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CONTINUOUS W^*-ALGEBRAS ARE NON-NORMAL

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Let A be a W^*-algebra with center Z on separable Hilbert space H, and let B be a W^*-subalgebra of A. B is *full* in A if it contains Z. B is *normal* in A if $B = B^{cc}$, where $B^c = B' \cap A$. Clearly if B is normal in A then B is full in A. We say A is *normal* if B is normal in A for every full W^*-subalgebra B of A.

Every type I factor is normal [5, Lemma 11.2.2], as is every type I W^*-algebra [3, p. 287, Exercise 13b]. Every type II factor is non-normal [4, Theorem 3]. The author has recently shown that every type II W^*-algebra is non-normal [8, Theorem 7]. In [1, Theorem 16] A. Connes has announced that every type III factor is non-normal. In proving this, he uses the fact [2, Lemma 1.6.4] that if A is a type III factor it has a proper W^*-subalgebra P such that $P^c = \mathbb{C}$, so that $P^{cc} = A \neq P$. In this paper we use this result together with direct integral theory to show that no type III W^*-algebra, and hence no continuous W^*-algebra, is normal. We express our gratitude to A. Connes for making his work available to us before publication.

We begin with two lemmas which let us characterize the condition $P^c = \mathbb{C}$ in a measurable way in order to apply direct integral theory. If K is a separable Hilbert space and $\{x_i\}$ is a sequence of vectors dense in the unit ball of K, then $d(S, T) = \sum_{i,j=1}^{\infty} 2^{-i-j} |((S - T)x_j, x_i)|$ defines a metric on $B(K)$ which coincides with the weak operator topology on bounded subsets of $B(K)$ [6, Lemma I.4.8]. For $S, T \in B(K)$, we denote $ST - TS$ by $[S, T]$. We set $W(S) = d(S, 0)$ and $W(S, T) = W([S, T])$. We let S denote the unit ball of $B(K)$, taken with the strong*- operator topology [6, Definition I.4.10]. For any W^*-algebra A, A_1 denotes the unit ball in A.

The following simple lemma is essential to our argument.

LEMMA 1. *If $S_n \to S$ strong*-, then $W(S_n - S, T) \to 0$ uniformly in T for $|T| \leqq 1$.*

PROOF. Since $S_n \to S$ strong*-, it follows that [6, Lemma I.4.11] $\sum_{i=1}^{\infty} 2^{-i} |(S_n - S)x_i| + \sum_{i=1}^{\infty} 2^{-i} |(S_n^* - S^*)x_i| \to 0$. Hence

$$W(S_n - S, T) = \sum_{i,j=1}^{\infty} 2^{-i-j} |([S_n - S, T]x_j, x_i)|$$
$$\leqq \sum_{i,j=1}^{\infty} 2^{-i-j}(|((S_n - S)x_j, T^*x_i)| + |(Tx_j, (S_n^* - S^*)x_i)|)$$
$$\leqq \sum_{i,j=1}^{\infty} 2^{-i-j}(|T^*x_i||(S_n - S)x_j| + |Tx_j||(S_n^* - S^*)x_i|)$$
$$\leqq \sum_{i,j=1}^{\infty} 2^{-i-j}|(S_n - S)x_j| + \sum_{i,j=1}^{\infty} 2^{-i-j}|(S_n^* - S^*)x_i|$$
$$= \sum_{j=1}^{\infty} 2^{-j}|(S_n - S)x_j| + \sum_{i=1}^{\infty} 2^{-i}|(S_n^* - S^*)x_i| \to 0 .$$

q.e.d.

Now let A be a factor on K, and let P be a W^*-subalgebra of A. For each integer $m > 0$, let $A^{(m)} = \{T \in A_1 | W(T - \lambda_k I) \geqq 1/m$ for each $\lambda_k\}$, where $\{\lambda_k\}$ is a dense sequence in $\mathcal{C}$. Let $\{S_j\}$ be a set of generators for P, where $\{S_j\} \subset P_1$ and $S_j^* = S_k$ for some k. Finally, let $\{T_k^{(m)}\}$ be a sequence which is strong*- dense in $A^{(m)}$.

LEMMA 2. *$P^c = \mathcal{C}$ if and only if for each m there is an integer n such that* (*) $\sup_j W(T_k^{(m)}, S_j) \geqq 1/n$ *for every* $T_k^{(m)}$.

PROOF. If $P^c \neq \mathcal{C}$, there is some m and some $T \in A^{(m)}$ such that $[T, S_j] = 0$ for every S_j. Also there is a sequence $T_{k_r}^{(m)}$ chosen from the $T_k^{(m)}$ such that $T_{k_r}^{(m)} \to T^{(m)}$ strong*-. By Lemma 1, $W(T_{k_r}^{(m)} - T^{(m)}, S_j) \to 0$ uniformly in S_j. Thus $W(T_{k_r}^{(m)}, S_j) = W(T_{k_r}^{(m)} - T^{(m)}, S_j) \to 0$ uniformly in S_j, whence (*) does not hold.

