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ON THE AVERAGES OF FOURIER-STIELTJES COEFFICIENTS

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Let m be a finite Borel measure (or, more generally, a pseudo-measure) on $[-\pi, \pi]$. If $m \sim (1/2)a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$, m has a canonical decomposition into an even measure μ and an odd measure ν defined by $\mu \sim (1/2)a_0 + \sum_1^\infty a_n \cos nx$, $\nu \sim \sum_1^\infty b_n \sin nx$. We shall assume for convenience $a_0 = 0$ and $m\{0\} = 0$. We consider the arithmetic means of the coefficients $\{a_n\}$ and $\{b_n\}$. Let

$$A_n = \frac{1}{n}\sum_{j=1}^{n} a_j,\ B_n = \frac{1}{n}\sum_{j=1}^{n} b_j ,$$

and the formal series operation H be defined by

$$(1) \qquad H(m) \sim \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx .$$

The operator H^* adjoint to H is defined for formal series by

$$(2) \qquad H^*(m) \sim \sum_{n=1}^{\infty} A_n^* \cos nx + \sum_{n=1}^{\infty} B_n^* \sin nx$$

where $A_n^* = \sum_n^\infty (a_j/j)$ and $B_n^* = \sum_n^\infty (b_j/j)$. The adjoint coefficients do not exist for all pseudo-measures or even measures. However, if

$$\sum_{n=1}^{\infty} b_n \sin nx \sim f \in L_1(-\pi, \pi) ,$$

the B_n^* surely exist ([16], p. 59).

The conjugates of (1) and (2) are

$$(1') \qquad \tilde{H}(m) \equiv (H(m))^\sim \sim \sum_{n=1}^{\infty} A_n \sin nx - \sum_{n=1}^{\infty} B_n \cos nx$$

and

$$(2') \qquad \tilde{H}^*(m) \equiv ((H^*)(m))^\sim \sim \sum_{n=1}^{\infty} A_n^* \sin nx - \sum_{n=1}^{\infty} B_n^* \cos nx .$$

The operator H was first studied by Hardy [3], who showed that for $1 < p < \infty$, $H(f)$ is the Fourier series of a function in $L_p(-\pi, \pi)$ if $f \in L_p(-\pi, \pi)$. H^* was studied by Bellman [1], Kawata [7], and Sunouchi [13] who showed that Hardy's theorem was true if H it replaced by H^*.

Since conjugation maps L_p onto itself, the corresponding results for $\tilde{H}$ and $\tilde{H}^*$ are immediate. For spaces not closed under the conjugation operator, however, the Fourier character of $\tilde{H}$ and $\tilde{H}^*$ is more delicate, and there is a failure of symmetry between even and odd measures.

We shall give conditions on m for the series in (1) and (2) and (1′) and (2′) to belong to various classes of functions. The integrability of these series depends on the behavior of m in a neighborhood of 0. We shall show among other results:

THEOREM A. *If m satisfies*

$$\int_{-\pi}^{\pi}\left(\log \frac{\pi}{|x|}\right)^{\alpha} d\,|m|\,(x) < \infty\,,$$

with $\alpha > 0$, then if m is even

$$H(m) \in L(\log^+ L)^{\alpha}, \qquad \tilde{H}(m) \in L(\log^+ L)^{\alpha-1}$$
$$H^*(m) \in L(\log^+ L)^{\alpha-1}, \ \tilde{H}^*(m) \in L(\log^+ L)^{\alpha-2}$$

whenever the exponent of $\log^+ L$ is nonnegative, and if m is odd

$$H(m) \in L(\log^+ L)^{\alpha}, \qquad \tilde{H}(m) \in L(\log^+ L)^{\alpha}$$
$$H^*(m) \in L(\log^+ L)^{\alpha-1}, \ \tilde{H}^*(m) \in L(\log^+ L)^{\alpha}$$

whenever the exponent of $\log^+ L$ is nonnegative. Whenever the range space is L, these results are the best possible in the sense that if m is positive on $[0, \pi]$, the condition is also necessary.

From the results in Theorem A we use the adjoint relation to obtain results on the growth of the H-functions at the origin which, in many cases, are also best possible. Our results extend, complement, and improve the results of Goes [3], Loo [10], and others [14], [8]. We shall also give sufficient conditions for the H series to represent functions in the Lorentz space Λ_{pq} [11], [5]. We have

THEOREM B. *Let $1 \leqq p < \infty, 1 \leqq q \leqq \infty$.*

(i) *If m satisfies*

$$\text{(0)} \qquad \int_{-\pi}^{\pi} t^{-1/p'} d\,|m|\,(t) < \infty\,, \qquad \frac{1}{p} + \frac{1}{p'} = 1\,,$$

$H(m)$, $H^*(m)$, and their conjugates are in every Λ_{pq}, $1 \leqq q \leqq \infty$. In particular, condition (0) is sufficient for $H(m)$, $H^*(m)$, and their conjugates to be in $\Lambda(p, 1)$.

