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ON PSEUDO-PRIME ENTIRE FUNCTIONS

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1. Introduction. Let $F(z)$ be an entire function. Then F is said to be pseudo-prime (E-pseudo prime) if and only if every factorization of the form $f(g)(z) = F(z)$ with f meromorphic (entire), g entire implies that either f is rational (polynomial) or g is a polynomial. The following three theorems were proved recently.

THEOREM A. (Ozawa [8]) *If F is an entire function of finite order with a finite Picard exceptional value, then F is E-pseudo prime.*

The above result has been generalized as follows:

THEOREM B. (Goldstein [3]) *Let $F(z)$ be an entire function of finite order such that $\delta(a, F) = 1$ for some $a \neq \infty$, where $\delta(a, F)$ denotes the Nevanlinna deficiency. Then F is E-pseudo prime.*

It was pointed out [3] that F must be of finite order, as is shown by the example $F = e^{e^z}$, $f(z) = g(z) = e^z$, where $\delta(0, F) = 1$ and $F = f(g)$. However, for functions of infinite order, the following result is known.

THEOREM C. (Ozawa [8]) *Let $L(z)$ be a transcendental entire function of order less than one and $p(z)$ a polynomial. Then the functional equation $f(g(z)) = L(z) \exp (p(z)e^z)$ has no pair of transcendental entire solutions f and g of finite order.*

In this paper we have improved these results and in particular we have extended Theorems B and C and some other results of Ozawa's (see e.g. [9]) to larger classes of entire functions. We shall prove the following results:

THEOREM 1. *Let $F(z)$ be an entire function of finite order ρ with $\delta(a, F) = 1$ for some $a \neq \infty$. (We note that $\rho > 0$ [11]). Let $H(z)$ be an entire function of order less than ρ and let $p(z)$ be a non-constant polynomial. Then $H(z)p(F(z))$ is E-pseudo prime.*

THEOREM 2. *Let $L(z)$ be a transcendental entire function of order less than k (k an integer > 0) having at least one zero and let $H(z)$ be an entire function ($\not\equiv 0$) of order less than k. If $S(z)$ is any entire function of order less than k which is not a polynomial of degree k, then*

$F(z) = L(z)\exp(H(z)e^{z^k} + S(z))$ *is pseudo-prime.*

THEOREM 3. *Let* L, H *and* $S(z)$ *be three transcendental entire functions of order less than one. Then* $L(z)\exp(H(z)e^z + S(z))$ *is prime if* L *can not be expressed in the form* $L(z) = [K(z)]^m$ *for some entire function* $K(z)$ *and some integer* $m \geqq 2$.

2. **Preliminaries.** It is assumed throughout the paper that the reader is familiar with the fundamental concept of Nevanlinna's theory of meromorphic functions and its standard symbols such as $T(r, f)$, $N(r, f)$ etc.

LEMMA 1. (Picard-Borel Theorem [7, p. 262]) *For a non-constant meromorphic function* f *there are at most two values of* a *for which the counting function* $N(r, a)$ [*or* $n(r, a)$] *is of lower order* (*class, type*) *than the characteristic* $T(r, f)$.

LEMMA 2. *Let* f *be a transcedental meromorphic function and* $a_i(z)$ $(i = 1, 2 \cdots, n)$ *be meromorphic functions satisfying*

$$T(r, a_i(z)) = o\{T(r, f)\}$$

as $r \to \infty$ *for* $i = 1, 2, \cdots, n$.

Assume that

$$f^n(z) + a_1(z)f^{n-1}(z) + a_2(z)f^{n-2}(z) + \cdots + a_n(z) = g(z)$$

and that

$$N(r, f) + N\left(r, \frac{1}{g}\right) = o\{T(r, f)\}$$

as $r \to \infty$ outside a set of r values of finite measure. Then

$$g(z) = \left(f + \frac{a_1(z)}{n}\right)^n .$$

REMARK. This is a special case of the Tumura-Clunie theorem (see [6, pp. 68–73]).

