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COMPLEX HYPERSURFACES OF $P_n(C) \times P_n(C)$

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Recently, Simons [7] has established a fundamental formula for the Laplacian of the length of the second fundamental tensor of a submanifold of a Riemannian manifold and has obtained an application to a minimal hypersurface of a sphere. Ogiue [6] and others then obtained an important application of the formula of Simons' type to a complex submanifold of a complex space form.

On the other hand, Ludden and Okumura [3] obtained a remarkable application of the formula of Simons' type to a hypersurface of constant mean curvature immersed in the product $S^n \times S^n$ of two n-spheres.

In this paper we deal with complex hypersurfaces immersed in a Kaehler manifold $P_n(C) \times P_n(C)$ by a similar method.

In §1, we review some fundamental formulas for a complex hypersurface M of the product $P_n(C) \times P_n(C)$ of two complex projective n-spaces and obtain a result: The scalar curvature ρ of M satisfies $\rho \leqq 2n^2$. If the equality holds, then the tangent space of M is invariant under an almost product structure on $P_n(C) \times P_n(C)$ (for simplicity, we say that M is an invariant hypersurface), and M is a totally geodesic hypersurface of $P_n(C) \times P_n(C)$ (Proposition 1.1).

In §2, using the formulas obtained in §1 we establish an integral formula of Simons' type and obtain results: A totally geodesic hypersurface, and a compact Kaehler hypersurface of $P_n(C) \times P_n(C)$ satisfying

$$\int_M \left(\frac{2n+1}{2n-1}\varphi^2 - (n+1)\varphi\right) dM \geqq 4\int_M ||\nabla^* H||^2 dM$$

are invariant hypersurfaces, where $\varphi = 2\,\mathrm{trace}\, H^2$ (Theorems 2.1 and 2.2).

In §3, we consider an invariant hypersurface of $P_n(C) \times P_n(C)$ and obtain a result: A compact invariant Kaehler hypersurface M of $P_n(C) \times P_n(C)$ is a totally geodesic hypersurface, $\varphi \equiv (n+1)/3$ or $\varphi(x) > (n+1)/3$ at some $x \in M$ (Theorem 3.1).

Moreover, using a fact that a complete invariant Kaehler hypersurface of $P_n(C) \times P_n(C)$ is the product of $P_n(C)$ and a hypersurface of $P_n(C)$ (Theorem 3.3), we obtain the main results: A) If $\varphi \leqq (n+1)/3$, then $M = P_{n-1}(C) \times P_n(C)$ or $n = 2$ and $M = Q_1(C) \times P_2(C)$, where $Q_1(C)$

is a complex quadric. B) $P_{n-1}(C) \times P_n(C)$ is the only totally geodesic hypersurface of $P_n(C) \times P_n(C)$. C) $P_{n-1}(C) \times P_n(C)$ and $Q_{n-1}(C) \times P_n(C)$ are the only compact invariant Kaehler hypersurfaces of $P_n(C) \times P_n(C)$ with constant scalar curvature, where $Q_{n-1}(C)$ is the complex quadric (Theorems 3.5, 3.6 and 3.7).

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1. Complex hypersurfaces of $P_n(C) \times P_n(C)$. Let $P_n(C)$ be a complex projective n-space with the Fubini-Study metric of constant holomorphic sectional curvature 1. Consider the Riemannian product $P_n(C) \times P_n(C)$. We denote by $\bar{P}$ and $\bar{Q}$ the projections of the tangent space of $P_n(C) \times P_n(C)$ to each component respectively. We put

$$\bar{F} = \bar{P} - \bar{Q} . \tag{1.1}$$

Then the Riemannian metric on $P_n(C) \times P_n(C)$ is given by

$$\bar{g}(\bar{X}, \bar{Y}) = g'(\bar{P}\bar{X}, \bar{P}\bar{Y}) + g'(\bar{Q}\bar{X}, \bar{Q}\bar{Y}) ,$$

where g' is the Kaehler metric of $P_n(C)$. Then we have

$$\bar{P} + \bar{Q} = I , \tag{1.2}$$

$$\bar{P}^2 = \bar{P} , \quad \bar{Q}^2 = \bar{Q} , \tag{1.3}$$

$$\bar{P}\bar{Q} = \bar{Q}\bar{P} = 0 , \tag{1.4}$$

$$\bar{F}^2 = I , \tag{1.5}$$

$$\operatorname{trace} \bar{F} = 0 , \tag{1.6}$$

$$\bar{g}(\bar{F}\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{F}\bar{Y}) , \tag{1.7}$$

$$\bar{\nabla}_{\bar{X}}\bar{F} = 0 , \tag{1.8}$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to $\bar{g}$. We call $\bar{F}$ an *almost product structure* on $P_n(C) \times P_n(C)$.