(ii) *If $f \in \Lambda_{pq}$ and, in particular, if f is in weak L_p, then*

$$t^{1/p'} H(f),\ t^{1/p'} H^*(f),\ t^{1/p'} \tilde{H}(f), \ \text{ and } \ t^{1/p'} \tilde{H}^*(f) \in L_\infty\,.$$

The result (i) improves Petersen's results [12] for Λ_{pq} when $q = 1$, since $\int_{-\pi}^{\pi} t^{-1/p'} f(t)dt < \infty$ implies $f \in \Lambda(p, 1)$ ([16], vol. II, p. 124). Furthermore, we may use interpolation theorems to infer Petersen's and also Hardy's and Bellman's and Kawata's theorems from (i). Both (i) and (ii) give improvements on some results of Izumi [6]. Theorems A and B as other related theorems will be proved in the theorems and lemmas below.

We begin by writing the H-operators as integral operators generated by kernels $K(x, t)$.

We note that

$$A_n = \frac{1}{\pi n}\int_{-\pi}^{\pi} D_n(t)d\mu(t) = \frac{1}{\pi n}\int_{-\pi}^{\pi} \frac{\sin nt}{2\tan t/2} d\mu(t) + \frac{1}{\pi n}\frac{1}{2}\int_{-\pi}^{\pi} \cos ntd\mu(t)\ ,$$

$$B_n = \frac{1}{\pi n}\int_{-\pi}^{\pi} \tilde{D}_n(t)d\nu(t) = \frac{1}{\pi n}\int_{-\pi}^{\pi} \frac{1-\cos nt}{2\tan t/2} d\nu(t) + \frac{1}{\pi n}\frac{1}{2}\int_{-\pi}^{\pi} \sin nt\, d\nu\ ,$$

where $D_n(t) = \sum_{j=0}^{n} \cos jt$ is the Dirichlet kernel and $\tilde{D}_n(t)$ is its conjugate ([16], 5.2, p. 50). Therefore, if $\chi_t(x)$ is the characteristic function of $[-|t|, |t|]$,

$$\begin{aligned} A_n - \frac{a_n}{2\pi n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\left(\int_{-\pi}^{\pi} \chi_t(x)\cos nx\, dx\right)d\mu(t) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\chi_t(x)d\mu(t)\right)\cos nx\, dx\ . \\ B_n - \frac{b_n}{2\pi n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\tan t/2}\left(\int_{-\pi}^{\pi} \operatorname{sgn}(x)\,\chi_t(x)\sin nx\, dx\right)d\nu(t) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} \frac{\operatorname{sgn}(x)}{2\tan t/2}\chi_t(x)d\nu(t)\right)\sin nx\, dx\ . \end{aligned}$$

So that, formally, the series (1) is the Fourier series of the function $H(m)$ defined by

$$(3)\qquad \begin{aligned} H(m)(x) = \frac{1}{2\pi}\Bigg[\int_{-\pi}^{\pi} &\frac{1}{|2\tan t/2|}\chi_t(x)d\mu(t) \\ &+ \int_{-\pi}^{\pi} \frac{\operatorname{sgn}(x)}{2\tan t/2}\chi_t(x)d\nu(t) + \tilde{F}(x)\Bigg]\ , \end{aligned}$$

where $F(x) = \int_{-\pi}^{x} dm(t)$ and the above integrals may be regarded as function valued integrals, improper around $t = 0$ or, for each fixed x as the integrals

$$\int_{|t|>|x|} \frac{1}{|2\tan t/2|}d\mu(t) \quad \text{and} \quad \operatorname{sgn}(x)\int_{|t|>|x|} \frac{1}{2\tan t/2}d\nu(t)$$

which are finite for $x \neq 0$. From (3) we immediately obtain for the adjoint transformation H^*

$$(4) \qquad H^*(m_1)(t) = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\chi_t(x)d\mu_1(x) + \int_{-\pi}^{\pi} \frac{\operatorname{sgn}(x)}{2\tan t/2}\chi_t(x)d\nu_1(x) - \tilde{F}_1(x)\right] .$$

The last term is obtained by observing that the operation $m \rightarrow \tilde{F}$ is a skew symmetrc multiplication operation.

It is immediately verifiable from the definition of H that $H(\tilde{m}) = (H(m))^\sim$. Thus, $\tilde{H}(m) = H(\tilde{m})$, we obtain from (3)

$$(5) \qquad \tilde{H}(m)(x) = \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\tilde{\chi}_t(x)d\mu(t) + \int_{-\pi}^{\pi} \frac{1}{2\tan t/2}[\operatorname{sgn}(x)\chi_t(x)]^\sim d\nu(t) - F(x)\right\}$$

which is valid whenever the function valued integrals in (3) and (5) converge in normed space because conjugation is a closed operator. We observe that the equation $\tilde{H}(m) = H(\tilde{m})$ implies by taking adjoints

$$(\tilde{H}(m_1))^* = (H(m_1))^* = (H^*(m_1))^\sim \equiv \tilde{H}^*(m_1)$$

Therefore $(H^*)^\sim$ is the adjoint of $\tilde{H}$, so we have from (5)

$$(6) \qquad \tilde{H}^*(m_1)(t) = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\tilde{\chi}_t(x)d\nu_1(x) + \int_{-\pi}^{\pi} \frac{1}{2\tan t/2}[\operatorname{sgn}(x)\chi_t(x)]^\sim d\mu_1(x) + F_1(t)\right] .$$

Our first theorem gives sufficient conditions for the function valued integrals in (3) and (4) to exist in some Banach spaces.