LEMMA 3 [6, p. 47]. *If* f *is a transcendental meromorphic function and* $a_1(z), a_2(z), a_3(z)$ *are distinct meromorphic functions satisfying for* $i = 1, 2$ *and* 3

$$T(r, a_i(z)) = o\{T(r, f)\}, \quad as \quad r \longrightarrow \infty$$

then

$$\{1 + o(1)\}T(r, f) \leqq \sum_{i=1}^{3} N\left(r, \frac{1}{f - a_i(z)}\right) + o\{T(r, f)\},$$

as $r \to \infty$ outside a set of r values of finite measure.

LEMMA 4 [4]. *Let p be a non-constant polynomial of degree m and h, k be two entire functions of order less than m with $h \not\equiv 0$, $k \not\equiv$ constant. If $he^p + k$ has a factorization $he^p + k = f(g)$ with f and g nonlinear and entire, then f is transcendental and g is a polynomial of degree no greater than m.*

3.1. Proof of Theorem 1. (This argument is a sight modification of Goldstein's proof of Theorem 3. We include this modification for the readers convenience.) For an entire function F of finite order with $\delta(a, F) = 1$ Edrei and Fuchs [2, pp. 281–283] proved that there is a connected path consisting of circular arcs and line segments which may be written as $\Gamma = l_1 \cup \gamma_2 \cup l_2 \cup \gamma_3 \cup \cdots$ where $\{\gamma_j\}$ are arcs on $|z| = r_j$, $(r_j \to \infty)$ each of angular measure not less than $2\pi/3\rho$ (ρ, a fixed integer depending on the order of the function F), and $\{l_j\}$ are segments which join the points $r_j e^{i\theta_j}$ of γ_j and $r_{j+1}e^{i\theta_{j+1}}$ of γ_{j+1}; $j = 1, 2\cdots$ and such that for $z \in \Gamma$ the following estimate holds:

$$\log|F(z)| \leqq \frac{-\pi}{16} T(r, F) \quad (|z| = r > r_0)\,. \tag{1}$$

In the proof of Theorem B, Goldstein proved that if (1) holds for such a path Γ for an entire function of finite order F then F is pseudo-prime.

Now we show that the inequality (1) holds for $p(F)$ with $-\pi/16$ replaced by a *different constant*. In fact, if we suppose (without loss of generality) $p(0) = 0$, $p(z) = c_0 z^m + c_1 z^{m-1} + \cdots + \cdots$, with $c_0 \neq 0$, then (1) becomes

$$\begin{aligned} \log|p(F(z))| &\leqq \left\{\frac{-\pi m}{16} + o(1)\right\} T(r, F) \\ &\leqq \left\{\frac{-\pi m}{16\, m} + o(1)\right\} T(r, p(F))\,, \end{aligned} \tag{2}$$

for $z \in \Gamma$ with $|z| > r_0$.

Now by the assumption that the order of $H(z)$ is less than the order of F, we have, in fact, that the order of H is less than the lower order of F, since $\delta(a, F) = 1$, implies that the order and lower order of F are the same, see e.g. [6, p. 105]. It follows that the logarithm of the maximal modulus of H grows much slower than $T(r, F)$. More precisely,

$$\frac{\log M(r, H)}{T(r, F)} \to 0 \quad \text{as} \quad r \to \infty\,.$$

Thus, it is clear that (1) is satisfied when F is replaced by $H(z)p(F)$ provided that at the same time the quantity $-\pi/16$ is replaced by $-\pi/16 + \varepsilon$

for some small number $\varepsilon > 0$. The remainder of the proof will be exactly the same as in the proof of Theorem B. Theorem 1 is thus proved.

3.2. Proof of Theorem 2. First we prove that F is E-pseudo prime. Suppose that there exist two transcendental entire functions f and g such that

$$f(g(z)) = F(z) = L(z) \exp(H(z)e^{z^k} + S(z)) . \tag{3}$$

We shall deal with the two cases (i) $\rho(g) \geqq k$ and (ii) $\rho(g) < k$ separately.

