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COMPACTNESS OF A CLASS OF VOLTERRA OPERATORS

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Abstract. A necessary and sufficient condition is given for integral operators of the form $Tf(x) = h(x)\int_0^x k(t)f(t)dt$ to be compact from $L^p[0, 1]$ into $L^s[0, 1]$, $1 < p \leqq s < +\infty$.

Recently, D. W. Boyd and J. A. Erdös [1] established necessary and sufficient conditions for certain integral operators to map $L^p[0, 1]$ into itself. It turns out that this result was proven earlier in the form of a generalized Hardy's inequality (see Muckenhoupt [4] for a simple proof and earlier references). Using their methods and a theorem of Ando, we give necessary and sufficient conditions for the operators to be compact.

The integral operators are of the form

$$Tf(x) = h(x)\int_0^x k(t)f(t)dt$$

where h and k are measurable functions on [0, 1]. Following Boyd and Erdös, we define

$$H(x) = \left\{\int_x^1 |h(t)|^s dt\right\}^{1/s}$$

$$K(x) = \left\{\int_0^x |k(t)|^q dt\right\}^{1/q}$$

and

$$\nu(T) = \sup_{0<x<1} H(x)K(x) ,$$

where $1/p + 1/q = 1$, $0 < p \leqq s < +\infty$. We shall also need certain projection operators defined by $P_D f(x) = f(x)\chi_D(x)$ where χ_D is the characteristic function of a measurable set D in [0, 1].

Our results depend on the following theorems.

THEOREM A (see [1] and [4]). *T is a bounded operator from $L^p[0, 1]$*

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into $L^s[0, 1]$, $1 < p \leqq s < +\infty$, *if and only if* $\nu(T) < +\infty$, *and, in fact,*

$$\nu(T) \leqq \| T \| \leqq A_{p,s}\nu(T) .$$

THEOREM B (Ando [2]; also see [3], p. 92). *A linear regular integral operator from* L^p *to* L^s, *where* $1 < p \leqq s < +\infty$, *is compact, if and only if*

$$\lim_{mD^*+mD\to 0} \| P_{D^*}TP_D \|_{L^p\to L^s} = 0 .$$

NOTE. A regular integral operator is an integral operator from L^p to L^s for which the absolute value of the kernel also defines an integral operator from L^p to L^s (e.g. see [3]). By Theorem A, the operators under consideration are regular.

We have the following theorem.

THEOREM 1. *The operator* T *is compact from* $L^p[0, 1]$ *to* $L^s[0, 1]$, $1 < p \leqq s < +\infty$, *if and only if* $\nu(T) < +\infty$ *and*

$$(*) \qquad \lim_{x\to 0} H(x)K(x) = \lim_{x\to 1} H(x)K(x) = 0 .$$

PROOF. Suppose that $H(x)K(x)$ satisfies condition $(*)$. Observe that the operator $P_{D^*}TP_D$ is one of our operators with kernel $\chi_{D^*}(x)h(x)k(t)\chi_D(t)$. Let $H_{D^*}(x)$ and $K_D(x)$ be the H and K corresponding to this kernel.

Let $\varepsilon > 0$ be given. Choose x_0 and x_1, $0 < x_0 < x_1 < 1$, so that (i) $H(x)K(x) < \varepsilon/A_{p,s}$ whenever $0 < x < x_0$ or $x_1 < x < 1$, and (ii) $H(x_1) \neq 0$. Choose δ so that $K_D(x_0) < \varepsilon/(A_{p,s}H(x_1))$ whenever $mD < \delta$. Then for $x_0 \leqq x \leqq x_1$, $H_{D^*}(x)K_D(x) \leqq H(x_1)K_D(x_0) < \varepsilon/A_{p,s}$. Thus, for $0 < x < 1$, $H_{D^*}(x)K_D(x) < \varepsilon/A_{p,s}$. By Theorem A, we have $\| P_{D^*}TP_D \| < \varepsilon$ whenever $mD < \delta$, and the sufficiency follows from Theorem B.

Suppose that $\nu(T) < +\infty$ and $(*)$ does not hold. We consider the case $\lim_{x\to 0} H(x)K(x) \neq 0$. Then there exists a sequence of points $\{\xi_i\}_{i=1}^{\infty}$, $0 < \xi_i < 1$, $\xi_i \downarrow 0$, and positive constants C_1 and C_2 such that

$$C_1 \leqq H(\xi_i)K(\xi_i) \leqq C_2 .$$

Let $D_j^* = D_j = (0, \xi_j)$ and $i > j$. Then, by Theorem A,

$$\| P_{D_j^*}TP_{D_j} \| \geqq \sup_{0<x<1} H_{D_j^*}(x)K_{D_j}(x) \geqq H_{D_j^*}(\xi_i)K_{D_j}(\xi_i) .$$

But,

$$\begin{aligned} [H_{D_j^*}(\xi_i)K_{D_j}(\xi_i)]^s &= \int_{\xi_i}^{\xi_j} | h(x) |^s dx \left(\int_0^{\xi_i} | k(x) |^q dx\right)^{s/q} \\ &= H(\xi_i)^s K(\xi_i)^s \left[1 - \frac{H(\xi_j)^s}{H(\xi_i)^s}\right] \\ &\geqq C_1^s[1 - (H(\xi_j)^s)/(H(\xi_i)^s)] . \end{aligned}$$

Observing that $\lim_{x\to 0} K(x)=0$ and $\lim_{\xi_i\downarrow 0} K(\xi_i)H(\xi_i)\neq 0$ implies $\overline{\lim}_{\xi_i\downarrow 0} H(\xi_i)=+\infty$, we obtain

$$\sup_{\xi_i} H_{D_j^*}(\xi_i)K_{D_j}(\xi_i) \geqq C_1 .$$

Hence, $\| P_{D_j^*}TP_{D_j}\| \geqq C_1 > 0$, independent of j. Therefore, by Theorem B, T is not compact.

The case $\lim_{x\to 1} H(x)K(x) \neq 0$ is handled in an analogous way by choosing $\xi_i \uparrow 1$ and $D_j^* = D_j = (\xi_j, 1)$. The theorem is proven.

For the case $1 < s < p < +\infty$, neither $\nu(T) < +\infty$ nor $\nu(T) < +\infty$ and $\lim_{x\to 0} H(x)K(x) = \lim_{x\to 1} H(x)K(x) = 0$ is sufficient for the boundedness of T. Indeed, the kernel $k(t) = (1-t)^{-1/q-1/s}[\log(\log 1/(1-t)+1)+1]^{-1/s}$ applied to the L^p function $(1-t)^{-1/p}[\log 1/(1-t)+1]^{-1/s}$ provides the counterexample.

REFERENCES

[1] D. W. BOYD AND J. A. ERDÖS, Norm inequalities for a class of Volterra operators.

[2] T. ANDO, On compactness of integral operators, Nederl. Akad. Wetensch. Proc., Ser. A-65, Indag. Math., 24 (1962), 235–239.

[3] M. A. KRASNOSEL'SKII, P. P. ZABREIKO, E. J. PUSTYL'NIK, AND P. E. SOBOLEVSKII, Integral Operators in Spaces of Summable Functions, Izdat. "Nauk", Moscow, 1966.

[4] B. MUCKENHOUPT, Hardy's inequality with weights, Studia Mathematica, 44 (1972), 31–38.

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