THEOREM 1. *Let m be a finite Borel measure on $[-\pi, \pi]$ and T_i and T_i^*, $i = 0, 1$, be the mutually adjoint integral transforms of m defined by*

$$(7) \qquad \begin{aligned} T_i(m)(x) &= \pi(\operatorname{sgn} x)^i \int_{-\pi}^{\pi} \frac{(\operatorname{sgn} t)^{i+1}}{2\tan t/2}\chi_t(x)dm(t) \\ T_i^*(m_1)(t) &= \frac{\pi(\operatorname{sgn} t)^{i+1}}{2\tan t/2}\int_{-\pi}^{\pi}(\operatorname{sgn} x)^i\chi_t(x)dm_1(x) . \end{aligned} \qquad i = 0, 1 ,$$

Let $\alpha \geqq 0$ and

$$\phi_\alpha(u) = u(\log^+ u)^\alpha .$$

Then if

$$\int_{-\pi}^{\pi}\left(\log\frac{\pi}{|w|}\right)^{\alpha} d\,|m|\,(w) < \infty\,, \tag{8}$$

we have

$$\int_{-\pi}^{\pi}\phi_\alpha(|\,T_i m(x)\,|)dx < \infty\,, \qquad \text{if } \alpha \geqq 0\,, \tag{9}$$

$$\int_{-\pi}^{\pi}\phi_{\alpha-1}(|\,T_i^*(m)(t)\,|)dt < \infty\,, \qquad \text{if } \alpha \geqq 1\,. \tag{10}$$

PROOF. We may assume without loss of generality that $\int_{-\pi}^{\pi} d|m| = 1$. Since $\phi_\alpha(u)$ is a convex increasing function of u on $[0, \infty)$, we may apply Jensen's inequality to the absolute values of the quantities in (7) and obtain

$$\begin{aligned}\phi_\alpha(|\,T_i m(x)\,|) &\leqq \int_{-\pi}^{\pi}\phi_\alpha\left(\frac{\pi}{|2\tan t/2|}\chi_t(x)\right)d\,|m|\,(t)\,,\\ \phi_{\alpha-1}(|\,T_i^* m(t)\,|) &\leqq \int_{-\pi}^{\pi}\phi_{\alpha-1}\left(\frac{\pi}{|2\tan t/2|}\chi_t(x)\right)d\,|m|\,(x)\,.\end{aligned} \qquad i = 1, 2\,,$$

Integrating these inequalities and applying Fubini's theorem, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi}\phi_\alpha(|\,T_i m(x)\,|)dx &\leqq \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\phi_\alpha\left(\frac{\pi}{|2\tan t/2|}\chi_t(x)\right)dx\right)d\,|m|\,(t)\,,\\ \int_{-\pi}^{\pi}\phi_{\alpha-1}(|\,T_i^* m(t)\,|)dt &\leqq \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\phi_{\alpha-1}\left(\frac{\pi}{2\tan t/2}\chi_t(x)\right)dt\right)d\,|m|\,(x)\,.\end{aligned} \tag{11}$$

Evaluating the inner integrals, we obtain for the first

$$\begin{aligned}&\int_{-\pi}^{\pi}\phi_\alpha\left(\frac{\pi}{|2\tan t/2|}\chi_t(x)\right)dx\\ &\quad = \int_{-t}^{t}\frac{\pi}{|2\tan t/2|}\left(\log^+\frac{\pi}{|2\tan t/2|}\right)^{\alpha}d\alpha \cong \left(\log\frac{\pi}{|t|}\right)^{\alpha}\end{aligned}$$

as $t \to 0$, and for the second

$$\begin{aligned}&\int_{-\pi}^{\pi}\phi_{\alpha-1}\left(\frac{\pi}{|2\tan t/2|}\,\chi_{[-|x|,|x|]^c}(t)d(t)\right.\\ &\quad = 2\int_{|x|}^{\pi}\phi_{\alpha-1}\left(\frac{\pi}{|2\tan t/2|}\right)dt \cong 2\int_{|x|}^{\pi}\frac{\pi}{t}\left(\log\frac{\pi}{t}\right)^{\alpha-1}dt\\ &\quad = \frac{-2}{\alpha}\left(\log\frac{\pi}{t}\right)^{\alpha}\Big|_{|x|}^{\pi} \cong \left(\log\left|\frac{\pi}{x}\right|\right)^{\alpha}.\end{aligned} \tag{12}$$

Using these estimates in (11), we see that (9) and (10) hold because of (8).

We designate by $\mathscr{L}_\alpha$ the Orlicz space of those functions f on

$[-\pi, \pi]$ satisfying $\int_{-\pi}^{\pi} \phi_\alpha(|f(x)|)dx < \infty$ ([16], p. 170) and by $\mathscr{M}_\alpha(1/x)$, the space of those measures m on $[-\pi, \pi]$ satisfying $\int_{-\pi}^{\pi} \log(\pi/|x|)d\,|m|\,(x) < \infty$. $\mathscr{L}_\alpha(1/x)$ shall denoto the subset of $\mathscr{M}_\alpha(1/x)$ consisting of absolutely continuous measures. We note that $\mathscr{L}_1 = L\log^+ L$ and $\mathscr{L}_0 \equiv L$.