If $P^c = \mathcal{C}$ and (*) does not hold, there is an integer m and a sequence $T_{k_r}^{(m)}$ chosen from the $T_k^{(m)}$ such that $\sup_j W(T_{k_r}^{(m)}, S_j) \to 0$. In particular, $W(T_{k_r}^{(m)}, S_j) \to 0$ for each S_j. We may assume (by the weak compactness of the unit ball in $B(K)$) that $T_{k_r}^{(m)} \to T$ weakly, with $T \in A^{(m)}$ ($A^{(m)}$ is the intersection of weakly closed sets and hence is weakly closed). Clearly $[T, S_j] = 0$ for all S_j. Hence $T \in P^c$ and $P^c \neq \mathcal{C}$. Thus if $P^c = \mathcal{C}$, (*) must hold. q.e.d.

We now recall some needed facts about direct integral theory (see [6] and [7] for details). If A on H has direct integral decomposition into factors given by

$$A = \int_{\Lambda} \oplus A(\lambda)\mu(d\lambda)$$

with K the underlying separable Hilbert space of H there is a sequence of operators B_n in A_1 such that $\{B_n(\lambda)\}$ is strong*- dense in $A(\lambda)_1$ μ-a.e. and such that the $B_n(\lambda)$ are strong*- continuous in λ [7, Lemma 1.5].

Also, by a standard construction technique, we can find for each m a sequence $\{T_k^{(m)}\} \in A_1$ such that $\{T_k^{(m)}(\lambda)\}$ is strong*- dense in $A(\lambda)^{(m)}$ μ-a.e. and such that $T_k^{(m)}(\lambda)$ is strong*- continuous in λ for all m and k μ-a.e. (see [7, Lemma 3.5] for a similar construction).

LEMMA 3. *Let* $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$ *be a* W^**-algebra such that for* μ*-a.e.* λ $A(\lambda)$ *has a proper* W^**-subalgebra* $P(\lambda)$ *such that* $P(\lambda)^c = \mathbb{C}$. *Then* A *is non-normal.*

PROOF. Let S_∞ denote the Cartesian product of a countable number of copies of S. Consider the set of $[\lambda, S_j, R]$ contained in $\Lambda \times S_\infty \times S$ defined by the following conditions.

(i) $S_j \in A(\lambda)$ for each j.

(ii) $S_{2(j+1)} = S_{2j+1}^*$ for each j.

(iii) For every m there is an n such that $\sup_j W(T_k^{(m)}(\lambda), S_j) \geqq 1/n$ for every k.

(iv) $[S_j, R] = 0$ for every j.

(v) For some r, $[B_r(\lambda), R] \neq 0$.

By Lemma 2 conditions (i) through (iii) guarantee that the W^*-subalgebra $P(\lambda)$ generated by the S_j satisfies $P(\lambda)^c = \mathbb{C}$. Conditions (iv), (v) show that $P(\lambda) \neq A(\lambda)$. Clearly these countably many conditions define a Borel subset of $\Lambda \times S_\infty \times S$ whose projection on Λ differs from Λ by a μ-null set because of our hypothesis concerning the $A(\lambda)$. Hence we may construct [6, Lemma I.4.7] μ-measurable functions $S_j(\lambda)$ and $R(\lambda)$ such that $[\lambda, S_j(\lambda), R(\lambda)]$ satisfy conditions (i) through (v) μ-a.e.. Let P be the W^*-subalgebra of A generated by the operators $S_j = \int_\Lambda \oplus S_j(\lambda)\mu(d\lambda)$ and by Z. Clearly $P = \int_\Lambda \oplus P(\lambda)\mu(d\lambda)$, and $P^{cc} = \int_\Lambda \oplus A(\lambda)\mu(d\lambda) = A$, but $P \neq A$, since $R \in P'$ but $R \notin A'$, where $R = \int_\Lambda \oplus R(\lambda)\mu(d\lambda)$. Since P contains Z by construction, A is non-normal. q.e.d.

COROLLARY 4. *Every type* III W^**-algebra* A *is non-normal.*

PROOF. If A is type III, then $A(\lambda)$ is type III μ-a.e., and by the result of Connes [2, Lemma 1.6.4] $A(\lambda)$ satisfies the hypothesis of Lemma 3 μ-a.e.. q.e.d.

THEOREM 4. *A* W^**-algebra* A *on* H *is normal if and only if it is of type* I.

PROOF. $A = A_{\mathrm{I}} \oplus A_{\mathrm{II}} \oplus A_{\mathrm{III}}$, where A_{I} is of type I, etc.. The result follows from [3, p. 287, Exercise 13b], [8, Theorem 7], and Corollary 4. q.e.d.

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