since $W(t, x, \tau_k)$ satisfies the Lipschitz condition with respect to x, we obtain

$$\begin{aligned} W'_{(2)}(t, x, \tau_k) &\leqq W'_{(1,\tau_k)}(t, x, \tau_k) + |g(t, x) - f_{\tau_k}(t, x)| \\ &\leqq V'_{(1)}(t + \tau_k, x, \tau_k) + |g(t, x) - f_{\tau_k}(t, x)| . \end{aligned}$$

From this and the condition (iv), it follows that

$$W'_{(2)}(t, x, \tau_k) \leqq \delta_k ,$$

where

$$\delta_k = \sup \{|f_{\tau_k}(t, x) - g(t, x)|; (t, x) \in M(\tau_k, T, \varepsilon)\} ,$$

and hence

$$W(t, y(t), \tau_k) - W(0, y(0), \tau_k) \leqq \delta_k t \quad \text{for} \quad t \in [0, t_2] .$$

The condition (ii) implies that

$$a(|x - u_{\tau_k}(t)|) \leqq W(t, x, \tau_k) \leqq |x - u_{\tau_k}(t)| \quad \text{for} \quad (t, x) \in M(\tau_k, T, \varepsilon) .$$

Therefore, it holds that

$$a(|y(t) - u_{\tau_k}(t)|) \leqq |y(0) - u_{\tau_k}(0)| + \delta_k t \quad \text{for} \quad t \in [0, t_2] .$$

Since $y(0) = v(0) = \lim_{k\to\infty} u_{\tau_k}(0)$ and $\lim_{k\to\infty} \delta_k = 0$, we have

$$a(|y(t) - v(t)|) \leqq 0 \quad \text{for} \quad t \in [0, t_2] ,$$

and hence

$$y(t) = v(t) \quad \text{for} \quad t \in [0, t_2] .$$

Necessity. We remark that $|x| < B^*$ if $|x - u(t)| \leqq \varepsilon$. For $(t, x) \in N(\tau, T, \varepsilon)$ and $t > \tau$, denote by $Z(t, x, \tau)$ the family of all functions $z(s)$, which are continuous on $[\tau, t]$, with the properties that their derivatives are continuous except for finite number of values of s and that $z(\tau) = u(\tau)$, $z(t) = x$ and $|z(s) - u(s)| \leqq \varepsilon$ for $s \in [\tau, t]$. For any $\tau \in I$ and any $(t, x) \in N(\tau, T, \varepsilon)$, set

$$V(t, x, \tau) = \begin{cases} \inf\limits_{z \in Z(t,x,\tau)} \displaystyle\int_\tau^t \left| \frac{dz}{ds}(s) - f(s, z(s)) \right| ds, & \text{if} \quad t > \tau , \\ |x - u(\tau)|, \quad \text{if} \quad t = \tau . \end{cases} \tag{4}$$

V is continuous on $N(\tau, T, \varepsilon)$ and satisfies the conditions (i), (iii) and (iv). Moreover, it holds that

(ii)′ $V(t, x, \tau) \leqq |x - u(t)|$ for all $(t, x) \in N(\tau, T, \varepsilon)$ and $V(t, x, \tau) > 0$ if $|x - u(t)| > 0$.

See p. 5–8 in [5] for the proof of these. Therefore, it remains only to prove the first inequality in (ii).

Set

$$a(\tau, r) = \inf\{V(t, x, \tau); (t, x) \in Q(\tau, r)\},$$

where $0 < r < \varepsilon$, $\tau \in I$ and

$$Q(\tau, r) = \{(t, x); t \in [\tau, \tau + T], |x - u(t)| = r\}.$$

Since $Q(\tau, r)$ is a compact set, there exists a $(t_0, x_0) \in Q(\tau, r)$ where V attains $a(\tau, r)$, so that $a(\tau, r) > 0$. We shall prove

$$\inf_{\tau \in I} a(\tau, r) \equiv a(r) > 0 \quad \text{for} \quad 0 < r < \varepsilon.$$

