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CENTRAL APPROXIMATION PROCESSES

To Prof. Dr. Hellmuth Kneser, on his 75th birthday, 16 April, 1973

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1. Introduction. The first investigations which are concerned with the saturation behaviour of approximation processes of the convolution type on general locally compact *abelian* topological groups are due to H. Buchwalter [7]. His brief sketch was followed by the recent papers [17, 18], [42] which contain a detailed and extended treatment of the locally compact abelian case by means of the quasimeasure concept. Inspired by the results of S. Pawelke [35], the study of saturation theory on compact (*not* necessarily abelian) groups was initiated by the junior author [16]. Some of the results of his thesis were announced in [15]. The methods employed in these papers are based on the Peter-Weyl decomposition theorem and remain to a certain extent in the *algebraic* domain.

On the other hand the constructive approximation theory on Lie groups has made some progress in the past years. In this connection we shall refer the reader to the papers by P. L. Butzer-H. Johnen [9], H. Johnen [25–27], R. A. Mayer [29–33] and D. L. Ragozin [37–40]. It is the primary objective of the present paper to apply the general methods as developed in [16] to the study of *the saturation behaviour of central approximation processes on compact Lie groups G.* In particular, the approximations by the heat-diffusion semigroup and by the Poisson semigroup which both are, as is well known, closely connected with hypoelliptic partial differential equations, will be treated (Sections 4 and 5). Since it is shown in [15, 16] that the techniques to deduce our saturation theorem for central approximation processes on compact Lie groups (Theorems 2 and 3 in Section 3 infra) apply also for zonal approximation processes on compact homogeneous spaces associated with Riemannian symmetric pairs, we shall deal in Section 6 with the interplay between zonal and central approximation processes on the compact Euclidean unit sphere S_3. Finally, Section 7 is devoted to an investigation of the approximation processes of the de La Vallée Poussin type and of the Fejér type on the special unitary group $\mathbf{SU}(2)$.

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2. Notation, Terminology and Preliminaries. Throughout this paper G will denote a *real compact connected Lie group* with neutral element $1 \in G$ and Haar measure ν which is assumed to be normalized by $\nu(G) = 1$. Let $\mathscr{E}(G) = \bigcap_{m \geq 0} \mathscr{E}_C^m(G)$ be the complex vector space of all infinitely differentiable complex-valued functions on G. We shall provide $\mathscr{E}(G)$ with the topology of uniform convergence of the functions and of all their derivatives. Then the strong topological dual $\mathscr{E}'(G)$ of the Fréchet space $\mathscr{E}(G)$ consists of all *complex distributions* on G. The space $\mathscr{E}'(G)$ is a complex locally convex topological algebra with respect to the (jointly continuous) convolution product

$$(R, S) \rightsquigarrow R * S$$

of distributions. Moreover, the Dirac measure ε_1 forms the neutral element of $\mathscr{E}'(G)$ and the mapping

$$S \rightsquigarrow S^*$$

which is the transpose of the continuous linear mapping

$$\mathscr{E}(G) \ni f \rightsquigarrow (f^*: G \ni x \rightsquigarrow \bar{f}(x^{-1}) \in \boldsymbol{C}) \in \mathscr{E}(G)\,,$$

is an involution of $\mathscr{E}'(G)$. The involutive topological $\boldsymbol{C}$-algebra $\mathscr{E}'(G)$ is said to be *the group algebra* of G.

In the sequel, $\mathscr{M}(G) = \mathscr{E}'^0(G)$ will denote the involutive Banach algebra of all complex Radon measures on G under convolution and $\mathscr{A}(G)$ will stand for the complex involutive Banach algebras $\mathrm{L}^p(G) = \mathrm{L}_C^p(G; \nu)$, $p \in [\![1, +\infty[\![$, and $\mathscr{E}^0(G)$ simultaneously. Thus the injections of the sequence

$$\mathscr{A}(G) \to \mathscr{M}(G) \to \mathscr{E}'(G)$$

are monomorphisms with respect to the category of involutive topological $\boldsymbol{C}$-algebras. It will be convenient to denote the canonical norms of $\mathscr{A}(G)$ and $\mathscr{M}(G)$ by the same symbol $\|\cdot\|$.

It is well known (J. Dieudonné [14]) that $\mathrm{L}^2(G)$ is a complex (complete) Hilbert algebra with respect to the standard scalar product. In view of the spectral theorem of Ambrose-Gurevič let the external Hilbert sum

$$\mathrm{L}^2(G) = \bigoplus_{\lambda \in \Lambda(G)} \mathfrak{a}_\lambda \tag{1}$$

be *the spectral decomposition* of $\mathrm{L}^2(G)$. Using the picturesque terminology of J. Dieudonné [13] for some ring theoretic concepts, *the feet* $\{\mathfrak{a}_\lambda \mid \lambda \in \Lambda(G)\}$

are minimal self-adjoint two-sided ideals of $L^2(G)$ and also of $\mathscr{M}(G)$. Moreover, each foot $\mathfrak{a}_\lambda(\lambda \in \Lambda(G))$ is a lame (complete) Hilbert subalgebra of $L^2(G)$ which has a (self-adjoint) *identity element* u_λ and the (finite) dimension

$$\dim_C \mathfrak{a}_\lambda = u_\lambda(1) = n_\lambda^2 \qquad (\lambda \in \Lambda(G)) .$$

Here the natural number

$$n_\lambda = \| u_\lambda \| = \operatorname{long}_{L^2(G)}(\mathfrak{a}_\lambda)$$

denotes the *longitude* of the foot $\mathfrak{a}_\lambda$. In the following $\mathfrak{a}_{\lambda_0}$ will always denote the *trivial* foot of dimension $n_{\lambda_0}^2 = 1$ which consists of the constant complex-valued functions on G. It should be observed that *the socle* of the ring $L^2(G)$, i.e. the (algebraic) direct sum

$$\mathfrak{S}_{L^2(G)} = \coprod_{\lambda \in \Lambda(G)} \mathfrak{a}_\lambda ,$$

which is a vector subspace of $\mathscr{E}(G)$, is an everywhere norm dense two-sided ideal of $\mathscr{A}(G)$. Moreover, according to the Peter-Weyl theorem, there exists a bijection η of the set $\{\mathfrak{a}_\lambda \mid \lambda \in \Lambda(G)\}$ onto the dual object $\hat{G}$ of G such that *the group character*

$$\chi_\lambda = \frac{1}{n_\lambda} u_\lambda$$

associated with the foot $\mathfrak{a}_\lambda$ and the equivalence class $\eta(\mathfrak{a}_\lambda) \in \hat{G}$ of continuous irreducible unitary representations of G are interrelated according to the equation

$$\operatorname{Tr}(\eta(\mathfrak{a}_\lambda)) = \bar{\chi}_\lambda$$

for any parameter $\lambda \in \Lambda(G)$.

