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# STOCHASTIC INTEGRAL OF $L_2$-FUNCTIONS WITH RESPECT TO GAUSSIAN PROCESSES

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**1. Introduction.** Let $X$ be a Brownian motion process on a probability space $(\Omega, \mathfrak{B}, P)$ and the unit interval $D = [0, 1] \subset R$. According to the well-known theory of stochastic integral of real valued functions of the class $L_2(D)$ with respect to $X$ (see for instance J. L. Doob [2] and K. Itô [3]) the stochastic integral $I(f) = \int_D f\,dX$ exists and is a random variable in the class $L_2(\Omega)$ for every $f \in L_2(D)$. Furthermore the expectation of $I(f)$ is equal to 0 for every $f \in L_2(D)$ and the covariance of $I(f)$ and $I(g)$ is equal to the inner product $(f, g)$ for every $f, g \in L_2(D)$, the probability distribution of $I(f)$ is the normal distribution $N(0, \|f\|^2)$ and $\{I(f), f \in L_2(D)\}$ is a Gaussian system of random variables. These results were obtained by making use of the stochastic independence, and in particular the orthogonality, of increments of the Brownian motion process. The objective of this paper is to show that similar results can be obtained for stochastic integral with respect to a large class of Gaussian processes with covariance functions satisfying certain continuity and smoothness conditions which includes the Brownian motion process as an example. These conditions are given below.

We assume that our Gaussian process $X$ on $(\Omega, \mathfrak{B}, P)$ and $D$ has a vanishing mean function on $D$ and has a covariance function $\Gamma(s, t)$, $(s, t) \in D \times D$, which satisfies the following conditions:

1° $\Gamma$ is continuous on $D \times D$.

2° $\partial^2\Gamma/\partial s^2$ and $\partial^2\Gamma/\partial t\partial s$ exist and are bounded on the two open triangles $T_1 = \{(s, t) \in D \times D; s \in (0, t) \text{ and } t \in (0, 1)\}$ and $T_2 = \{(s, t) \in D \times D; s \in (t, 1) \text{ and } t \in (0, 1)\}$.

These are the conditions assumed by G. Baxter [1] to obtain his celebrated strong limit theorem for Gaussian processes. These conditions imply in particular that

$$\gamma(t) = \lim_{s \uparrow t} \frac{\Gamma(s, t) - \Gamma(t, t)}{s - t} - \lim_{s \downarrow t} \frac{\Gamma(s, t) - \Gamma(t, t)}{s - t}$$

is a bounded and continuous function on $(0, 1)$. In addition to 1° and 2° we also assume

3° $\partial^2\Gamma/\partial t\partial s$ is continuous on $T_1 \cup T_2$ except possibly on a subset with Lebesgue measure 0.

In what follows we write $L_2(D)$ for the collection of real valued Lebesgue measurable functions which are square integrable over $D$ and write $\mathfrak{L}_2(D)$ for the Hilbert space of equivalence classes of functions in $L_2(D)$ modulo a.e. equality on $D$. We distinguish likewise between $L_2(\Omega)$ and $\mathfrak{L}_2(\Omega)$. Let $C(D)$ be the collection of real valued continuous functions on $D$ and let $\mathfrak{C}(D)$ be the collection of those elements of $\mathfrak{L}_2(D)$ each of which has a version in $C(D)$. Thus $\mathfrak{C}(D)$ is a dense linear subspace of $\mathfrak{L}_2(D)$. We shall write $(\cdot, \cdot)$ and $\|\cdot\|$ for inner product and norm in both $\mathfrak{L}_2(D)$ and $\mathfrak{L}_2(\Omega)$ since there will be no ambiguity from the context. We shall also use the same notation for both an element in $\mathfrak{L}_2(D)$ (or $\mathfrak{L}_2(\Omega)$) and any of its versions in $L_2(D)$ (or $L_2(\Omega)$) to avoid cumbersomeness in notation. The symbol $m_L$ stands for the Lebesgue measures in $R^1$ and $R^2$.

In Theorem 1, §2 we show that under the assumption of the conditions 1°, 2° and 3° on the covariance function of the Gaussian process the stochastic integral $I(f)=\int_D f\,dX$ exists as the limit in $\mathfrak{L}_2(\Omega)$ of sequences of Riemann-Stieltjes sums of $f$ with respect to $X$ for every $f \in C(D)$. In Theorem 2, §2 we show that the stochastic integral with respect to the Gaussian process has all the properties of the stochastic integral with respect to a Brownian motion process mentioned above except that the covariance of $I(f)$ and $I(g)$ for $f$, $g \in C(D)$ is now given by

$$(I(f), I(g)) = \int_D f(t)g(t)\gamma(t)m_L(dt) + \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t)) .$$