In case (i), from the hypotheses that $\rho(L) < k$ and that L has at least one zero we conclude by virtue of Lemma 1 that f has one and only one zero, say a, of multiplicity n ($n \geqq 1$). Thus we can express f as

$$f(z) = (z - a)^n e^{\alpha(z)} \tag{4}$$

where α is an entire function.

From (3) and (4) we have

$$L(z) = (g - a)^n e^{\alpha(g) - H(z)e^{z^k} - S(z)} .$$

Hence,

$$L(z) = (g - a)^n e^{\beta(z)} \tag{5}$$

where $\beta(z)$ is an entire function, and

$$\alpha(g) - \beta(z) = H(z)e^{z^k} + S(z) . \tag{6}$$

In view of (5) one can conclude readily that β must be an entire function of zero order. For otherwise g would be of infinite order and composed with f would grow much faster than F, a contradiction. Hence, applying Lemma 4 we see that identity (6) cannot hold unless α is a polynomial and the order of g is equal to k. It follows from (5) that β is a polynomial of degree k, therefore $\beta(z) + S(z) \not\equiv 0$, and hence α must be linear by Lemma 2. We rewrite equation (6) as

$$\alpha(g) = H(z)e^{z^k} + S(z) + \beta(z) . \tag{7}$$

We note that $S(z) + \beta(z)$ is an entire function of order less than k and is never equal to a constant. Now since α is linear, set $\alpha(z) = bz + c$. Then by applying Lemma 3 with $f(z) = bg(z) + c$, $a_1(z) \equiv ab + c$, $a_2(z) \equiv S(z) + \beta(z) + c$, and $a_3 \equiv \infty$, we would have for $r \to \infty$ outside a set of finite measure,

$$\begin{aligned} T(r, bg(z) + c) &\leqq N\left(r, \frac{1}{\alpha(g) - a_1(z)}\right) + N\left(r, \frac{1}{\alpha(g) - a_2(z)}\right) \\ &\quad + N(r, \alpha(g)) + o\{T(r, \alpha(g))\} \\ &= o\{T(r, \alpha(g))\} . \end{aligned} \tag{8}$$

This of course is impossible.

In case (ii), by using a result of Edrei and Fuchs [1] we conclude first that the exponent of convergence of the zeros of f is zero. Thus f can be expressed as

$$f(z) = \pi(z)e^{\alpha_1(z)} \tag{9}$$

where $\alpha_1(z)$, $\pi(z)$ are entire functions and the order of $\pi(z)$ is zero.

From this we have

$$\begin{aligned} f(g(z)) &= \pi(g(z))e^{\alpha_1(g(z))} \\ &= L(z)\exp(H(z)e^{z^k} + S(z)) . \end{aligned} \tag{10}$$

Hence,

$$L(z) = \pi(g(z))e^{\beta_1(z)} \tag{11}$$

and

$$\alpha_1(g(z)) = \beta_1(z) + H(z)e^{z^k} + S(z) , \tag{12}$$

where $\beta_1(z)$ is an entire function.

Since $\rho(g) < k$ and $\rho(\pi) = 0$, one can conclude from (11) by an application of a result of Polya [12, Theorem 2, pp. 12–13] that the order of β_1 is less than k. For otherwise the order of $L(z)$ would be infinite, which contradicts the hypothesis that L is of finite order. Furthermore since $\rho(g) < k$ it follows from (12) that α_1 cannot be a polynomial. But then $H(z)e^{z^k} + \beta_1(z) + S(z)$ has a factorization $\alpha_1(g)$ with both α_1 and g being transcendental entire. This is impossible again according to Lemma 4 unless $\beta_1(z) + S(z)$ is a constant. But then one can apply Theorem B to conclude that (12) is impossible to hold. Thus anyway we have proved that F is E-pseudo prime. Now we show F is pseudo-prime. Suppose there exist f meromorphic and g entire such that $F = f(g)$. We shall show that if g is transcendental then f has to be a rational function.