Choose an orientation on G and equip G with a (fixed) Riemannian structure such that the associated metric $\boldsymbol{g}$ is invariant by both right and left translations. Since G may be identified with the Riemannian homogeneous manifold $G \times G$ modulo the diagonal of $G \times G$ such a choice is indeed possible. Let $\Delta = \Delta_g$ be the Laplace-Beltrami operator on G with respect to the Riemannian metric $\boldsymbol{g}$. Then

$$\Delta \colon \mathscr{E}(G) \ni f \rightsquigarrow \operatorname{div}_g(\operatorname{grad}_g f) = * \mathrm{d} * \mathrm{d} f$$

where $*$ denotes the Hodge star with respect to the oriented Riemannian structure of G. It is a well known consequence of Schur's lemma that (1) represents a discrete spectral decomposition of $L^2(G)$ with respect to the self-adjoint positive-definite elliptic differential operator $-\Delta$. For the corresponding eigenvalues ζ_λ $(\lambda \in \Lambda(G))$ we have

$$\zeta_{\lambda_0} = 0\,, \qquad \zeta_\lambda > 0 \quad \text{for} \quad \lambda \in \Lambda(G)\backslash\{\lambda_0\}$$

and the countable family

$$\operatorname{Spec}(G, \boldsymbol{g}) = \{\zeta_\lambda \mid \lambda \in \Lambda(G)\}$$

is said to be *the* (*discrete*) *spectrum* of the compact connected oriented Riemannian manifold $(G, \boldsymbol{g})$. In particular, the space of all harmonic functions $G \to \boldsymbol{C}$ coincides with the trivial foot $\mathfrak{a}_{\lambda_0}$.

3. **Approximation Processes on $\mathscr{A}(G)$.** Let T be a non-empty *directed* set of parameters. A family $(I_t)_{t\in T}$ of continuous $\boldsymbol{C}$-linear mappings I_t: $\mathscr{A}(G) \to \mathscr{A}(G)$ is said to form an *approximation process* on the complex Banach space $\mathscr{A}(G)$ if it converges pointwise with respect to the section filter of the directed set T towards the identity automorphism of $\mathscr{A}(G)$, i.e. if

$$\lim_{t\in T} \| I_t(f) - f \| = 0$$

holds for each function $f \in \mathscr{A}(G)$.

As usual $\boldsymbol{R}_+^* =]\!]0, +\infty[\![= \boldsymbol{R}_+\backslash\{0\}$ will denote the open positive real half-line.

DEFINITION 1. The approximation process $(I_t)_{t\in T}$ on $\mathscr{A}(G)$ is said to have *the saturation structure* $(\varphi; \mathscr{A}(G); V)$ iff the following conditions are satisfied:

(I) There exists a mapping φ: $T \to \boldsymbol{R}_+^*$ fulfilling $\lim_{t\in T} \varphi(t) = 0$ such that $f \in \mathscr{A}(G)$ and

$$\| I_t(f) - f \| = o(\varphi(t)) \qquad (t \in T) \tag{2}$$

imply

$$f \in \mathfrak{a}_{\lambda_0}\,.$$

(II) There exists a vector subspace V of $\mathscr{A}(G)$ such that $f \in \mathscr{A}(G)$ and the condition

$$\| I_t(f) - f \| = O(\varphi(t)) \qquad (t \in T) \tag{3}$$

imply $f \in V$.

(III) Conversely, if $f \in V$ then f satisfies the condition (3).

In this case, V is called *the Favard space* of the saturation structure $(\varphi; \mathscr{A}(G); V)$.

After having reviewed these general concepts, let us now turn to an important special class of approximation processes (of the convolution type) on $\mathscr{A}(G)$. We shall agree to retain the preceding notations. In addition, denote by $\mathscr{Z}\mathscr{M}(G)$ *the center* of the complex involutive Banach algebra $\mathscr{M}(G)$ so that $\mathscr{Z}\mathscr{M}(G)$ is a closed involutive subalgebra of $\mathscr{M}(G)$. Observe that $\mathscr{A}(G)$ is in particular a Banach $\mathscr{Z}\mathscr{M}(G)$-module.

DEFINITION 2. A family $(\mu_t)_{t\in T}$ of complex Radon measures which belong to the convolution algebra $\mathscr{Z}\mathscr{M}(G)$ is said to form a *central approximate unit for* $\mathscr{A}(G)$ if the relations

$$\mu_t(G) = 1 \text{ for any } t \in T$$

and

$$\lim_{t\in T} \| \mu_t * f - f \| = 0$$

hold for any function $f \in \mathscr{A}(G)$.

Define for any parameter $t \in T$ the continuous linear mapping

$$I_t\colon \mathscr{A}(G) \ni f \rightsquigarrow \mu_t * f \in \mathscr{A}(G) .$$

Then $(I_t)_{t\in T}$ is said to be a *central approximation process which is generated on* $\mathscr{A}(G)$ *by the approximate unit* $(\mu_t)_{t\in T}$.

REMARK 1. Since $\mathscr{A}(G)$ is a Banach $\mathrm{L}^1(G)$-module, any central approximate unit for $\mathrm{L}^1(G)$ is also a central approximate unit for $\mathscr{A}(G)$. See E. Hewitt-K. A. Ross [22], Chapter VIII.

Before presenting a general saturation theorem for central approximation processes, we shall make a few observations and introduce some further concepts. Since each foot $\mathfrak{a}_\lambda$ is a two-sided ideal of the complex Banach algebra $\mathscr{M}(G)$ it follows that $\mathscr{Z}\mathscr{M}(G) * u_\lambda$ is a vector subspace of the center $\mathscr{Z}\mathfrak{a}_\lambda$ for each $\lambda \in \Lambda(G)$. On the other hand, since $\mathfrak{a}_\lambda$ is a *simple* complex algebra, its center $\mathscr{Z}\mathfrak{a}_\lambda$ is isomorphic with the field $\boldsymbol{C}$. Denoting by X the Gelfand functor, i.e. the contravariant functor from the category of complex commutative Banach algebras with identity to the category of compact topological spaces which assigns to each object of the first category its space of characters, we obtain the following

THEOREM 1. *For any* $\lambda \in \Lambda(G)$ *define*

$$c_\lambda\colon \mathscr{Z}\mathscr{M}(G) \ni \mu \rightsquigarrow \frac{1}{n_\lambda^2}\int_G \bar{u}_\lambda(x)\,\mathrm{d}\,\mu(x) \in \boldsymbol{C} .$$

Then we have $c_\lambda \in X(\mathscr{Z}\mathscr{M}(G))$ *and the identity*

$$\mu * u_\lambda = c_\lambda(\mu) \cdot u_\lambda$$

obtains for any measure $\mu \in \mathscr{Z}\mathscr{M}(G)$.

REMARK 2. Y. Kawada [28] has shown that the identity

$$X(\mathscr{Z}\mathrm{L}^1(G)) = \{c_\lambda \mid \mathscr{Z}\mathrm{L}^1(G) \mid \lambda \in \Lambda(G)\}$$

obtains. See also E. Hewitt-K. A. Ross [22], Chapter VII.

DEFINITION 3. A central approximate unit $(\mu_t)_{t\in T}$ for $\mathscr{A}(G)$ is said to be a (central) *pro-saturation measure of type* $(\varphi; \psi)$ with respect to $\mathscr{A}(G)$ iff the following requirements are satisfied:

There exists a function $\varphi: T \longrightarrow \boldsymbol{R}_+^*$ satisfying $\lim_{t\in T}\varphi(t) = 0$ and a function $\psi: \Lambda(G) \longrightarrow \boldsymbol{C}$ where

$$\psi(\lambda) \neq 0 \quad \text{for} \quad \lambda \in \Lambda(G)\backslash\{\lambda_0\}$$

in such a way that

$$\lim_{t\in T}\frac{c_\lambda(\mu_t) - 1}{\varphi(t)} = \psi(\lambda)$$

holds for all $\lambda \in \Lambda(G)$.