For a Brownian motion process for which $\Gamma(s, t) = \min\{s, t\}$ we have $\gamma = 1$ on $(0, 1)$ and $\partial^2\Gamma/\partial t\partial s = 0$ on $T_1 \cup T_2$ so that the right side of the equality above reduces to $(f, g)$. In Theorem 3, §2 we show that if $f$ is continuous and of bounded variation on $D$ and if every sample function $X(\cdot, \omega)$, $\omega \in \Omega$, of $X$ is continuous on $D$ then for a.e. $\omega \in \Omega$, the stochastic integral $I(f)=\int_D f\,dX$ is equal to the Riemann-Stieltjes integral $\int_0^1 f(t)dX(t, \omega)$. In §3, we extend the definition of the stochastic integral from $C(D)$ to $L_2(D)$ by limiting processes utilizing the denseness of $\mathfrak{C}(D)$ in $\mathfrak{L}_2(D)$. Finally in Theorem 4, §3 we show that the stochastic integral, now extended to $L_2(D)$, preserves all the above mentioned properties. Not the step functions on $D$ as in the case of the stochastic integral with respect to a Brownian motion process but our $\mathfrak{C}(D)$ is the appropriate dense linear subspace of $\mathfrak{L}_2(D)$ on which to start defining a stochastic integral with respect to our Gaussian process.

**2. Stochastic integral for $f \in C(D)$.** To prove the convergence of the Riemann-Stieltjes sums of $f \in C(D)$ with respect to $X$ in the Hilbert space $\mathfrak{L}_2(\Omega)$ we require two lemmas concerning real valued functions $\Gamma(s, t)$, $(s, t) \in D \times D$, which satisfies 1°, 2° and 3° of § 1. The substance of the first of these two lemmas, namely Lemma 3, is already given in G. Baxter [1]. Here we state it in a form which is suitable to our subsequent application. Rather than proving Lemma 3 here we state the crucial steps for the proof as Lemma 1 and Lemma 2. Our Lemma 4 is the principal lemma for Theorem 1.

LEMMA 1. *Let $f$ be a real valued function which is defined and differentiable on $(a, b)$ with $|f'(t)| \leqq B$ for $t \in (a, b)$. Then* $\lim_{t\downarrow a} f(t)$ *and* $\lim_{t\uparrow b} f(t)$ *exist and satisfy*

$$\left|\lim_{t\downarrow a} f(t) - f(c)\right| \leqq B(c - a) \quad \text{and} \quad \left|\lim_{t\uparrow b} f(t) - f(c)\right| \leqq B(b - c)$$

*for* $c \in (a, b)$.

LEMMA 2. *Let $f$ be a real valued function which is defined and differentiable on $(a, b)$. If $\alpha = \lim_{t\downarrow a} f'(t)$ and $\beta = \lim_{t\uparrow b} f'(t)$ exist and are finite then $\lim_{t\downarrow a} f(t)$ and $\lim_{t\uparrow b} f(t)$ also exist and are finite. If we define $f(a) = \lim_{t\downarrow a} f(t)$ and $f(b) = \lim_{t\uparrow b} f(t)$ then the right hand derivative of $f$ at $a$ and the left hand derivative of $f$ at $b$ exist and are equal to $\alpha$ and $\beta$ respectively.*

LEMMA 3. *Let $\Gamma(s, t)$ be a real valued function defined on the open triangle $T_1 = \{(s, t) \in D \times D; s \in (0, t)$ and $t \in (0, 1)\}$. If $\partial^2\Gamma/\partial s^2$ and $\partial^2\Gamma/\partial t\partial s$ exist and are bounded by $B \geqq 0$ on $T_1$ and $\Gamma(t, t)$ is defined by $\Gamma(t, t) = \lim_{s\uparrow t} \Gamma(s, t)$ then the left side derivative*

$$(1) \qquad \gamma_1(t) = \lim_{s\uparrow t} \frac{\Gamma(s, t) - \Gamma(t, t)}{s - t}$$

*exists and is finite for every $t \in (0, 1)$. Furthermore $\gamma_1$ is bounded and continuous on $(0, 1)$ and satisfies*

$$(2) \qquad \gamma_1(t) = \lim_{s\uparrow t} \frac{\partial\Gamma}{\partial s}(s, t)$$

$$(3) \qquad \left|\gamma_1(t) - \frac{\partial\Gamma}{\partial s}(s, t)\right| \leqq B(t - s)$$

$$(4) \qquad |\gamma_1(t') - \gamma_1(t'')| \leqq 2B|t' - t''| \quad \text{for } t', t'' \in (0, 1)\ .$$

*Parallel statements concerning*

$$\gamma_2(t) = \lim_{s\downarrow t} \frac{\Gamma(s, t) - \Gamma(t, t)}{s - t}$$

*hold on the open triangle* $T_2 = \{(s, t) \in D \times D; s \in (t, 1)$ *and* $t \in (0, 1)\}$.