We shall only consider the case when f has exactly one pole with multiplicity n, say ($n \geqq 1$). Hence we can express f as

$$f(w) = \frac{h(w)}{(w - a)^n} \tag{13}$$

and hence

$$g(z) = e^{\alpha(z)} + a \tag{14}$$

where h and α are entire functions.

Thus

$$f(g)(z) = \frac{h_1(e^{\alpha(z)})}{e^{n\alpha(z)}} = h_2(\alpha(z)) \tag{15}$$

where $h_1(w) = h(w + a)$, $h_2(w) = h_1(e^w)e^{-nw}$.

We have already proved that F must be E-pseudo-prime and we conclude that either (a) $\alpha(z) = Q(z)$ a polynomial or (b) the left factor $h_2(w)$ is non-constant polynomial.

If case (a) holds, then we have

$$\begin{aligned} h_1(e^{Q(z)}) &= f(g)(z) \cdot e^{nQ} \\ &= L(z)e^{Hz^k} + S + nQ \,. \end{aligned} \tag{16}$$

It follows that either $e^{Q(z)}$ reduces to a polynomial or h is a polynomial. The former case is impossible, hence we conclude that h_1 is a polynomial. But then the left side of (16) is of finite order and right side of infinite order, a contradiction. Thus case (a) is ruled out. In case (b) we have

$$h_2(w) = \frac{h_1(e^w)}{e^{nw}} \,. \tag{17}$$

Clearly, the above expression can be a polynomial if and only if $h_1(e^w) = c_1 e^{nw}$, i.e., $h_1(w) = c_1 w^n$ a monomial, where c_1 is a non-zero constant. Then $h_2(w)$ reduces to a constant, a contradiction. Thus, we conclude that f cannot have a pole. Thus F does not possess any non-entire left factor and we have proved that F is pseudo-prime.

3.3. Proof of Theorem 3. Set $F(z) = L(z)\exp(H(z)e^z + S(z))$. Then according to Theorem 2, the only possible non-trivial factorization of $F(z)$ is either of the form (i) $F(z) = p(f(z))$ or of the form (ii) $F(z) = f(p(z))$ for some non-linear polynomial $p(z)$ and transcendental entire function $f(z)$ (which must be of infinite order). Again according to the Picard -Borel Theorem in case (i) p must assume the form $p(z) = c(z - a)^n$ for some constants $c \neq 0$, a, and integer $n \geqq 2$. But then $L(z)$ would have the form $L(z) = [K(z)]^m$ for some integer $m \geqq 2$ and some entire function $K(z)$, contradicting the hypothesis. In case (ii), set $f(z) = \Pi(z)e^{\alpha(z)}$ with $\Pi(z)$ being the canonical product formed with the zero of f. Clearly, the exponent of convergence of $\Pi(z)$ is less than $1/d$ (d is the degree of $p(z)$) and hence the order of $\Pi(z)$ is less than one. Therefore, from $f(p(z)) = L(z)\exp(H(z)e^z + S(z))$ we have

$$\alpha(p(z)) = H(z)e^z + S(z) + c_0 \,, \tag{18}$$

where c_0 is a constant. But according to a result of Goldstein [4, Corollary of Theorem 6, p. 503] $H(z)e^z + S(z) + c_0$ is a prime function, thus case (ii) is also ruled out. This completes the proof of the theorem.

4. Final Remark. In Theorem 2, the condition that L must have at least one zero cannot be removed from the statement. A counter example is given by $k = 2$, $H = \sin z$, $L = e^z$. Then $F = L(z) \exp (H(z)e^{z^k}) = e^z \exp (\sin z\, e^{z^2})$ has a factorization $f(g)$ where $g(z) = z + (\sin z)e^{z^2}$, $f(z) = e^z$.

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