REMARK 3. Let K be a closed subgroup of G, with normalized Haar measure ν_K, such that (G, K) forms a Riemannian *symmetric pair*. See, for instance, R. R. Coifman-G. Weiss [12]. Then (G, K) is a Gelfand pair, i.e. the convolution algebra $\mathscr{M}(K\backslash G/K)$ of all complex Radon measures on G which are bi-invariant with respect to K, is a commutative involutive Banach algebra with the identity element ν_K. Denote by

$$\mathrm{L}^2(G/K) = \bigoplus_{\lambda\in\Lambda_K(G)} \mathfrak{H}_\lambda(G/K)$$

the spectral decomposition of the complex Hilbert space $\mathrm{L}^2(G/K)$ which is deduced from (1) and by $(\omega_\lambda)_{\lambda\in\Lambda_K(G)}$ the family of associated *zonal spherical functions*. Then the mapping

$$c_\lambda: \mathscr{M}(K\backslash G/K) \ni \mu \rightsquigarrow \frac{1}{n_\lambda}\int_G \bar{\omega}_\lambda(x)\,\mathrm{d}\,\mu(x) \in \boldsymbol{C}$$

is an element of the space $X(\mathscr{M}(K\backslash G/K))$ for any $\lambda \in \Lambda_K(G)$ and we have

$$\mu * \omega_\lambda = c_\lambda(\mu)\cdot\omega_\lambda$$

for all measures $\mu \in \mathscr{M}(K\backslash G/K)$.

Thus, by a generalization of Definition 3, the concept of *zonal pro-saturation measure* and, similarly, of *zonal approximation process* can be considered for symmetric pairs. Details of a saturation theory which is generalized in this direction may be found in [16] and [19]. Clearly, if the subgroup K is the diagonal of $G \times G$, the notion of central pro-saturation measure for G and of zonal pro-saturation measure for $(G \times G)/K$ coalesce. We shall return to this question in Example (ii) of Section 6 infra.

We shall assign to $\mathscr{A}(G)$ a complex Banach space $\mathscr{B}(G)$ according to the following rule:

$$\mathscr{B}(G) = \begin{cases} \mathscr{C}^0(G) & \text{if } \mathscr{A}(G) = \mathrm{L}^1(G) ; \\ \mathrm{L}^1(G) & \text{if } \mathscr{A}(G) = \mathscr{C}^0(G) ; \\ \mathrm{L}^{p'}(G) & \text{if } \mathscr{A}(G) = \mathrm{L}^p(G) \text{ and } p \in \,]1, +\infty[\,. \end{cases}$$

As usual, p' denotes the dual exponent of p. Under these circumstances, we shall refer to $(\mathscr{A}(G), \mathscr{B}(G))$ as an *adapted pair*. Bearing in mind this terminology, we establish the following saturation theorem.

THEOREM 2. *Let $(I_t)_{t \in T}$ be an approximation process which is generated on $\mathscr{A}(G)$ by the central pro-saturation measure $(\mu_t)_{t \in T}$ of type $(\varphi; \psi)$ with respect to $\mathscr{A}(G)$. Suppose that $(\mathscr{A}(G), \mathscr{B}(G))$ is an adapted pair and define*

$$\mathscr{V}_\psi(\mathscr{A}(G)) = \{f \in \mathscr{A}(G) \mid \psi(\lambda) \cdot u_\lambda * f = u_\lambda * \rho,\ \rho \in \mathscr{B}'(G),\ \lambda \in \Lambda(G)\} \,.$$

Then $(I_t)_{t \in T}$ has the saturation structure $(\varphi; \mathscr{A}(G); V)$ where the Favard space V satisfies the condition

$$V \subset \mathscr{V}_\psi(\mathscr{A}(G)) \,.$$

PROOF. (I) Suppose that the function $f \in \mathscr{A}(G)$ satisfies the condition (2), i.e. that

$$\lim_{t \in T} \frac{\| \mu_t * f - f \|}{\varphi(t)} = 0$$

holds. Then we obtain

$$\lim_{t \in T} \frac{c_\lambda(\mu_t) - 1}{\varphi(t)} \cdot u_\lambda * f = 0 \,,$$

hence

$$\psi(\lambda) \cdot u_\lambda * f = 0$$

for any $\lambda \in \Lambda(G)$. It follows $u_\lambda * f = 0$ for $\lambda \in \Lambda(G) \backslash \{\lambda_0\}$. Let $g = f - u_{\lambda_0} * f$. Then $g \in \mathrm{L}^1(G)$ and $u_\lambda * g = 0$ for each index $\lambda \in \Lambda(G)$. A familiar limiting argument yields $g = 0$. Hence $f = u_{\lambda_0} * f \in \mathfrak{a}_{\lambda_0}$.

(II) Suppose that $f \in \mathscr{A}(G)$ satisfies the condition (3). For brevity, define the function

$$Q_t(f) = \frac{1}{\varphi(t)} (\mu_t * f - f) \in \mathscr{A}(G)$$

for any $t \in T$. The task will be to show that the existence of a section S of the directed set T and the existence of the upper bound

$$\sup_{t \in S} \| Q_t(f) \| < +\infty$$

imply the existence of an element $\rho \in \mathscr{B}'(G)$ which verifies the identity

$$\psi(\lambda) \cdot u_\lambda * f = u_\lambda * \rho \tag{4}$$

for each $\lambda \in \Lambda(G)$. We provide $\mathscr{B}'(G)$ with the strong dual topology $\beta(\mathscr{B}'(G), \mathscr{B}(G))$. Since $(\mathscr{A}(G), \mathscr{B}(G))$ is an adapted pair, there exists a natural isometric isomorphism of the complex involutive Banach algebra $\mathscr{A}(G)$ onto a closed subalgebra of $\mathscr{B}'(G)$. Let $\mathscr{A}(G)$ be identified with its image under this isomorphism. Owing to the Alaoglu theorem, there exists a cofinal subset T_0 of S and an element $\rho \in \mathscr{B}'(G)$ such that

$$\lim_{t \in T_0} Q_t(f) = \rho$$

holds with respect to the weak dual topology $\sigma(\mathscr{B}'(G), \mathscr{B}(G))$. In particular, for any $\lambda \in \Lambda(G)$, the identity

$$\lim_{t \in T_0} u_\lambda * Q_t(f) = u_\lambda * \rho \tag{5}$$

holds with respect to the topology of pointwise convergence. On the other hand, the equality

$$\lim_{t \in T_0} u_\lambda * Q_t(f) = \psi(\lambda) \cdot u_\lambda * f \tag{6}$$

obtains pointwise for any $\lambda \in \Lambda(G)$. Combining (5) and (6) we obtain the desired identity (4).

This completes the proof of the saturation theorem.—

REMARK 4. Theorems 4, 5 and Theorems 7, 8 infra will show that, in general, the "set theoretic estimate" $\mathscr{V}_\psi(\mathscr{A}(G))$ for the Favard space V can *not* be improved.

In order to establish a further characterization of the space $\mathscr{V}_\psi(\mathscr{A}(G))$, we shall recall the concept of relative completion in the sense of E. Gagliardo [21], N. Aronszajn-E. Gagliardo [1] and H. Berens [4].