LEMMA 4. *Let $\Gamma$ be a real valued positive definite symmetric function on $D \times D$ which satisfies the conditions* $1°$, $2°$ *and* $3°$ *of* §1 *and let*

$$(1) \qquad \gamma(t) = \gamma_1(t) - \gamma_2(t) \quad \textit{for } t \in (0, 1)$$

*where $\gamma_1$ and $\gamma_2$ are as defined in Lemma 3. For a partition $\mathfrak{P}$ of $D$ given by $0 = a_0 < a_1 < \cdots < a_q = 1$ let*

$$(2) \qquad \Delta\Gamma_{k,l} = \Gamma(a_k, a_l) - \Gamma(a_{k-1}, a_l) - \Gamma(a_k, a_{l-1}) + \Gamma(a_{k-1}, a_{l-1})$$

*for $k, l = 1, 2, \cdots, q$ and let*

$$(3) \qquad S_0(\mathfrak{P}) = \sum_{k=1}^{q} \Delta\Gamma_{k,k}$$

$$(4) \qquad S_1(\mathfrak{P}) = \sum_{\substack{k,l=1 \\ k \neq l}}^{q} \Delta\Gamma_{k,l}$$

*and*

$$(5) \qquad S_2(\mathfrak{P}) = \sum_{\substack{k,l=1 \\ k \neq l}}^{q} |\Delta\Gamma_{k,l}| \, .$$

*Then for every $\varepsilon > 0$ there exists some $\eta > 0$ such that*

$$(6) \qquad \left| S_0(\mathfrak{P}) - \int_D \gamma(t) m_L(dt) \right| < \varepsilon$$

$$(7) \qquad \left| S_1(\mathfrak{P}) - \int_{D \times D} \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) m_L(d(s, t)) \right| < \varepsilon$$

*and*

$$(8) \qquad \left| S_2(\mathfrak{P}) - \int_{D \times D} \left| \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) \right| m_L(d(s, t)) \right| < \varepsilon$$

*whenever $|\mathfrak{P}| < \eta$ where*

$$(9) \qquad |\mathfrak{P}| = \max_{k=1,\cdots,q} (a_k - a_{k-1}) \, .$$

PROOF. With regard to (6) observe that the boundedness and the continuity of $\gamma$ on $(0, 1)$ imply first the Lebesgue integrability of $\gamma$ on $D$, then the convergence of the improper Riemann integral $\int_0^1 \gamma(t) dt$ and finally the equality

$$(10) \qquad \int_0^1 \gamma(t) dt = \int_D \gamma(t) m_L(dt) \, .$$

Applying (3) and (4) of Lemma 3 we have

$$
\begin{aligned}
(11) \qquad \Delta\Gamma_{k,k} &= \{\Gamma(a_k, a_k) - \Gamma(a_{k-1}, a_k)\} - \{\Gamma(a_k, a_{k-1}) - \Gamma(a_{k-1}, a_{k-1})\} \\
&= \left\{\frac{\partial\Gamma}{\partial s}(a_k^*, a_k) - \frac{\partial\Gamma}{\partial s}(a_k^{**}, a_{k-1})\right\}(a_k - a_{k-1}) \\
&= \{\gamma_1(a_k) - \gamma_2(a_k) + O(a_k - a_{k-1})\}(a_k - a_{k-1}) \\
&= \{\gamma(a_k) + O(a_k - a_{k-1})\}(a_k - a_{k-1})
\end{aligned}
$$

where $a_k^*, a_k^{**} \in (a_{k-1}, a_k)$. Thus from the convergence of $\int_0^1 \gamma(t)dt$, for every $\varepsilon > 0$ there exists some $\eta > 0$ such that

$$\left|S_0(\mathfrak{P}) - \int_0^1 \gamma(t)dt\right| < \varepsilon \quad \text{whenever } |\mathfrak{P}| < \eta .$$

This and (10) prove (6).