THEOREM 2. (a) *The condition $m \in \mathscr{M}_\alpha(1/x)$ is sufficient for $H(m) \in \mathscr{L}_\alpha$ and $\tilde{H}(m) \in \mathscr{L}_{\alpha-1}$. When $\alpha = 1$, this condition is also necessary for $\tilde{H}(m) \in L$ if m is even and positive.*

(b) *The condition $m \in \mathscr{M}_\alpha(1/x)$ is sufficient for $H^*(m) \in \mathscr{L}_{\alpha-1}$, $\alpha \geqq 1$. If $\alpha = 1$ and the even and odd parts of m are positive, the condition is also necessary.*

(c) *The condition $m \in \mathscr{M}_\alpha(1/x)$ is sufficient for $\tilde{H}^*(m) \in \mathscr{L}_{\alpha-2}$, $\alpha \geqq 2$. If $\alpha = 2$ and m is even and positive, the condition is also necessary.*

PROOF. For (a) we apply Theorem 1 to the decomposition (3) of $H(m)$ and observe that since $\tilde{F}(x)$ is in L_2 (it is actually exponentially integrable), it is in $\mathscr{L}_\alpha$ for all α. Therefore, $H(m) \in \mathscr{L}_\alpha$ which implies $\tilde{H}(m) \in \mathscr{L}_{\alpha-1}$ by ([16], p. 296). To prove the sufficiency parts of (b) and (c), we apply Theorem 1 to the decomposition (4) and argue as in (a). To prove the necessity part in (b) we note the order of the evaluation in (12), and that if μ and ν are positive in $[0, \pi]$, we have that $H^*(m)$ is essentially positive for $t \geqq 0$, for by (4),

$$\begin{aligned} H^*(m)(t) &= H^*(\mu)(t) + H^*(\nu)(t) \\ &= \int_{-\pi}^{\pi} \frac{1}{|2\tan t/2|}\chi_t(x)d\mu(x) + \int_{-\pi}^{\pi} \frac{1}{2\tan t/2}\chi_t(x)d\,|\nu|\,(x) + \tilde{F}(t) \end{aligned}$$

with $\tilde{F}(t) \in L_2$. Hence the first two terms can be rewritten for $t \geqq 0$ as

$$\int_{-\pi}^{\pi} \frac{1}{2\tan t/2}\chi_t(x)d\,|\,m\,|\,(x)\ .$$

Thus, by Fubini's theorem,

$$\begin{aligned} \int_0^{\pi} H^*(m)(t)dt &= \int_0^{\pi}\left(\int_{-\pi}^{\pi} \frac{1}{2\tan t/2}\chi_t(x)d\,|\,m\,|\,(x)\right)dt + \int_0^{\pi}|\tilde{F}(t)|\,dt \\ &\int_{-\pi}^{\pi} \log\left|\frac{\pi}{x}\right| d\,|m|\,(x) + O(1) = \infty \end{aligned}$$

if $m \notin \mathscr{M}_1(1/x)$ so that $H^*(m) \notin L[-\pi, \pi]$.

We postpone the proofs of the necessity parts of (a) and (c). (See Theorem 4(a).)

REMARKS. The results for $\tilde{H}$ for $\alpha = 1$ in (a) are an improvement on Loo's Theorems 3 and 4 [10]. For $\alpha = 1$, the sufficiency part of (b) is Loo's Theorems 5 and 6. The necessity part shows that his results are best possible. The result for $\tilde{H}$ in part (c) is a considerable improvement on Loo's Theorems 7 and 8. Kinukawa and Igari [8] have remarked that in case m is odd, the condition $m \in \mathscr{M}(1/x)$ is sufficient for $\tilde{H}^*(m) \in L$.

We turn now to some dual results. We recall that the topological dual of $L \log^+ L$ is the exponentially integrable functions $\exp(L)$ ([9], p. 217). That of $\mathscr{L}_\alpha(1/x)$ is L_∞.

THEOREM 3. *If* $h \in \exp(L)$,

(a) $$\frac{1}{\log \pi/|x|} H^*(h) \in L_\infty$$

(b) $$\frac{1}{\log^2 \pi/|x|} H(h) \in L_\infty .$$

If $h \in L_\infty$,

(c) $$\frac{1}{\log \pi/|x|} \tilde{H}^*(h) \in L_\infty$$

(d) $$\frac{1}{\log^2 \pi/|x|} \tilde{H}(h) \in L_\infty .$$

(e) *If h is odd, the results* (c) *and* (d) *are best possible, but not if h is even.*

(f) *If h is even or odd, the results* (a) *and* (b) *are best possible.*

PROOF. By Theorem 2(a), $H(h_1) \in L \log^+ L$ if $h_1 \in \mathscr{L}_1(1/x)$. Thus, by the closed graph theorem, H must satisfy

$$\| H(h_1) \|_{L\log^+ L} \leqq B \int_{-\pi}^{\pi} \log \frac{\pi}{|x|} | h_1(x) | dx , \qquad h_1 \in \mathscr{L}_1\left(\frac{1}{x}\right) .$$