Let $(X; \|\cdot\|_X)$, $(Y; \|\cdot\|_Y)$ be complex Banach spaces. Suppose that the inclusion $X \subset Y$ holds and that the canonical injection $X \hookrightarrow Y$ is continuous and has norm $\leqq 1$. *The completion* $[X, Y]_0$ *of* X *relative to* Y is the complex vector space of all limits with respect to the norm topology of Y of those sequences $(x_n)_{n \geqq 1}$ in X which are bounded with respect to $\|\cdot\|_X$. The natural norm

$$[X, Y]_0 \ni y \rightsquigarrow \inf_{(x_n)_{n \geqq 1}} \{\sup_{n \geqq 1} \| x_n \|_X \mid \lim_{n \geqq 1} \| y - x_n \|_Y = 0\} = \| y \|_0$$

turns $[X, Y]_0$ into a complex Banach space. It is to be emphasized that the identity $[X, Y]_0 = X$ holds in the case when the space X is *reflexive.*

Let $(\mathscr{A}(G), \mathscr{B}(G))$ be an adapted pair and $\psi: \Lambda(G) \to \boldsymbol{C}$ be any mapping. The graph norm

$$\mathscr{V}_\psi(\mathscr{A}(G)) \ni f \rightsquigarrow \| f \|_{\mathscr{V}} = \| f \| + \| \rho \|$$

where $\rho \in \mathscr{B}'(G)$ is the unique element associated with f according to the equations

$$\psi(\lambda) \cdot u_\lambda * f = u_\lambda * \rho \qquad (\lambda \in \Lambda(G)) ,$$

turns $\mathscr{V}_\psi(\mathscr{A}(G))$ into a complex Banach space. In a similar way, define the complex vector space

$$\mathscr{W}_\psi(\mathscr{A}(G)) = \{f \in \mathscr{A}(G) \mid \psi(\lambda) \cdot u_\lambda * f = u_\lambda * g,\ g \in \mathscr{A}(G),\ \lambda \in \Lambda(G)\} .$$

Then $\mathscr{W}_\psi(\mathscr{A}(G))$ is a Banach space when equipped with the graph norm

$$\mathscr{W}_\psi(\mathscr{A}(G)) \ni f \rightsquigarrow \| f \|_{\mathscr{W}} = \| f \| + \| g \| .$$

Now $\mathscr{V}_\psi(\mathscr{A}(G))$ admits the following characterization:

THEOREM 3. *Retain the above notations and assumptions - then the identity*

$$\mathscr{V}_\psi(\mathscr{A}(G)) = [\ \mathscr{W}_\psi(\mathscr{A}(G)), \mathscr{A}(G)]_0$$

obtains.

PROOF. Let us suppose that $\mathscr{A}(G) = \mathrm{L}^1(G)$ holds and let $f \in \mathscr{V}_\psi(\mathrm{L}^1(G))$ be fixed. In view of a result due to E. Hewitt-K. A. Ross [22], Chapter VII, it is possible to construct a central approximate unit $(h_n \nu)_{n \geqq 1}$ for $\mathrm{L}^1(G)$ such that $h_n \in \mathfrak{S}_{\mathrm{L}^2(G)}$ and $\| h_n \nu \| = 1$ for any integer $n \geqq 1$. Let $\rho \in \mathscr{M}(G)$ be the Radon measure associated with f such that $\psi(\lambda) u_\lambda * f = u_\lambda * \rho$ holds for any $\lambda \in \Lambda(G)$. This being so, it then follows that

$$h_n * f \in \mathscr{W}_\psi(\mathrm{L}^1(G)) \qquad (n \geqq 1) ;$$

$$\sup_{n \geqq 1} \| h_n * f \|_{\mathscr{W}} \leqq \| f \| + \| \rho \| ;$$

$$\lim_{n \to \infty} \| h_n * f - f \| = 0 .$$

Thus $f \in [\mathscr{W}(\mathrm{L}^1(G)), \mathrm{L}^1(G)]_0$.

Conversely, let $f \in [\mathscr{W}(\mathrm{L}^1(G)), \mathrm{L}^1(G)]_0$ be given. Choose sequences $(f_n)_{n \geqq 1}$ and $(g_n)_{n \geqq 1}$ in the space $\mathrm{L}^1(G)$ such that

$$\lim_{n \to \infty} \| f_n - f \| = 0 ;$$

$$\psi(\lambda) \cdot u_\lambda * f_n = u_\lambda * g_n \qquad (n \geqq 1,\ \lambda \in \Lambda(G)) ;$$

$$\sup_{n \geqq 1} \| f_n \|_{\mathscr{W}} = \sup_{n \geqq 1} (\| f_n \| + \| g_n \|) < +\infty .$$

By virtue of the Alaoglu theorem, the subsets of $\mathscr{M}(G)$ which are bounded with respect to the strong dual topology are relatively sequentially compact with respect to the vague topology. This fact implies the existence of a subsequence $(g_{n_k}\nu)_{k\geq 1}$ of the sequence $(g_n\nu)_{n\geq 1}$ and of a Radon measure $\rho \in \mathscr{M}(G)$ such that

$$\lim_{k\to\infty} g_{n_k}\nu = \rho$$

holds with respect to the vague topology of $\mathscr{M}(G)$. Consequently, the equations

$$\lim_{k\to\infty} u_\lambda * g_{n_k} = u_\lambda * \rho \qquad (\lambda \in \Lambda(G))$$

and

$$\lim_{k\to\infty} \psi(\lambda) u_\lambda * f_{n_k} = u_\lambda * \rho \qquad (\lambda \in \Lambda(G))$$

hold with respect to the topology of pointwise convergence. In particular

$$\psi(\lambda) u_\lambda * f = u_\lambda * \rho$$

obtains for any $\lambda \in \Lambda(G)$, i.e. $f \in \mathscr{V}_\psi(\mathrm{L}^1(G))$ and $\|f\|_0 = \|f\|_{\mathscr{V}}$.

Summing up, we have proved the equality $\mathscr{V}_\psi(\mathscr{A}(G)) = [\mathscr{W}_\psi(\mathscr{A}(G)), \mathscr{A}(G)]_0$ in the case $\mathscr{A}(G) = \mathrm{L}^1(G)$. If $\mathscr{A}(G) = \mathscr{E}^0(G)$, the proof follows in a similar fashion. Finally, for $\mathscr{A}(G) = \mathrm{L}^p(G)$ and $p \in]\!]1, +\infty[\![$, the statement of the theorem is immediate from the reflexivity of the spaces $\mathrm{L}^p(G)$.—

In the special case when $G = \boldsymbol{T}$ is the one-dimensional torus group, a proof of Theorem 3 based on the convergence theorem for Fejér means may be found in the monograph by P. L. Butzer-R. J. Nessel [10], Chapter 10.

4. **The Heat-Diffusion Approximation Process.** Let us agree to retain the notations and conventions introduced in the preceding sections; in particular $\boldsymbol{R}_+^*$ denotes the open positive real half-line, equipped with the induced Lebesgue measure dt.