Regarding (7) note that since $\partial^2\Gamma/\partial t\partial s$ is bounded on $T_1 \cup T_2$ and is continuous there except on a subset of Lebesgue measure 0, it is Lebesgue integrable on $D \times D$ and its improper Riemann integral on $T_1 \cup T_2$ converges to the Lebesgue integral on $D \times D$, i.e.,

$$(12) \qquad \iint_{T_1 \cup T_2} \frac{\partial^2\Gamma}{\partial t\partial s}(s, t)dsdt = \int_{D\times D} \frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t)) .$$

From the existence and boundedness of $\partial^2\Gamma/\partial s^2$ on $T_1 \cup T_2$, we have for $k \neq l$

$$
\begin{aligned}
(13) \qquad \Delta\Gamma_{k,l} &= \left\{\frac{\partial\Gamma}{\partial s}(a_k^*, a_l) - \frac{\partial\Gamma}{\partial s}(a_k^{**}, a_{l-1})\right\}(a_k - a_{k-1}) \\
&= \left\{\frac{\partial\Gamma}{\partial s}(a_k^*, a_l) - \frac{\partial\Gamma}{\partial s}(a_k^*, a_{l-1}) + O(a_k - a_{k-1})\right\}(a_k - a_{k-1}) \\
&= \left\{\frac{\partial^2\Gamma}{\partial t\partial s}(a_k^*, a_l^*)(a_l - a_{l-1}) + O(a_k - a_{k-1})\right\}(a_k - a_{k-1})
\end{aligned}
$$

where $a_k^*, a_k^{**} \in (a_{k-1}, a_k)$ and $a_l^* \in (a_{l-1}, a_l)$. Then from the convergence of the improper Riemann integral of $\partial^2\Gamma/\partial t\partial s$ on $T_1 \cup T_2$, for $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left|S_1(\mathfrak{P}) - \iint_{T_1 \cup T_2} \frac{\partial^2\Gamma}{\partial t\partial s}(s, t)dsdt\right| < \varepsilon \quad \text{whenever } |\mathfrak{P}| < \eta .$$

This and (12) prove (7).

Finally (8) can be proved as (7) by means of

$$|\Delta\Gamma_{k,l}| = \left\{\left|\frac{\partial^2\Gamma}{\partial t\partial s}(a_k^*, a_l^*)\right|(a_l - a_{l-1}) + O(a_k - a_{k-1})\right\}(a_k - a_{k-1})$$

which is implied by (13).

Below we list some properties of Gaussian systems which we need in proving our theorems. To begin with, a system of random variables $\{X_\alpha, \alpha \in A\}$ on a probability space $(\Omega, \mathfrak{B}, P)$ where $A$ is an arbitrary index set is called a Gaussian system if every linear combination of members of the system is normally distributed, or equivalently, if for any $\{\alpha_1, \cdots, \alpha_n\} \subset A$ the probability distribution of the $n$-dimensional random vector $(X_{\alpha_1}, \cdots, X_{\alpha_n})$ is an $n$-dimensional (possibly degenerate) normal distribution. When $A$ is a subset of $R$ the Gaussian system is called a Gaussian process. It is obvious that every subsystem of a Gaussian system is itself a Gaussian system. On the other hand the collection of all linear combinations of members of a Gaussian system is again a Gaussian system. It is well known that if a sequence of $n$-dimensional normal distributions on $R^n$ converges to a probability distribution on $R^n$ then this limit probability distribution too is an $n$-dimensional normal distribution and furthermore the mean vectors and the covariance matrices of the sequence converge componentwise to those of the limit probability distribution. From this follows that the collection of all limits of convergence in probability of sequences in a Gaussian system is again a Gaussian system. It follows also that a sequence in a Gaussian system converges in the $\mathfrak{L}_2(\Omega)$ sense to a random variable on $(\Omega, \mathfrak{B}, P)$ if and only if the sequence converges in probability to the random variable. Thus the closed linear subspace in the Hilbert space $\mathfrak{L}_2(\Omega)$ spanned by a Gaussian system $\{X_\alpha, \alpha \in A\}$, i.e., the closure in $\mathfrak{L}_2(\Omega)$ of the collection of all linear combinations of members of the Gaussian system, is then a Hilbert space as well as a Gaussian system. We call this closed subspace of $\mathfrak{L}_2(\Omega)$ the Gaussian space generated by the Gaussian system and designate it by $\mathfrak{G}\{X_\alpha, \alpha \in A\}$.

THEOREM 1. *Let $X$ be a Gaussian process on a probability space $(\Omega, \mathfrak{B}, P)$ and $D = [0, 1]$ with a vanishing mean function whose covariance function $\Gamma$ satisfies the conditions* 1°, 2° *and* 3° *of* §1. *Let $\mathfrak{P}^n$ be a partition of $D$ given by*

$$0 = a_{n,0} < a_{n,1} < \cdots < a_{n,q(n)} = 1$$

*for $n = 1, 2, \cdots$, and $\lim_{n\to\infty} |\mathfrak{P}^n| = 0$ where $|\mathfrak{P}^n|$ is the maximum of the lengths of the subintervals by the partition $\mathfrak{P}^n$. Corresponding to $\mathfrak{P}^n$ and a collection of $q(n)$ real numbers*

$$\alpha^n = \{\alpha_{n,k} \in [a_{n,k-1}, a_{n,k}], k = 1, 2, \cdots, q(n)\} ,$$