If $\| h_1 \|_{\mathscr{L}_1(1/x)} = 1$,

$$B \| h \|_{\exp(L)} \geqq \left| \int_{-\pi}^{\pi} H(h_1)(x) h(x) dx \right| = \left| \int_{-\pi}^{\pi} H^*(h)(t) h_1(t) dt \right|$$
$$= \left| \int_{-\pi}^{\pi} \frac{1}{\log \pi/|t|} H^*(h)(t) h_1(t) \log \frac{\pi}{|t|} dt \right| .$$

Therefore,

$$\left\| \frac{1}{\log \pi/|t|} H^*(h) \right\|_\infty = \sup \left\{ \left| \int_{-\pi}^{\pi} \frac{1}{\log \pi/|t|} H^*(h)(t) h_1(t) \log \frac{\pi}{|t|} \right| dt,\ \| h_1 \|_{\mathscr{L}_1(1/x)} = 1 \right\}$$
$$\leqq B \| h \|_{\exp(L)} .$$

This proves (a). The proofs of (b), (c), and (d) are similar.

It is known that $\tilde{H}$ and $\tilde{H}^*$ map even L_∞ into L_∞ [14], [8], (see also Theorem 4 below). We consider the 2π periodic odd function g defined by $g(t) = (1/2)(\pi - t)$, $t \in [-\pi, \pi)$. Since $g \sim \sum_1^\infty (\sin nt)/n$ and

$$\tilde{g} = \log \frac{1}{|2 \sin t/2|} ,$$

$$\begin{aligned}\tilde{H}(g)(x) &= H(\tilde{g})(x) \\ &= \int_x^\pi \frac{-1}{2 \tan t/2} \log \left| 2 \sin \frac{t}{2} \right| dt + \sum_1^\infty \frac{\cos nx}{n^2} \cong \log^2 \frac{1}{t} + O(1)\end{aligned}$$

and

$$\begin{aligned}\tilde{H}^*(g)(t) = H^*(\tilde{g})(t) &= \frac{-1}{2 \tan t/2} \int_{-t}^t \log \left| 2 \sin \frac{x}{2} \right| dx - \sum_1^\infty \frac{\cos nt}{n^2} \\ &= \frac{-1}{t} \int_0^t \log x dx + O(1) = \log \frac{1}{|t|} + O(1) ,\end{aligned}$$

which shows that (c) and (d) are best possible for odd h.

To prove (f), we note that the g above is even and exponentially integrable, and that from the proof of (e), we have

$$H^*(\tilde{g}) = \tilde{H}^*(g) = \log \frac{1}{|t|} + O(1)$$

and

$$H(\tilde{g}) = \tilde{H}(g) = \log^2 \frac{1}{|t|} + O(1) .$$

For odd h let $h(t) = g_1(t) = \operatorname{sgn} t\tilde{g}(t)$. Then $g_1(t)$ is odd and exponentially integrable. From (3) and (4), we obtain

$$H(g_1)(x) = \operatorname{sgn} xH(\tilde{g})(x) + O(1)$$

and

$$H^*(g_1)(t) = \operatorname{sgn} tH^*(\tilde{g}) + O(1) .$$

REMARKS. (d) is Loo's Theorem 15 [10]. It has been observed by Wang [15] that $\tilde{H}(g) \cong \log^2 1/t$. (c) implies Loo's Theorems 11 and 12.

Theorem 1 gave estimates for $H(m)$ and $H^*(m)$ which do not depend on the evenness or oddness of m so that the results in Theorem 2 concerning $\tilde{H}(m)$ and $\tilde{H}^*(m)$ do not distinguish between even and odd m. However, if m is specialized to even or odd in some cases considerably improved results can be obtained for $\tilde{H}(m)$ and $\tilde{H}^*(m)$. Our method will be to make estimates of the conjugate kernels appearing in (5). Explicitly, these are

$$\left(\frac{1}{|2\tan t/2|}\chi_t(x)\right)^{\sim} = \frac{1}{\pi}\frac{1}{|2\tan t/2|}\log\left|\frac{\sin((x+t)/2)}{\sin((x-t/)2)}\right| = K_1(x, t) ,$$

$$\left(\frac{\operatorname{sgn} x}{2\tan t/2}\chi_t(x)\right)^{\sim} = \frac{1}{\pi}\,\frac{1}{2\tan t/2}\log\left|\frac{\sin^2 x/2}{\sin^2 t/2 - \sin^2 x/2}\right| = K_2(x, t) .$$

LEMMA 1. *Let α be a nonnegative integer,*

(a) $$\int_{-\pi}^{\pi}|K_1(x, t)|\,dx \cong \log\left|\frac{\pi}{t}\right|$$

(b) $$\int_{-\pi}^{\pi}\phi_\alpha(|K_1(x, t)|)dt = O\left(\log\left|\frac{\pi}{x}\right|\right)^{\alpha}$$

(c) $$\int_{-\pi}^{\pi}\phi_\alpha(|K_2(x, t)|)dx = O\left(\log\left|\frac{\pi}{t}\right|\right)^{\alpha}$$

(d) $$\int_{-\pi}^{\pi}(|K_2(x, t)|)dt \cong \left(\log\left|\frac{\pi}{x}\right|\right)^{2} .$$

When $\alpha = 0$, the estimates in (a) and (d) are the best possible. For the proof of Lemma 1, we shall need

LEMMA 2. *With $a > 0$ and α a nonnegative integer, let*

$$h_\alpha(a) = \int_2^{1/a}\frac{|\log u|^{\alpha}}{u^2\sqrt{1-a^2u^2}}du ,$$

then $h_\alpha(a) = O(1)$ as $a \to 0$.