To prove this, suppose that there exists an r_0, $0 < r_0 < \varepsilon$, such that $a(r_0) = 0$. By the definition, it holds that

$$\lim_{k \to \infty} V(t_k, x_k, \tau_k) = 0$$

for some sequence $\{\tau_k\}$, $\tau_k \in I$, and some $(t_k, x_k) \in Q(\tau_k, r_0)$. If we set $s_k = t_k - \tau_k$, we have $s_k \in [0, T]$ and

$$\lim_{k \to \infty} W(s_k, x_k, \tau_k) = 0, \tag{5}$$

where $W(t, x, \tau_k)$ is defined by (3). For $(t, x) \in M(\tau, T, \varepsilon)$ and $t > 0$, set

$$Y(t, x, \tau) = \{z_\tau; z \in Z(\tau + t, x, \tau)\}.$$

From (4) and the definition of W, we have

$$W(t, x, \tau) = \begin{cases} \inf\limits_{y \in Y(t,x,\tau)} \displaystyle\int_0^t \left| \frac{dy}{ds}(s) - f_\tau(s, y(s)) \right| ds, & \text{if} \quad t > 0, \\ |x - u_\tau(0)|, & \text{if} \quad t = 0. \end{cases} \tag{6}$$

We shall show that $\liminf_{k \to \infty} s_k \equiv \sigma > 0$. Since $|f_{\tau_k}(t, x)| \leqq L$ for some $L = L(B + \varepsilon)$, all $(t, x) \in \overline{M(\tau_k, T, \varepsilon)}$ and all $k = 1, 2, \cdots$, it follows from (6) that

$$W(s_k, x_k, \tau_k) \geqq |x_k - u_{\tau_k}(0)| - Ls_k$$

(see p. 6 Lemma 1. 2 in [5]). With the aid of inequalities

$$\begin{aligned} |x_k - u_{\tau_k}(0)| &\geqq |x_k - u_{\tau_k}(s_k)| - |u_{\tau_k}(s_k) - u_{\tau_k}(0)| \\ &\geqq r_0 - Ls_k, \end{aligned}$$

we have

$$W(s_k, x_k, \tau_k) \geqq r_0 - 2Ls_k .$$

In view of (5), we have $\sigma > 0$.

Therefore, from (5) and (6), there exist $y_k \in Y(s_k, x_k, \tau_k)$ such that

$$\lim_{k\to\infty}\int_0^{s_k}\left|\frac{dy_k}{ds}(s) - f_{\tau_k}(s, y_k(s))\right|ds = 0 . \tag{7}$$

Hence, there exists a subsequence of $\{y_k\}$ converging uniformly on any compact subset of $[0, \sigma)$. In the following, by renumbering, we shall denote subsequences and their original sequences by the same notations. Since $H(f)$ is compact, $\{f_{\tau_k}\}$ has a subsequence converging uniformly on any compact subset of $I \times S_{B^*}$. Let $y(t)$ and $g(t, x)$ be limit functions of $\{y_k(t)\}$ and $\{f_{\tau_k}(t, x)\}$ respectively. By standard arguments, we have

$$y(t) - y(0) - \int_0^t g(s, y(s))ds = 0 \quad \text{for} \quad t \in [0, \sigma) ,$$

so that $y(t)$ is a solution of (2).

On the other hand, choosing a subsequence, we can assume that $\{u_{\tau_k}\}$ converges to some $v \in H(u)$ uniformly on any compact interval of I. v is clearly a solution of (2).