*let the Riemann-Stieltjes sum of $f \in C(D)$ with respect to $X$ be defined by*

$$S(f, \mathfrak{P}^n, \alpha^n)(\omega) = \sum_{k=1}^{q(n)} f(\alpha_{n,k})\{X(a_{n,k}, \omega) - X(a_{n,k-1}, \omega)\} \quad \text{for } \omega \in \Omega .$$

*Then* $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\}$ *is a Cauchy sequence in* $\mathfrak{L}_2(\Omega)$. *Furthermore the element in* $\mathfrak{L}_2(\Omega)$ *to which this Cauchy sequence converges is determined by* $f$ *independently of the sequences* $\{\mathfrak{P}^n, n = 1, 2, \cdots\}$ *and* $\{\alpha^n, n = 1, 2, \cdots\}$.

PROOF. Let us show that $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\} \subset \mathfrak{S}\{X(t, \cdot), t \in D\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrarily given. From the uniform continuity of $f$ on $D$ there exists $\delta > 0$ such that

$$|f(t') - f(t'')| < \varepsilon \quad \text{whenever } t', t'' \in D \quad \text{and} \quad |t' - t''| < \delta .$$

Let $N$ be so large that whenever $n \geqq N$ we have $|\mathfrak{P}^n| < \delta/2$ as well as $|\mathfrak{P}^n| < \eta$ where the positive number $\eta$ is as prescribed in Lemma 4 for our $\varepsilon$. Now let $m, n \geqq N$ and let $0 = a_0 < a_1 < \cdots < a_q = 1$ be the partition $\mathfrak{P}^{m,n}$ of $D$ obtained by superposition of $\mathfrak{P}^m$ and $\mathfrak{P}^n$. Then

$$S(f, \mathfrak{P}^m, \alpha^m) - S(f, \mathfrak{P}^n, \alpha^n) = \sum_{k=1}^{q} c_k\{X(a_k) - X(a_{k-1})\}$$

where $c_k \in R$ and $|c_k| < \varepsilon$ for $k = 1, 2, \cdots, q$. Thus

$$\begin{aligned} &\| S(f, \mathfrak{P}^m, \alpha^m) - S(f, \mathfrak{P}^n, \alpha^n) \|^2 \\ &= \sum_{k=1}^{q} \sum_{l=1}^{q} c_k c_l (X(a_k) - X(a_{k-1}), X(a_l) - X(a_{l-1})) \\ &= \sum_{k=1}^{q} \sum_{l=1}^{q} c_k c_l \Delta\Gamma_{k,l} \\ &\leqq \varepsilon^2 \sum_{k=1}^{q} \sum_{l=1}^{q} |\Delta\Gamma_{k,l}| \end{aligned}$$

where $\Delta\Gamma_{k,l}$ is as given by (2) of Lemma 4. Note that from the positive definiteness of $\Gamma$ we have $\Delta\Gamma_{k,k} \geqq 0$ for $k = 1, 2, \cdots, q$.

Now since $|\mathfrak{P}^{m,n}| < \eta$ we have by (6) of Lemma 4

$$\sum_{k=1}^{q} \Delta\Gamma_{k,k} < \int_D \gamma(t) m_L(dt) + \varepsilon$$

and by (8) of Lemma 4

$$\sum_{\substack{k,l=1 \\ k \neq l}}^{q} |\Delta\Gamma_{k,l}| < \int_{D\times D} \left| \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) \right| m_L(d(s, t)) + \varepsilon .$$

Thus

$$\begin{aligned} &\| S(f, \mathfrak{P}^m, \alpha^m) - S(f, \mathfrak{P}^n, \alpha^n) \|^2 \\ &\leqq \varepsilon^2 \left\{ \int_D \gamma(t) m_L(dt) + \int_{D\times D} \left| \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) \right| m_L(d(s, t)) \right\} + 2\varepsilon^3 . \end{aligned}$$

This establishes the fact that $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\}$ is a Cauchy sequence in $\mathfrak{L}_2(\Omega)$.

To show that the element in $\mathfrak{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{\mathfrak{P}^n\}$ and $\{\alpha^n\}$ with $\lim_{n\to\infty}|\mathfrak{P}^n| = 0$, let $\{\mathfrak{Q}^n\}$ and $\{\beta^n\}$ be another pair of such sequences. Then since the sequence $|\mathfrak{P}^1|, |\mathfrak{Q}^1|, |\mathfrak{P}^2|, |\mathfrak{Q}^2|, \cdots$ converges to 0, the sequence

$$S(f, \mathfrak{P}^1, \alpha^1), S(f, \mathfrak{Q}^1, \beta^1), S(f, \mathfrak{P}^2, \alpha^2), S(f, \mathfrak{Q}^2, \beta^2), \cdots$$

is a Cauchy sequence in $\mathfrak{L}_2(\Omega)$ and the two subsequences $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\}$ and $\{S(f, \mathfrak{Q}^n, \beta^n), n = 1, 2, \cdots\}$ converge to the same element in $\mathfrak{L}_2(\Omega)$.