PROOF. Let $v = au$. The integral $h_\alpha(a)$ then becomes

$$a\int_{2a}^{1}\frac{(\log v/a)^{\alpha}}{v^2\sqrt{1-v^2}}dv \leqq a\left[2\int_{2a}^{1/2}\frac{1}{v^2}\left(\log\left(\frac{v}{a}\right)^{\alpha}\right)dv + 4\left(\log\frac{1}{a}\right)\int_{1/2}^{1}\frac{dv}{\sqrt{1-v^2}}\right] .$$

Since

$$\int_{2a}^{1/2}\frac{1}{v^2}\left(\log\frac{v}{a}\right)^{i}dv = \frac{1}{2a}(\log 2)^{i} - 2\left(\log\frac{1}{2a}\right)^{i} + i\int_{2a}^{1/2}\frac{1}{v^2}\left(\log\frac{v}{a}\right)^{i-1}dv ,$$

we see by induction that the first integral in the square brackets is $O(1/a)$. Since the second is also, we have the required result.

PROOF OF LEMMA 1. Since K_1 and K_2 are either even or odd in each variable, we may in each case assume that (x, t) is in the first quadrant and perform integrations of $\phi_\alpha(|K_i|)$ over the interval $[0, \pi]$.

(a) We make the substitution $u = \tan t/2 \cot x/2 = a \cot x/2$ in the integral in (a) and obtain

$$\int_0^\pi |K_1(x, t)|\,dx = \frac{1}{\pi}\int_0^\infty \log\left|\frac{u+1}{u-1}\right|\frac{du}{a^2+u^2} \tag{13}$$

$$= \int_0^{1/2A} + \int_{1/2A}^2 + \int_2^\infty = J_1 + J_2 + J_3\,,$$

A having been chosen so that

$$A \leqq u \log\left|\frac{u+1}{u-1}\right| \leqq A_1 \quad \text{if} \quad u > 2$$

(14) and

$$A \leqq \frac{1}{u}\log\left|\frac{u+1}{u-1}\right| \leqq A_1 \quad \text{if} \quad u < \frac{1}{2}\,.$$

By using (14) we can easily see that $J_2 = O(1)$, $J_3 = O(1)$, and

$$J_1 \cong \int_0^{1/2A}\frac{udu}{a^2+u^2} \cong \log\frac{1}{t}\,.$$

(b) If we make the substitution $u = \cot x/2 \tan t/2 = a \tan t/2$ and expand the α power,

$$\int_0^\infty \phi_\alpha(|K_1(x, t)|)dt$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{a}{u}\log\left|\frac{u+1}{u-1}\right|\left(\log^+\left(\frac{a}{2\pi}\frac{1}{u}\log\left|\frac{u+1}{u-1}\right|\right)\right)^\alpha\frac{a}{u^2+a^2}du$$

$$\leqq \frac{1}{\pi}\int_0^\infty \frac{1}{u}\log\left|\frac{u+1}{u-1}\right|\cdot\left|\log\frac{a}{2\pi}\frac{1}{u}\log\left|\frac{u+1}{u-1}\right|\right|^\alpha du$$

$$\leqq \frac{1}{\pi}\sum_{i=0}^\infty c_i I_i \cdot O\,(\log(a))^{\alpha-i} = O\left(\log\frac{1}{t}\right)^\alpha$$

since by using (14), we see

$$I_i = \int_0^\infty \frac{1}{u}\log\left|\frac{u+1}{u-1}\right|\cdot\left|\log\frac{1}{u}\log\left|\frac{u+1}{u-1}\right|\right|^i du$$

$$= \int_0^{1/2} + \int_{1/2}^2 + \int_2^\infty = O(1) + O(1) + O(1)\,.$$

(c) If we substitute $u = (\sin x/2)/(\sin t/2) = (\sin x/2)/a$ and proceed as before, we obtain

$$\int_0^\pi \phi_\alpha(|K_2(x, t)|)dx$$

$$\leqq \frac{2\cos t/2}{\pi}\sum_{i=0}^\alpha c_i\cdot O\left(\log\frac{1}{2\pi a}\right)^{\alpha-i}\int_0^{1/a}\frac{\log\left|\frac{u^2}{1-u^2}\right|\left|\log\left|\log\left|\frac{u^2}{1-u^2}\right|\right|\right|^i}{\sqrt{1-a^2u^2}}du$$

$$\leqq B\sum_{i=0}^\alpha c_i\cdot O\left(\log\frac{1}{t}\right)^{\alpha-i} I_i\,.$$

But

$$I_i = \int_0^{1/a} = \int_0^2 + \int_2^{1/a} \leqq \frac{1}{\sqrt{1-a^2/4}} \int_0^2 k(h)du + B_1 \int_2^{1/a} \frac{1}{u^2} \frac{(2\log u/B)^i}{\sqrt{1-a^2u^2}} du$$

where $k(u)$ is integrable over $[0, 2]$ and B, B_1 are such that $B/u^2 \leqq \log|u^2/(1-u^2)| \leqq B_1/u^2$ if $u > 2$. The first integral is $O(1)$ and the second is $O(1)$ by Lemma 2.