We shall examine the relation of the solutions y and v of the system (2). Choosing a subsequence of $\{s_k\}$, if necessary, we can assume that $\lim_{k\to\infty} s_k = \sigma$. Then, if $s_0 \in [0, \sigma)$ is sufficiently close to σ, we have for k sufficiently large

$$|y_k(s_k) - y_k(s_0)| < \frac{r_0}{4} \quad \text{and} \quad |u_{\tau_k}(s_k) - u_{\tau_k}(s_0)| < \frac{r_0}{4} .$$

Obviously it holds that

$$\begin{aligned} |y(s_0) - v(s_0)| \geqq |x_k - u_{\tau_k}(s_k)| - \{&|y(s_0) - y_k(s_0)| \\ &+ |y_k(s_0) - y_k(s_k)| + |u_{\tau_k}(s_k) - u_{\tau_k}(s_0)| \\ &+ |u_{\tau_k}(s_0) - v(s_0)|\} . \end{aligned}$$

From these inequalities, we have

$$|y(s_0) - v(s_0)| \geqq \frac{r_0}{2} .$$

On the other hand, $y(0) = v(0)$ because $y_k(0) = u_{\tau_k}(0)$.

Therefore, v is not a unique solution to the right of the system (2). This is a contradiction. Hence it is proved that

$$a(r) > 0 \quad \text{for} \quad 0 < r < \varepsilon . \qquad \text{q.e.d.}$$

2. We shall show some examples of the functions V and an application of the theorem.

When f satisfies the Lipschitz condition such that

$$|f(t, x) - f(t, y)| \leqq K|x - y|$$

for some $K > 0$ and all (t, x), $(t, y) \in I \times S_{B^*}$, we set

$$V(t, x, \tau) = e^{-K(t-\tau)}|x - u(t)| \quad \text{for} \quad (t, x) \in N(\tau, T, \varepsilon) .$$

More generally, let f be inner product in the sense of Strauss and Yorke in [4], that is to say, f satisfies the condition

$$\langle x - y, f(t, x) - f(t, y)\rangle \leqq K|x - y|^2$$

for some $K > 0$ and all (t, x), $(t, y) \in I \times S_{B^*}$, where $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ and $|x| = \langle x, x\rangle^{1/2}$ for $x, y \in R^n$. We set in this case

$$V(t, x, \tau) = (2\varepsilon)^{-1}e^{-2K(t-\tau)}|x - u(t)|^2 \quad \text{for} \quad (t, x) \in N(\tau, T, \varepsilon) .$$

It is easy to check that these V fulfill the conditions (i), $\cdots$, (iv) in Theorem.

Applying Theorem, we can present a short proof of the following proposition, which corresponds to Lemma 6 in [6], though Yoshizawa has proved the lemma for functional differential systems.

PROPOSITION. *Let $T > 0$ be given. Then, for any $(v, g) \in H(u, f)$ v is a unique solution to the right of* (2) *if and only if for any small $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\tau \in I$, $|x - u(\tau)| < \delta(\varepsilon)$ and $|h(t)| < \delta(\varepsilon)$, we have*

$$|x(t) - u(t)| < \varepsilon \quad on \quad \tau \leqq t \leqq \tau + T ,$$

where $x(t)$ is a solution through (τ, x) of the system

$$\frac{dx}{dt} = f(t, x) + h(t) \tag{8}$$

and $h(t)$ is continuous on I.

PROOF. Sufficiency can be proved by standard arguments. We shall only show the proof of necessity. According to Theorem, there exist continuous functions V satisfying the conditions (i), $\cdots$, (iv). We remark that $a(r)$ in (ii) can be replaced by an increasing positive definite continuous function. Define $\delta(\varepsilon)$ by the relation

$$a^{-1}(\delta(\varepsilon)(1 + T)) < \varepsilon . \tag{9}$$

Obviously it holds that

$$V'_{(8)}(t, x(t), \tau) \leqq V'_{(1)}(t, x(t), \tau) + |h(t)| \leqq \delta(\varepsilon) .$$

Therefore, we have

$$a(|x(t) - u(t)|) \leqq |x(\tau) - u(\tau)| + \delta(\varepsilon)(t - \tau)\,.$$

From this and (9), it holds that

$$|x(t) - u(t)| \leqq \varepsilon \quad \text{for} \quad t \in [\tau, \tau + T]\,. \qquad \text{q.e.d.}$$

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