We first define by means of Spec$(G, \boldsymbol{g})$ the family of functions

$$q_\lambda\colon \boldsymbol{R}_+^* \ni t \rightsquigarrow \exp(-\zeta_\lambda t) \qquad (\lambda \in \Lambda(G))\,. \tag{7}$$

Let now the value $t \in \boldsymbol{R}_+^*$ be fixed and consider, following E. M. Stein [44, 45], the continuous endomorphism $\exp(\Delta t)$ of the complex Banach space $\mathrm{L}^2(G)$. Since the positivity of the resolvent of Δ on $\boldsymbol{R}_+^*$ implies that the restricted operator

$$\exp(\Delta t)\colon \mathscr{E}^0(G) \to \mathscr{E}^0(G)$$

is a *positive* linear mapping, there exists exactly one (positive) Radon measure $\mu_t \in \mathscr{M}(G)$ such that $\mu_t(G) = 1$ and

$$\exp(\Delta t)\colon \mathscr{E}^0(G) \ni f \rightsquigarrow f * \mu_t \in \mathscr{E}^0(G)$$

holds. By an extension to $L^2(G)$, we infer the identities

$$u_\lambda * \exp(\Delta t)f = \exp(\Delta t)(u_\lambda * f) \qquad (\lambda \in \Lambda(G))$$

for any function $f \in L^2(G)$. The spectral decomposition (1) then shows that

$$\exp(\Delta t)f = \sum_{\lambda \in \Lambda(G)} q_\lambda(t) u_\lambda * f$$

holds in the complete Hilbert algebra $L^2(G)$. Consequently, the kernel function

$$K = \sum_{\lambda \in \Lambda(G)} q_\lambda \otimes u_\lambda$$

is an element of the complex Fréchet space $L^2_{\mathrm{loc}}(\boldsymbol{R}^*_+ \times G)$.

As usual, let $\mathscr{D}(\boldsymbol{R}^*_+ \times G)$ be the complex vector space of all infinitely differentiable complex-valued functions on the manifold $\boldsymbol{R}^*_+ \times G$ whose support is compact. Provide the space $\mathscr{D}(\boldsymbol{R}^*_+ \times G)$ with the canonical $\mathscr{LF}$ topology and let $\mathscr{D}'(\boldsymbol{R}^*_+ \times G)$ be its strong topological dual which consists of all complex distributions on $\boldsymbol{R}^*_+ \times G$. Then we have

$$\mathscr{D}'(\boldsymbol{R}^*_+ \times G) = \mathscr{D}'(\boldsymbol{R}^*_+) \hat{\otimes} \mathscr{E}'(G)$$

and we may identify the Fréchet space $L^2_{\mathrm{loc}}(\boldsymbol{R}^*_+ \times G)$ with its image under the continuous injection

$$h \rightsquigarrow h \cdot (\mathrm{d}t \otimes \nu)\ .$$

Let the heat-diffusion operator $\partial/\partial t - \Delta$ be operating from the space $\mathscr{D}'(\boldsymbol{R}^*_+ \times G)$ into itself as a continuous linear mapping. We observe that for an arbitrary *finite* subset Λ_0 of $\Lambda(G)$ the function $K_0 = \sum_{\lambda \in \Lambda_0} q_\lambda \otimes u_\lambda$ which belongs to $\mathscr{E}(\boldsymbol{R}^*_+ \times G)$ satisfies the heat-diffusion equation $(\partial/\partial t - \Delta)K_0 = 0$. We obtain therefore the identity

$$\left(\frac{\partial}{\partial t} - \Delta\right)K = 0 \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^*_+ \times G)\,.$$

On the other hand it is a well known fact that the parabolic differential operator $\partial/\partial t - \Delta$ is *hypoelliptic*: see, for instance, F. Trèves [46], Chapter 7. It follows that the singular support of K in $\boldsymbol{R}^*_+ \times G$ is empty, i.e.

$$K \in \mathscr{E}(\boldsymbol{R}^*_+ \times G) = \mathscr{E}(\boldsymbol{R}^*_+) \hat{\otimes} \mathscr{E}(G)\ .$$

Therefore we have

$$\mu_t = K(t, \cdot)\nu \qquad (t \in \boldsymbol{R}^*_+)\ ,$$

i.e. the Radon measure μ_t admits the central function $K(t, \cdot)$ as its Lebesgue-Nikodym density with respect to the base ν.
From the fact that $(\exp(\Delta t))_{t \in \boldsymbol{R}_+}$ forms a strongly continuous one-parameter Markovian semigroup in $\mathscr{A}(G)$ which admits Δ as its infinitesimal generator (see too N. Dunford-J. T. Schwartz [20], Chapter XIV), we infer that the generating family $(\mu_t)_{t \in \boldsymbol{R}_+^*}$ is a *central* approximate unit for $\mathscr{A}(G)$. Moreover we have the following result.

THEOREM 4. *The heat-diffusion approximation process* $(\exp(\Delta t))_{t \in \boldsymbol{R}_+^*}$ *on* $\mathscr{A}(G)$ *is central. It has the saturation structure* $(\varphi; \mathscr{A}(G); \mathscr{V}_\psi(\mathscr{A}(G)))$ *where the mappings* φ, ψ *are defined in the following way*:

$$\varphi\colon \boldsymbol{R}_+^* \ni t \rightsquigarrow t\ ; \qquad \psi\colon \Lambda(G) \ni \lambda \rightsquigarrow -\zeta_\lambda\ .$$

PROOF. For the character $c_\lambda \in X(\mathscr{Z}\mathscr{M}(G))$ defined in Theorem 1 we obtain in the present case obviously

$$c_\lambda(\mu_t) = q_\lambda(t)$$

for any pair $(\lambda, t) \in \Lambda(G) \times \boldsymbol{R}_+^*$. It follows that $(\mu_t)_{t \in \boldsymbol{R}_+^*}$ forms a central pro-saturation measure of type $(\varphi; \psi)$ where the functions φ and ψ are defined as indicated above. Let V denote the Favard space of $(\exp(\Delta t))_{t \in \boldsymbol{R}_+^*}$ with respect to $\mathscr{A}(G)$. By virtue of Theorem 2 which ensures the validity of conditions (I) and (II) in Definition 1, and in view of Theorem 3 it suffices to prove that the inclusion

$$[\,\mathscr{W}_\psi(\mathscr{A}(G)), \mathscr{A}(G)]_0 \subset V$$

holds.

(III) Let any function $f \in [\,\mathscr{W}_\psi(\mathscr{A}(G)), \mathscr{A}(G)]_0$ be given. Choose sequences $(f_n)_{n \geqq 1}$ and $(g_n)_{n \geqq 1}$ in $\mathscr{A}(G)$ such that

$$\lim_{n \to \infty} \|f_n - f\| = 0\ ;$$

$$\psi(\lambda) \cdot u_\lambda * f_n = u_\lambda * g_n \qquad (n \geqq 1,\ \lambda \in \Lambda(G))\ ;$$

$$\sup_{n \geqq 1} \|f_n\|_{\mathscr{W}} = \sup_{n \geqq 1} (\|f_n\| + \|g_n\|) = M < +\infty\ .$$

A direct computation shows that the identity

$$\exp(\Delta t) f_n - f_n = \int_0^t \exp(\Delta \tau) g_n \mathrm{d}\tau$$

obtains for any $n \geqq 1$ and $t \in \boldsymbol{R}_+^*$. It follows

$$\|\exp(\Delta t) f - f\| \leqq M \cdot t$$

for $t \in \boldsymbol{R}_+^*$. Thus we have

$$\|\exp(\Delta t) f - f\| = O(\varphi(t)) \qquad (t \longrightarrow 0+)\ ,$$

i.e. $f \in V$ and the proof is complete.—

Let Dom (Δ) denote the domain of the infinitesimal generator Δ of the heat-diffusion semigroup $(\exp(\Delta t))_{t \in \boldsymbol{R}_+}$ in $\mathscr{A}(G)$. Taking into account a general theorem due to H. Berens [4], we obtain furthermore the following result.