(d) Substitute $u = (\sin t/2)/(\sin x/2) = (\sin t/2)/a$ in the integral. Since the integrand is uniformly integrable on $[0, 2]$ for small a, we obtain

$$\int_0^\pi |K_2(x, t)|dt = O(1) + \int_2^{1/a} \frac{1}{u} |\log|u^2 - 1||dy \cong \left(\log\frac{1}{a}\right)^2. \tag{15}$$

THEOREM 4. *Let* $\alpha \geqq 0$.

(a) *If m is even and positive, the conditions $m \in \mathscr{M}_1(1/x)$, $m \in \mathscr{M}_2(1/x)$ are necessary for $\tilde{H}(m) \in L$ and $\tilde{H}^*(m) \in L$, respectively.*

(b) *If m is odd, the condition $\int_{-\pi}^{\pi} (\log|\pi/w|)^\alpha d|m|$ is sufficient for $\tilde{H}(m) \in L(\log^+ L)^\alpha$ and $\tilde{H}^*(m) \in L(\log^+ L)^\alpha$.*

PROOF. To prove the sufficiency part (b), apply Jensen's inequality and Fubini's theorem to the appropriate expression in equation (5) or (6) and use Lemma 1 (b) and (c). (Cf. the proof of Theorem 1.) The necessity part (a) also follows from Lemma 1 (a) and (d), and the fact that K_1 and K_2 are positive in the first quadrant. (Cf. the proof of the necessity part of Theorem 2(b).)

COROLLARY 1. (a) *If f is an even bounded function, $\tilde{H}(f)$ and $\tilde{H}^*(f)$ are bounded functions.*

(b) *If f is even and exponentially integrable, $(1/\log \pi/|x|)\tilde{H}^*(f)$ and $(1/\log \pi/|x|)\tilde{H}(f)$ are bounded.*

PROOF. (a) is proved by Theorem 4(b) with $\alpha = 0$ and a simple duality argument. For (b) use Theorem 4(b) with $\alpha = 1$, and the argument of Theorem 3.

Theorems 2 and 4 prove Theorem A.

REMARKS. The result for $\tilde{H}(m)$ with $\alpha = 0$ and m odd was obtained by Kinukawa and Igari [8] for functions and by G. Goes [3] for measures. After completing the research for this paper, the author discovered that the result for $\tilde{H}^*(m)$ with $\alpha = 0$ and m odd was an unpublished result of G. Goes who obtained it using sequence space techniques and announced it in 1970 in colloquia at the University of Chicago and Illinois Institute of Technology. Corollary 1(a) is the result of Turán [14] and

Kinukawa and Igari [8]. Part (b) improves and supplements Theorem 3 (c), (d), and (e) for even functions. This corollary may be compared with Loo's Theorems 12 and 17 [10], which it improves.

We now consider sufficient conditions for $H(m)$ and $H^*(m_1)$ to belong to the Lorentz space Λ_{pq}. The Lorentz space Λ_{pq}, $p, q \geqq 1$, is the collection of all measurable f on $[-\pi, \pi]$ such that $\|f\|_{pq}^* < \infty$ where, denoting the nondecreasing rearrangement of $|f|$ by f^*([5], p. 253),

$$\|f\|_{pq}^* = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty \left(t^{1/p}f^*(t)\right)^q \dfrac{dt}{t}\right)^{1/q} & \text{if } \quad 0 < p < \infty, 0 < q < \infty \\ \sup\limits_{t>0} t^{1/p}f^*(t) & \text{if } \quad 0 < p \leqq \infty, q = \infty\ . \end{cases}$$

We shall use well known results about these spaces, all of which can be found in Hunt's monograph [5].

To prove Theorem B we need two lemmas.

LEMMA 3. *Let* $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$, *and*

$$F_i(t) = F_i(x, t) = \frac{(\operatorname{sgn} x)^i}{(2\tan t/2)^{i+1}}\chi_t(x), \qquad i = 0, 1\ ,$$

then $\|F_i(t)\|_{L_{pq}}^* \cong Bt^{-1/p'}$ *where* $1/p' + 1/p = 1$.

PROOF. For fixed t,

$$|F_i(x, t)|^* = \left|\frac{1}{2\tan t/2}\chi_{[0,2t]}\right|^* = \frac{1}{|2\tan t/2|}\chi_{[0,2t]}(x)$$

since, as functions of x, $|F_i(x, t)|$ and $|(1/(2\tan(t/2))\chi_{[0\ 2t]}(x)|$ are equi-distributed and the latter is nonincreasing over $[0, \infty)$ and is thus its own nonincreasing rearrangement and also that of the equi-distributed $F_i(x, t)$. Therefore, for $q < \infty$,

$$\begin{aligned}\|F_i(t)\|_{pq}^* &= \left(\frac{q}{p}\int_0^\infty (|F_i(x, t)|^* x^{1/p})^q \frac{dx}{x}\right)^{1/q} \\ &= \frac{1}{2|\tan t/2|}\left(\frac{q}{p}\int_0^{2t} x^{q/p-1}\right)^{1/q} = \frac{1}{2|\tan t/2|}\left(x^{q/p}\Big|_0^{2t}\right)^{1/q} = \frac{(2t)^{1/p}}{2|\tan t/2|} \\ &\sim Bt^{-(1-1/p)} = Bt^{-1/p'}\ .\end{aligned}$$