DEFINITION 1. For $f \in C(D)$ we define the stochastic integral $I(f)$ of $f$ with respect to the Gaussian process $X$ to be the element in $\mathfrak{L}_2(\Omega)$ to which the sequence $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\}$ in Theorem 1 converges.

THEOREM 2. *Let the covariance function $\Gamma$ of the Gaussian process $X$ in Theorem 1 satisfy the conditions $1°$, $2°$ and $3°$ of §1. For $f, g \in C(D)$ and $\alpha, \beta \in R$, the stochastic integral $I$ satisfies the following*:

$$(1) \qquad I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$

$$(2) \qquad E[I(f)] = 0$$

$$(3) \qquad (I(f), I(g)) = \int_D f(t)g(t)\gamma(t)m_L(dt) + \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t))$$

$$(4) \qquad \| I(f) \|^2 = \int_D [f(t)]^2\gamma(t)m_L(dt) + \int_{D\times D} f(s)f(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t))$$

(5) $\qquad I(f)$ *is distributed by* $N(0, \| I(f) \|^2)$

(6) $\qquad \{I(f), f \in C(D)\}$ *is a Gaussian system of random variables*

*where $\gamma$ in* (3) *and* (4) *is as defined by* (1) *of Lemma* 4.

PROOF. Let $\{\mathfrak{P}^n\}$ and $\{\alpha^n\}$ be as given in Theorem 1. Then (1) is immediate since

$$S(\alpha f + \beta g, \mathfrak{P}^n, \alpha^n) = \alpha S(f, \mathfrak{P}^n, \alpha^n) + \beta S(f, \mathfrak{P}^n, \alpha^n)$$

(2) follows from

$$E[S(f, \mathfrak{P}^n, \alpha^n)] = \sum_{k=1}^{q} f(a_k)E[X(a_k) - X(a_{k-1})] = 0$$

and

$$E[I(f)] = (I(f), 1) = \lim_{n\to\infty} (S(f, \mathfrak{P}^n, \alpha^n), 1) = \lim_{n\to\infty} E[S(f, \mathfrak{P}^n, \alpha^n)] = 0 .$$

To prove (3) note that from the convergence of the sequence of

Riemann-Stieltjes sums to the stochastic integral in $\mathfrak{L}_2(\Omega)$ we have

$$(I(f), I(g)) = \lim_{n\to\infty} (S(f, \mathfrak{P}^n, \alpha^n), S(g, \mathfrak{P}^n, \alpha^n)) . \tag{7}$$

Let $\mathfrak{P}^n$ be given by $0 = a_0 < a_1 < \cdots < a_q = 1$ and let $\alpha^n$ be such that $\alpha_k = a_k$ for $k = 1, 2, \cdots, q$. We then have

$$S(f, \mathfrak{P}^n, \alpha^n) = \sum_{k=1}^{q} f(a_k)\{X(a_k) - X(a_{k-1})\}$$

and a similar expression for $S(g, \mathfrak{P}^n, \alpha^n)$. Since

$$(X(a_k) - X(a_{k-1}), X(a_l) - X(a_{l-1})) = \Delta\Gamma_{k,l}$$

where $\Delta\Gamma_{k,l}$ is as given by (2) of Lemma 4, we have

$$(S(f, \mathfrak{P}^n, \alpha^n), S(g, \mathfrak{P}^n, \alpha^n)) = \sum_{k=1}^{q} f(a_k)g(a_k)\Delta\Gamma_{k,k} + \sum_{\substack{k,l=1\\k\neq l}}^{q} f(a_k)g(a_l)\Delta\Gamma_{k,l} . \tag{8}$$

According to (11) in the proof of Lemma 4

$$\Delta\Gamma_{k,k} = \{\gamma(a_k) + O(a_k - a_{k-1})\}(a_k - a_{k-1}) .$$

Since $\gamma$ is bounded and continuous on $(0, 1)$ and both $f$ and $g$ are continuous on $D$, $fg\gamma$ is Lebesgue integrable on $D$ and the improper Riemann integral $\int_0^1 f(t)g(t)\gamma(t)dt$ converges to the Lebesgue integral. Thus

$$\lim_{n\to\infty} \sum_{k=1}^{q} f(a_k)g(a_k)\Delta\Gamma_{k,k} = \int_D f(t)g(t)\gamma(t)m_L(dt) . \tag{9}$$