COROLLARY. *The approximation process* $(\exp(\Delta t))_{t \in \boldsymbol{R}_+^*}$ *on* $\mathscr{A}(G)$ *has the relative completion*

$$V = [\mathrm{Dom}\,(\Delta), \mathscr{A}(G)]_0$$

as its Favard space.

5. The Poisson Approximation Process. We continue to follow the treatises of E. M. Stein [44, 45] in replacing the family $(q_\lambda)_{\lambda \in \Lambda(G)}$ in (7) by the functions

$$p_\lambda\colon \boldsymbol{R}_+^* \ni t \rightsquigarrow \exp(-\sqrt{\zeta_\lambda}\, t) \qquad (\lambda \in \Lambda(G)) .$$

Then the Poisson kernel

$$P = \sum_{\lambda \in \Lambda(G)} p_\lambda \otimes u_\lambda$$

associated with the compact Lie group G belongs to the space $\mathrm{L}^2_{\mathrm{loc}}(\boldsymbol{R}_+^* \times G)$. Denote by

$$\underline{\Delta} = \frac{\partial^2}{\partial t^2} + \Delta$$

the Laplace-Beltrami operator of the Riemannian manifold $\boldsymbol{R}_+^* \times G$ and assume that $\underline{\Delta}$ acts on the space $\mathscr{D}'(\boldsymbol{R}_+^* \times G)$ of distributions. Then P is a *harmonic* distribution on $\boldsymbol{R}_+^* \times G$, i.e.

$$\underline{\Delta} P = 0 \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}_+^* \times G) .$$

Therefore an application of Weyl's regularity lemma shows that P is a harmonic *function* on $\boldsymbol{R}_+^* \times G$. In particular, we have $P \in \mathscr{E}(\boldsymbol{R}_+^* \times G)$. Define the Radon measures

$$\pi_t = P(t, \cdot)\nu \qquad (t \in \boldsymbol{R}_+^*)$$

on G. Then the family $(\pi_t)_{t \in \boldsymbol{R}_+^*}$ forms a central approximate unit for $\mathscr{A}(G)$. The associated strongly continuous one-parameter Markovian semigroup $(\exp(-\sqrt{-\Delta}\, t))_{t \in \boldsymbol{R}_+}$ has $-\sqrt{-\Delta}$ as its infinitesimal generator.

THEOREM 5. *The Poisson approximation process* $(\exp(-\sqrt{-\Delta}\, t))_{t \in \boldsymbol{R}_+^*}$ *on* $\mathscr{A}(G)$ *is central. Define the mappings*

$$\varphi\colon \boldsymbol{R}_+^* \ni t \rightsquigarrow t\ ; \qquad \psi\colon \Lambda(G) \ni \lambda \rightsquigarrow -\sqrt{\zeta_\lambda}\ .$$

Then $(\exp(-\sqrt{-\Delta}\, t))_{t \in \boldsymbol{R}_+^*}$ *has the saturation structure* $(\varphi; \mathscr{A}(G); \mathscr{V}_\psi(\mathscr{A}(G))$

and the identity

$$\mathscr{V}_{\psi}(\mathscr{A}(G)) = [\mathrm{Dom}\,(-\sqrt{-\Delta}\,),\ \mathscr{A}(G)]_0$$

obtains for the associated Favard space.

The proof is based on Theorems 2 and 3 supra and follows by an application of the method outlined in Section 4. We omit the details.

REMARK 5. Let $\alpha \in]\!]0, 1[\![$. Needless to say, there are analogous results concerning the saturation behaviour of the central approximation processes $(\exp(-(-\Delta)^{\alpha}t))_{t \in \boldsymbol{R}_+^*}$ on $\mathscr{A}(G)$. A discussion of this case, of course, depends upon the concepts of *fractional* differentiation and integration. See, for instance, P. L. Butzer-H. Berens [8], Chapter II, and the recent paper by H. Bavinck [3]. We shall return to these problems within our general framework in a forthcoming paper.

6. Applications I. (i) Let G be an n-dimensional *abelian* compact connected Lie group. Then G is isomorphic with the Euclidean *torus* $\boldsymbol{T}^n = \boldsymbol{R}^n/\boldsymbol{Z}^n = (\boldsymbol{R}/\boldsymbol{Z})^n$ of dimension n. We shall identify the integral lattice $\boldsymbol{Z}^n$ with its dual one, i.e. with the lattice of all vectors $x \in \boldsymbol{R}^n$ such that the scalar product $(\lambda \mid x)$ belongs to $\boldsymbol{Z}$ for all $\lambda \in \boldsymbol{Z}^n$. Then we have

$$\Lambda(\boldsymbol{T}^n) = \widehat{\boldsymbol{T}}^n = \boldsymbol{Z}^n .$$

If $\boldsymbol{R}^n \ni x \rightsquigarrow \dot{x} \in \boldsymbol{T}^n$ denotes the canonical epimorphism, we obtain for any wave number $\lambda \in \boldsymbol{Z}^n$ in the present case

$$n_\lambda = 1 ;$$

$$u_\lambda = \chi_\lambda \colon \boldsymbol{T}^n \ni \dot{x} \rightsquigarrow \exp(2\pi i(\lambda \mid x)) \in \boldsymbol{C} ;$$

$$c_\lambda \colon \mathscr{M}(\boldsymbol{T}^n) \ni \mu \rightsquigarrow \mathscr{F}\mu(\lambda) \in \boldsymbol{C} ;$$

where $\mathscr{F} \colon \mathscr{M}(\boldsymbol{T}^n) \longrightarrow \mathscr{E}^0(\boldsymbol{Z}^n)$ stands for the classical Fourier transformation.

Let the manifold $\boldsymbol{R}^n$ be endowed with its canonical orientation and with its canonical Riemannian structure such that the corresponding metric $\boldsymbol{g}_0$ induces the Euclidean norm $|\cdot|$. Denote by $\boldsymbol{g}_0/\boldsymbol{Z}^n$ the unique flat Riemannian metric on $\boldsymbol{T}^n$ such that $(\boldsymbol{R}^n, \boldsymbol{g}_0)$ is the universal Riemannian covering manifold of $(\boldsymbol{T}^n, \boldsymbol{g}_0/\boldsymbol{Z}^n)$. Then we obtain

$$\mathrm{Spec}\,(\boldsymbol{T}^n, \boldsymbol{g}_0/\boldsymbol{Z}^n) = \{4\pi^2 |\lambda|^2 \mid \lambda \in \boldsymbol{Z}^n\}$$

and the heat-diffusion kernel K admits in the present case the following expansion in a *theta series*:

$$K \colon (\boldsymbol{R}_+^* \times \boldsymbol{T}^n) \ni (t, \dot{x}) \rightsquigarrow \sum_{\lambda \in \boldsymbol{Z}^n} \exp(2\pi i((\lambda \mid x) + 2\pi i t(\lambda \mid \lambda))) = \Theta_0(\tau_t, x) .$$

Here $\Theta_0(\tau_t, \cdot)$ denotes the classical theta function of characteristic 0 and of modulus

$$\tau_t = 4\pi i t \cdot 1_n \qquad (t \in \boldsymbol{R}_+^*)$$

where 1_n stands for the (n, n) identity matrix. Therefore τ_t belongs to the Siegel upper-half space of degree n: see, for instance, J. Igusa [24], Chapter I.