When $q = \infty$, we have

$$\|F_i(t)\|_{p,\infty}^* = \sup_{x>0} \frac{1}{2|\tan t/2|}\chi_{[0,\ 2t]}(x)\cdot x^{1/p} = \frac{1}{2\tan t/2}(2t)^{1/p} \cong Bt^{-1/p'}\ .$$

LEMMA 4. *Let* $G_i(x) = F_i(x, t)$ *where* F_i *is as in Lemma* 3 (*Then*

$G_i(x)$ is a function of t for each x). Then if $1 \leqq p < \infty, 1 \leqq q \leqq \infty$, $\|G_i(x)\|^*_{p,q} = O|x|^{-1/p'}$.

PROOF. For fixed x and $q < \infty$

$$|F_i(x, t)|^* = \frac{2}{2 \tan (2|x| + t)/2}\chi_{[0,2(\pi-x)]}(t) .$$

Therefore,

$$\|F_i(x, t)\|^*_{p,q} < B\left(\frac{q}{p}\int_0^{2(\pi-x)} \frac{t^{(q/p)-1}}{(2|x| + t)^q}dt\right)^{1/q} = B\left(\frac{q}{p}\int_{2|x|}^{2\pi} \frac{(y - 2|x|)^{(q/p)-1}}{y^q}dy\right)^{1/q}$$
$$= \left[\frac{(y - 2|x|)^{q/p}}{y^q}\Big|_{2x}^{\pi} + q\int_{2|x|}^{2\pi} \frac{(y - 2|x|)^{q/p}}{y^{q+1}}dy\right]^{1/q} \leqq \left[B_1\int_{2|x|}^{2\pi} \frac{y^{q/p}}{y^{q+1}}dy\right]^{1/q}$$
$$= \left(B_2 y^{(q/p)-q}\Big|_{2|x|}^{2\pi}\right)^{1/q}$$
$$= (O(1) + (2|x|)^{(q/p)-q})^{1/q} = O(1) + O(|x|^{(1/p)-1}) = O(|x|^{-1/p'}) .$$

When $q = \infty$, note that for the function $g(x, t) = t^{1/p}/(2|x| + t)$ (which is uniformly close to $t^{1/p}|F_i|^*$), we have $\|g(x, *)\|_\infty = O(|x|^{-1/p'})$.

We can now prove Theorem B.

PROOF. We recall the decompositions (3), (4) of $H(m)$, $H^*(m)$. We note that the third term is in every Λ_{pq} since $F(x) = \int_{-\pi}^{x} dm(t) \in L_\infty \subset \Lambda_{pq}$ so that $\tilde{F}(x) \in \Lambda_{pq}$. Also, the first and second terms $m(3)$ may be written as

$$\pi H(m)(x) - \frac{1}{2}\tilde{F}(x) = 2\int_0^{\pi} F_0(x, t)d\mu(t) + 2\int_0^{\pi} F_1(x, t)d\nu(t)$$
$$= 2\lim_{\varepsilon\to 0}\left(\int_\varepsilon^{\pi} F_0(t)d\mu(t) + \int_\varepsilon^{\pi} F_1(t)d\nu(t)\right) ,$$

where we regard the last integrals as improper (at zero) Λ_{pq} valued integrals. Each of these integrals converges in Λ_{pq} because for $\varepsilon_2 > \varepsilon_1$, we have, for example,[†]

$$\left\|\int_{\varepsilon_1}^{\pi} F_0(t)d\mu(t) - \int_{\varepsilon_2} F_0(t)d\mu(t)\right\|_{pq} = \left\|\int_{\varepsilon_1}^{\varepsilon_2} F_0(t)d\mu(t)\right\|_{pq}$$
$$\leqq B\int_{\varepsilon_1}^{\varepsilon_2}\|F_0(t)\|^*_{pq}d|\mu|(t) \leqq B_1\int_{\varepsilon_1}^{\varepsilon_2} t^{-1/p'}d|\mu|(t) < \eta$$

if ε_1 and ε_2 are sufficiently small by Lemma 3 and the hypotheses on m. Thus, the integral exists in Λ_{pq}. Therefore, $H(m) \in \Lambda_{pq}$ and, hence, also $\tilde{H}(m) \in \Lambda_{pq}$. A similar argument using Lemma 4 proves the result for

† The symbol $\|\ \ \|_{pq}$ represents the Banach space norm of Λ_{pq} for which we have $\|\ \ \|_{pq} \leqq B\|\ \ \|^*_{pq}$ ([5] p. 258).

H^*. This proves B(i). B(ii) now follows by a duality argument.

REMARKS. C. Georgakis [2] has given a necessary and sufficient condition on even $m \sim \sum_{n=1}^{\infty} a_n \cos nx$ for $\sum_{n=1}^{\infty} A_n \sin nx$ to represent an integrable function. Our criterion, which has been proved necessary only for positive even m, is different and may be useful in some applications where it is easier to verify.

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