Similarly for $k \neq l$, from (13) in the proof of Lemma 4 we have

$$\Delta\Gamma_{k,l} = \left\{\frac{\partial^2\Gamma}{\partial t\partial s}(a_k^*, a_l^*)(a_l - a_{l-1}) + O(a_k - a_{k-1})\right\}(a_k - a_{k-1})$$

where $a_k^*, a_k^{**} \in (a_{k-1}, a_k)$ and $a_l^* \in (a_{l-1}, a_l)$. Since $\partial^2\Gamma/\partial t\partial s$ is bounded on $T_1 \cup T_2$ and is continuous there except on a subset of Lebesgue measure 0 and since $f(s)g(t)$ is continuous on $D\times D$, $f(s)g(t)(\partial^2\Gamma/\partial t\partial s)(s, t)$ is Lebesgue integrable on $D \times D$ and the improper Riemann integral of the same on $T_1 \cup T_2$ converges to the Lebesgue integral, i.e.,

$$\iint_{T_1\cup T_2} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)dsdt = \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t)) .$$

Thus

$$\lim_{n\to\infty} \sum_{\substack{k,l=1\\k\neq l}}^{q} f(a_k)g(a_l)\Delta\Gamma_{k,l} = \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s, t)m_L(d(s, t)) . \tag{10}$$

From (7), (8), (9), (10) we have (3).

(4) is a particular case of (3).

(6) holds since $\{I(f), f \in C(D)\} \subset \mathfrak{G}\{X(t, \cdot), t \in D\}$. It implies in particular that $I(f)$ is normally distributed. Thus (5) holds in view of (2) and (4).

THEOREM 3. *If every sample function of the Gaussian process $X$ in Theorem 1 is continuous on $D$, then for every continuous function $f$ with bounded variation on $D$ we have*

$$I(f)(\omega) = \int_0^1 f(t) dX(t, \omega) \quad \text{for a.e. } \omega \in \Omega .$$

PROOF. Since $f$ is of bounded variation on $D$ and $X(\cdot, \omega)$ is continuous on $D$ for every $\omega \in \Omega$, the Riemann-Stieltjes integral $\int_0^1 f(t) dX(t, \omega)$ converges for every $\omega \in \Omega$. On the other hand with $\mathfrak{P}^n$ and $\alpha^n$ as in Theorem 1, the sequence $\{S(f, \mathfrak{P}^n, \alpha^n), n = 1, 2, \cdots\}$ converges to $I(f)$ in $\mathfrak{L}_2(\Omega)$ so that there exists a subsequence $\{n_m\}$ of $\{n\}$ such that

$$\lim_{n \to \infty} S(f, \mathfrak{P}^{n_m}, \alpha^{n_m})(\omega) = I(f)(\omega) \quad \text{for a.e. } \omega \in \Omega .$$

But for every $\omega \in \Omega$

$$\lim_{m \to \infty} S(f, \mathfrak{P}^{n_m}, \alpha^{n_m})(\omega) = \int_0^1 f(t) dX(t, \omega) .$$

Therefore our theorem holds.

## 3. Stochastic integral for $f \in L_2(D)$.

LEMMA 5. *For $f \in L_2(D)$ let $\{f_n, n = 1, 2, \cdots\} \subset C(D)$ be such that $\lim_{n \to \infty} \| f_n - f \| = 0$. Then $\{I(f_n), n = 1, 2, \cdots\}$ is a Cauchy sequence in $\mathfrak{L}_2(\Omega)$. Furthermore the element in $\mathfrak{L}_2(\Omega)$ to which our Cauchy sequence converges is independent of the sequence $\{f_n, n = 1, 2, \cdots\}$.*

PROOF. From the fact that $\mathfrak{C}(D)$ is a linear space and from (1) and (4) of Theorem 2, § 2 we have

$$\begin{aligned} \| I(f_m) - I(f_n) \|^2 &= \| I(f_m - f_n) \|^2 \\ &\leqq \int_D | f_m(t) - f_n(t) |^2 | \gamma(t) | \, m_L(dt) \\ &\quad + \int_{D \times D} | f_m(s) - f_n(s) | \, | f_m(t) - f_n(t) | \left| \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) \right| m_L(d(s, t)) \\ &\leqq A \| f_m - f_n \|^2 + B \| f_m - f_n \|_1^2 \end{aligned}$$

where

$$A = \sup_{(0,1)} | \gamma(t) | , \qquad B = \sup_{T_1 \cup T_2} \left| \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) \right|$$

and

$$\|f_m - f_n\|_1 = \int_D |f_m(t) - f_n(t)|\, m_L(dt) \leqq \|f_m - f_n\| .$$

Thus the fact that $\{f_n, n = 1, 2, \cdots\}$ is a Cauchy sequence in $\mathfrak{L}_2(D)$ implies that $\{I(f_n), n = 1, 2, \cdots\}$ is a Cauchy sequence in $\mathfrak{L}_2(\Omega)$.