An application of Theorems 4 and 5 in the present situation furnishes results due to R. J. Nessel-A. Pawelke [34]. If we assume $n = 1$, the Cauchy problem which is solved by the heat-diffusion semigroup $(\exp(\Delta t))_{t \in \boldsymbol{R}_+}$ in $\mathscr{A}(\boldsymbol{T})$ is known as the "*Fourier torus problem*" (A. Sommerfeld [43], Kapitel III). Detailed treatments of this problem and of the corresponding Dirichlet problem from the saturation theoretic point of view may be found in the monographs by P. L. Butzer-H. Berens [8] and P. L. Butzer-R. J. Nessel [10]. Also see [17] for a study of saturation on abelian quotient groups.

(ii) Now suppose that the compact connected Lie group G is *diffeomorphic with the standard sphere* $\boldsymbol{S}_{n-1}$ of an Euclidean space $\boldsymbol{R}^n (n \geqq 1)$. Since $\boldsymbol{S}_0$ is not connected and $\boldsymbol{S}_1$ is diffeomorphic with the torus $\boldsymbol{T}$ as treated in Example (i) supra, we may suppose $n \geqq 3$. Then, according to a well-known result due to H. Samelson [41], *only the choice* $n = 4$ *is possible*. Indeed, the Lie group SU(2), the group of unit quaternions, is diffeomorphic with the sphere $\boldsymbol{S}_3$ and SU(2) acts as a subgroup of the special orthogonal group SO(4) in a natural way transitively by rotations on $\boldsymbol{S}_3$. As usual, let

$$\Lambda(\mathrm{SU}(2)) = \frac{1}{2}\boldsymbol{N},$$

be the family of weights, where $\boldsymbol{N} = \boldsymbol{Z}_+$.

Let the compact manifold $\boldsymbol{S}_3$ be oriented to the exterior and let it be endowed with the Riemannian structure induced by the canonical structure of $\boldsymbol{R}^4$. Denote by $\boldsymbol{g}$ the Riemannian metric of $\boldsymbol{S}_3$. A direct computation (cf. M. Berger-P. Gauduchon-E. Mazet [6]) shows that

$$n_\lambda = 2\lambda + 1 \qquad \left(\lambda \in \frac{1}{2}\boldsymbol{N}\right);$$

$$\mathrm{Spec}\,(\mathrm{SU}(2), \boldsymbol{g}) = \left\{4\lambda(\lambda + 1) \mid \lambda \in \frac{1}{2}\boldsymbol{N}\right\}.$$

In order to obtain explicitly the units u_λ and the characters χ_λ $(\lambda \in (1/2)\boldsymbol{N})$ of the spectral decomposition of $\mathrm{L}^2(\mathrm{SU}(2))$, fix the North pole

$1=(0,0,0,1)\in \boldsymbol{R}^4$ of S_3. Then we may consider SO(3) as the stabilizer of 1 in the rotation group SO(4) and identify S_3 with the compact homogeneous manifold SO(4)/SO(3). Since (SO(4), SO(3)) is a Riemannian *symmetric pair*, we obtain for the unit u_λ the identity

$$u_\lambda = Z_1^{2\lambda} \qquad \left(\lambda \in \frac{1}{2}\boldsymbol{N}\right)$$

where $Z_1^{2\lambda}$ denotes *the zonal surface spherical harmonic* of degree 2λ with pole 1 on S_3. See, for instance, R. R. Coifman-G. Weiss [11, 12]. Thus, for any $\lambda \in (1/2)\boldsymbol{N}$, the character χ_λ takes the following form:

$$\chi_\lambda\colon S_3 \ni (x_j)_{1\leq j\leq 4} \rightsquigarrow \frac{1}{2\lambda+1} Z_1^{2\lambda}(x_4) = \prod\nolimits_{2\lambda}^{(\frac{1}{2},\frac{1}{2})}(x_4)\ .$$

Here $\prod_{2\lambda}^{(1/2,1/2)}(X) \in \boldsymbol{R}[X]$ denotes the *ultraspherical polynomial* (Gegenbauer polynomial) in the indeterminate X of degree 2λ with exponent 1/2 i.e. *the Čebyšev polynomial of the second kind*

$$\prod\nolimits_{2\lambda}^{(\frac{1}{2},\frac{1}{2})}(X) = \sum_{0\leq m\leq[\lambda]} (-1)^m \frac{(2\lambda-m)!}{m!(2\lambda-2m)!}(2X)^{2(\lambda-m)}\ .$$

The heat-diffusion kernel K and the Poisson kernel P admit therefore the following expansions in *Laplace series*:

$$K\colon \boldsymbol{R}_+^* \times S_3 \ni (t, (x_j)_{1\leq j\leq 4}) \rightsquigarrow \sum_{\lambda\in(1/2)\boldsymbol{N}} \exp(-4\lambda(\lambda+1)t)\cdot(2\lambda+1)\prod\nolimits_{2\lambda}^{(\frac{1}{2},\frac{1}{2})}(x_4)\ ;$$

$$P\colon \boldsymbol{R}_+^* \times S_3 \ni (t, (x_j)_{1\leq j\leq 4}) \rightsquigarrow \sum_{\lambda\in(1/2)\boldsymbol{N}} \exp(-2\sqrt{\lambda(\lambda+1)}t)\cdot(2\lambda+1)\prod\nolimits_{2\lambda}^{(\frac{1}{2},\frac{1}{2})}(x_4)\ .$$

Equip the sphere S_3 with the normalized Lebesgue surface measure which may be identified with the Haar measure ν of SU(2). Then an application of Theorems 4 and 5 to the special unitary group SU(2) yields results which are concerned with *the saturation behaviour of the approximation of functions $f\in \mathscr{A}(S_3)$ performed by Laplace expansions in Čebyšev polynomials of the second kind.*

At this juncture, bearing in mind the notion of zonal approximation process [19] as outlined in Remark 3 supra and the fact that the commutative complex Banach algebras $\mathscr{M}$(SO(3)\SO(4)/SO(3)) and $\mathscr{Z}\mathscr{M}$(SU(2)) may be identified, we shall summarize the preceding considerations in the following

THEOREM 6. *On the compact unit sphere S_3 in $\boldsymbol{R}^4$, the zonal approximation processes with respect to the compact homogeneous manifold* SO(4)/SO(3) *and the central approximation processes with respect to the special unitary group* SU(2) *coalesce.*

For further results concerning saturation theory on spheres we refer to S. Pawelke [35] and H. Berens-P. L. Butzer-S. Pawelke [5]. Also see S. Pawelke [36]. Details for saturation on general compact homogeneous spaces may be found in [15, 16].