To show that the element in $\mathfrak{L}_2(\Omega)$ to which the sequence $\{I(f_n), n = 1, 2, \cdots\}$ converges does not depend on the sequence $\{f_n, n = 1, 2, \cdots\}$ converging to $f$ in $\mathfrak{L}_2(D)$, let $\{g_n, n = 1, 2, \cdots\} \subset C(D)$ be another such sequence. Then $\{f_1, g_1, f_2, g_2, \cdots\}$ is a sequence from $C(D)$ which converges to $f$ in $\mathfrak{L}_2(D)$ so that $\{I(f_1), I(g_1), I(f_2), I(g_2), \cdots\}$ is a Cauchy sequence in $\mathfrak{L}_2(\Omega)$ and its subsequences $\{I(f_n), n = 1, 2, \cdots\}$ and $\{I(g_n), n = 1, 2, \cdots\}$ converge to the same element in $\mathfrak{L}_2(\Omega)$.

DEFINITION 2. For $f \in L_2(D)$ we define the stochastic integral $I(f)$ of $f$ with respect to the Gaussian processes $X$ to be the element in $\mathfrak{L}_2(\Omega)$ to which the sequence $\{I(f_n), n = 1, 2, \cdots\}$ in Lemma 5 converges.

THEOREM 4. *The stochastic integral $I(f)$, $f \in L_2(D)$, satisfies* (1), (2), (3), (4), (5) *of Theorem* 2. *Also $\{I(f), f \in L_2(D)\}$ is a Gaussian system of random variables.*

PROOF. To prove (1) let $\{f_n, n = 1, 2, \cdots\}$, $\{g_n, n = 1, 2, \cdots\} \subset C(D)$ be such that $\lim_{n\to\infty} \|f_n - f\| = 0$ and $\lim_{n\to\infty} \|g_n - g\| = 0$. Then

$$\lim_{n\to\infty} \|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| = 0$$

so that

$$I(\alpha f + \beta g) = \lim_{n\to\infty} I(\alpha f_n + \beta g_n) = \lim_{n\to\infty} \{\alpha I(f_n) + \beta I(g_n)\} = \alpha I(f) + \beta I(g) .$$

Also

$$E[I(f)] = (I(f), 1) = \lim_{n\to\infty} (I(f_n), 1) = \lim_{n\to\infty} E[I(f_n)] = 0$$

so that (2) holds.

As for (3) note that

$$(I(f), I(g)) = \lim_{n\to\infty} (I(f_n), I(g_n)) .$$

Then since (3) holds on $C(D)$, to show that it holds on $L_2(D)$ it suffices to show that

$$(7)\quad \lim_{n\to\infty} \int_D f_n(t) g_n(t) \gamma(t) m_L(dt) + \int_{D\times D} f_n(s) g_n(t) \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) m_L(d(s, t))$$
$$= \int_D f(t) g(t) \gamma(t) m_L(dt) + \int_{D\times D} f(s) g(t) \frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) m_L(d(s, t)) .$$

Now with $A$, $B$ and $\|\cdot\|_1$ as defined in the proof of Lemma 5 we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_D f_n(t)g_n(t)\gamma(t)m_L(dt) - \int_D f(t)g(t)\gamma(t)m_L(dt)\right| \\
&\quad \leqq \lim_{n\to\infty} A\int_D \{|g_n(t)|\,|f_n(t)-f(t)| + |f(t)|\,|g_n(t)-g(t)|\}m_L(dt) \\
&\quad \leqq \lim_{n\to\infty} A\{\|g_n\|\,\|f_n-f\| + \|f\|\,\|g_n-g\|\} = 0
\end{aligned}$$

and similarly

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_{D\times D} f_n(s)g_n(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s,t)m_L(d(s,t)) - \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s,t)m_L(d(s,t))\right| \\
&\quad \leqq \lim_{n\to\infty} B\int_{D\times D}\{|g_n(t)|\,|f_n(s)-f(s)| + |f(s)|\,|g_n(t)-g(t)|\}m_L(d(s,t)) \\
&\quad \leqq \lim_{n\to\infty} B\{\|g_n\|_1\,\|f_n-f\|_1 + \|f\|_1\,\|g_n-g\|_1\} = 0\,.
\end{aligned}$$

Thus (7) holds and this proves (3) and hence (4) also.

Since $\{I(f), f\in L_2(D)\}$ is contained in $\mathfrak{G}\{X(t,\cdot), t\in D\}$ it is a Gaussian system of random variables. This implies in particular that $I(f)$ is normally distributed. Its mean and variance are given by (2) and (4) respectively.

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