7. **Applications II.** In this section we shall continue to study central approximation processes on the Lie group SU(2). In order to adopt the conventions of R. A. Mayer [29, 30], let us now parametrize the spectral decomposition of $L^2(\mathbf{SU}(2))$ by the set

$$\Lambda(\mathbf{SU}(2)) = \boldsymbol{N}^* .$$

Henceforth we will write

$$k = 2\lambda + 1 \qquad \left(\lambda \in \frac{1}{2}\boldsymbol{N}\right)$$

for the weights of SU(2). Let us identify SU(2) and S_3 as in Section 6. Then we have

$$\left.\begin{array}{l} u_k\colon S_3 \ni (x_j)_{1\leqq j\leqq 4} \rightsquigarrow Z_1^{k-1}(x_4) \\ \chi_k\colon S_3 \ni (x_j)_{1\leqq j\leqq 4} \rightsquigarrow \prod_{k-1}^{(\frac{1}{2},\frac{1}{2})}(x_4) \end{array}\right. \qquad (k \in \boldsymbol{N}^*) .$$

(iii) In the algebra $\mathscr{Z}\mathscr{M}(\mathbf{SU}(2))$ define the sequence $(\mu_n)_{n\geqq 0}$ by letting

$$\mu_n = \sum_{1\leqq k\leqq n+1}\left[\binom{2n+2}{n+k+1}\Big/\binom{2n+2}{n+2}\right]u_k\nu \qquad (n \in \boldsymbol{N}) .$$

Obviously $\mu_n(\mathbf{SU}(2)) = 1$. Since $\mu_n \geqq 0$ (R. A. Mayer [30]) it follows

$$\|\mu_n\| = 1 \qquad (n \in \boldsymbol{N}) .$$

Consequently, the continuous linear mappings

$$V_n\colon L^1(\mathbf{SU}(2)) \ni f \rightsquigarrow \mu_n * f \in L^1(\mathbf{SU}(2)) \qquad (n \in \boldsymbol{N})$$

are contractions. Therefore $(V_n)_{n\geqq 0}$ forms an equicontinuous family which converges pointwise on the socle $\mathfrak{S}_{L^2(\mathbf{SU}(2))}$ towards the identity automorphism of $L^1(\mathbf{SU}(2))$. Since $\mathfrak{S}_{L^2(\mathbf{SU}(2))}$ is everywhere norm dense in $L^1(\mathbf{SU}(2))$, it follows from Remark 1 supra that $(\mu_n)_{n\geqq 1}$ is a central approximate unit for $\mathscr{A}(\mathbf{SU}(2))$ which generates *the approximation process* $(V_n)_{n\geqq 1}$ *of the de La Vallée Poussin type* on $\mathscr{A}(\mathbf{SU}(2))$. In the present case we obtain

$$c_k(\mu_n) = \begin{cases} \binom{2n+2}{n+k+1}\Big/\binom{2n+2}{n+2} & \text{if } 1 \leqq k \leqq n+1 ; \\ 0 & \text{if } n+1 < k ; \end{cases}$$

whence

$$\lim_{n\to\infty} \frac{c_k(\mu_n) - 1}{1/n} = 1 - k^2$$

for any $k \in N^*$. Therefore $(\mu_n)_{n\geq 1}$ forms a central pro-saturation measure of type $(n \rightsquigarrow 1/n, k \rightsquigarrow 1 - k^2)$ with respect to $\mathscr{A}(\mathrm{SU}(2))$.

THEOREM 7. *The approximation process* $(V_n)_{n\geq 1}$ *on* $\mathscr{A}(\mathrm{SU}(2))$ *of the de La Vallée Poussin type is central and has the saturation structure* $(\varphi; \mathscr{A}(\mathrm{SU}(2)); \mathscr{V}_\psi(\mathrm{SU}(2)))$ *where*

$$\varphi\colon N^* \ni n \rightsquigarrow \frac{1}{n}\,; \qquad \psi\colon N^* \ni k \rightsquigarrow 1 - k^2\,.$$

PROOF. Let V denote the Favard space of $(V_n)_{n\geq 1}$ with respect to $\mathscr{A}(\mathrm{SU}(2))$. In view of Theorems 2 and 3 it suffices to prove that the inclusion

$$[\mathscr{W}_\psi(\mathrm{SU}(2)), \mathscr{A}(\mathrm{SU}(2))]_0 \subset V$$

holds.

(III) Let the function $f \in [\mathscr{W}_\psi(\mathrm{SU}(2)), \mathscr{A}(\mathrm{SU}(2))]_0$ be given. There are sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ in the space $\mathscr{A}(\mathrm{SU}(2))$ such that

$$\lim_{n\to\infty} ||f_n - f|| = 0\,;$$

$$\psi(k)u_k * f_n = u_k * g_n \qquad (n \geqq 1,\, k \geqq 1)\,;$$

$$\sup_{n\geqq 1} ||f_n||_{\mathscr{W}} = \sup_{n\geqq 1} (||f_n|| + ||g_n||) = M < +\infty\,.$$

Since the identity

$$\psi(k)c_k(\mu_j) = -j(j+2)(c_k(\mu_j) - c_k(\mu_{j-1}))$$

holds for any pair $(k, j) \in N^* \times N^*$, we obtain for all integers $r \geqq m \geqq 1$:

$$V_m(f_n) - V_r(f_n) = \sum_{m+1\leqq j\leqq r} \frac{1}{j(j+2)} V_j(g_n) \qquad (n \in N^*)\,.$$

Thus the estimate

$$||V_m(f_n) - f_n|| \leqq M \cdot \sum_{m+1\leqq j} \frac{1}{j(j+2)} \leqq \varphi(m) \cdot M \qquad (m \in N^*)$$

obtains. Hence $f \in V$ and the proof is complete.

(iv) Finally, let

$$\mu_{n-1} = \Big(\sum_{1\leqq k\leqq n} k^2\Big)^{-1} \Big(\sum_{1\leqq k\leqq n} k\chi_k\Big)^2 \nu \qquad (n \in N^*)\,.$$

As in the preceding section it can be shown that the sequence $(\mu_{n-1})_{n\geqq 1}$

forms a central approximate unit for $\mathscr{A}(\mathrm{SU}(2))$. Since the identity

$$\mu_{n-1} = \sum_{1 \leq k \leq 2n-1} \left(1 - \frac{6(2n+1)^2[(1/2)k^2] - k^2(k^2-1)}{8kn(n+1)(2n+1)}\right) u_k \nu \qquad (n \in \boldsymbol{N}^*)$$

obtains (R. A. Mayer [30]), it follows

$$\lim_{n\to\infty} \frac{c_k(\mu_{n-1}) - 1}{1/(n-1)} = -\frac{3}{2}\frac{[(1/2)k^2]}{k}$$

for $k \in \boldsymbol{N}^*$.

THEOREM 8. *The central approximation process $(F_{n-1})_{n\geq 1}$ on $\mathscr{A}(\mathrm{SU}(2))$ of the Fejér type which is generated by the approximate unit $(\mu_{n-1})_{n\geq 1}$ for $\mathscr{A}(\mathrm{SU}(2))$ has the saturation structure $(\varphi; \mathscr{A}(\mathrm{SU}(2)))$; $V_\psi(\mathscr{A}(\mathrm{SU}(2)))$ where*

$$\varphi\colon \boldsymbol{N}^* \ni n \rightsquigarrow \frac{1}{n}, \qquad \psi\colon \boldsymbol{N}^* \ni k \rightsquigarrow -\frac{3}{2}\frac{[(1/2)k^2]}{k}.$$

Since the proof of Theorem 8 is similar to the preceding one, we may omit the details. For further results concerning summation processes on the special unitary group SU(2) we refer to R. A. Mayer [31–33].

8. **Conclusion.** Let us briefly review Sections 4, 5 and 6. As it was pointed out by E. M. Stein [45], his construction of the heat-diffusion semigroup $(\exp(\Delta t))_{t\in \boldsymbol{R}_+}$ and of the Poisson semigroup $(\exp(-\sqrt{-\Delta}\, t))_{t\in \boldsymbol{R}_+}$ which is based on the celebrated paper by G. A. Hunt [23], can be extended to *non-compact* connected Lie groups. In this connection also see R. Azencott [2].

On the other hand, it is a well known fact that the expansions in series of special functions require in general the study of non-compact Lie groups. For these reasons we shall return to saturation theory on Lie groups in a forthcoming paper. In particular we shall draw upon the theory of nuclear topological vector spaces.

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