The curvature tensor of $P_n(C)$ may be written as

$$\begin{aligned} R'(X', Y')Z' &= \frac{1}{4}\{g'(Y', Z')X' - g'(X', Z')Y' + g'(J'Y', Z')J'X' \\ &\quad - g'(J'X', Z')J'Y' + 2g'(X', J'Y')J'Z'\} , \end{aligned}$$

where J' denotes the complex structure of $P_n(C)$. We put

$$\bar{J}\bar{X} = J'\bar{P}\bar{X} + J'\bar{Q}\bar{X} .$$

Then we can easily see that

$$
\begin{aligned}
(1.9) \qquad & J'\bar{P} = \bar{P}\bar{J}\,, \quad J'\bar{Q} = \bar{Q}\bar{J}\,, \\
& \bar{F}\bar{J} = \bar{J}\bar{F}\,, \quad \bar{J}^2 = -I\,, \\
& \bar{g}(\bar{J}\bar{X}, \bar{J}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y})\,.
\end{aligned}
$$

Therefore the curvature tensor of $P_n(C) \times P_n(C)$ is given by

$$
\begin{aligned}
(1.10) \quad \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{1}{8}\{ & \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(\bar{J}\bar{Y}, \bar{Z})\bar{J}\bar{X} \\
& - \bar{g}(\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y} + 2\bar{g}(\bar{X}, \bar{J}\bar{Y})\bar{J}\bar{Z} + \bar{g}(\bar{F}\bar{Y}, \bar{Z})\bar{F}\bar{X} - \bar{g}(\bar{F}\bar{X}, \bar{Z})\bar{F}\bar{Y} \\
& + \bar{g}(\bar{F}\bar{J}\bar{Y}, \bar{Z})\bar{F}\bar{J}\bar{X} - \bar{g}(\bar{F}\bar{J}\bar{X}, \bar{Z})\bar{F}\bar{J}\bar{Y} + 2\bar{g}(\bar{F}\bar{X}, \bar{J}\bar{Y})\bar{F}\bar{J}\bar{Z}\}\,,
\end{aligned}
$$

from which we can easily see that $P_n(C) \times P_n(C)$ is an Einstein Kaehler manifold because of (1.6), (1.7) and (1.9) (See [8], [10]).

Now, let M be a complex hypersurface of $P_n(C) \times P_n(C)$, and B the differential of the immersion $\underline{i}$ of M into $P_n(C) \times P_n(C)$. Let g and J be the induced Riemannian metric and the induced complex structure on M, respectively, and ∇ denote the operator of covariant differentiation with respect to the Riemannian connection of g. Let X, Y and Z be tangent to M and N a unit normal vector. Then we have the following:

$$
\begin{aligned}
(1.11) \qquad & \bar{F}BX = BfX + u(X)N + \tilde{u}(X)\bar{J}N\,, \\
(1.12) \qquad & \bar{F}N = BU + \lambda N + \tilde{\lambda}\bar{J}N\,, \\
& g(U, X) = u(X)\,, \quad g(JU, X) = \tilde{u}(X)\,, \\
& \tilde{u}(X) = -u(JX)\,, \quad Jf = fJ\,, \quad \tilde{\lambda} = 0\,, \\
(1.13) \qquad & \bar{\nabla}_{BX}BY = B\nabla_X Y + h(X, Y)N + k(X, Y)\bar{J}N\,, \\
(1.14) \qquad & \bar{\nabla}_{BX}N = -BHX + s(X)\bar{J}N\,, \\
& h(X, Y) = g(HX, Y)\,, \quad k(X, Y) = g(JHX, Y) \\
& HJ = -JH\,, \quad \operatorname{trace} H = \operatorname{trace} HJ = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
(1.15) \quad & R(X, Y)Z \\
& = \frac{1}{8}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY \\
& \quad + 2g(X, JY)JZ + g(fY, Z)fX - g(fX, Z)fY \\
& \quad + g(fJY, Z)fJX - g(fJX, Z)fJY + 2g(fX, JY)fJZ\} \\
& \quad + \{h(Y, Z)HX - h(X, Z)HY\} + \{k(Y, Z)JHX - k(X, Z)JHY\}
\end{aligned}
$$

——Gauss equation,

$$
\begin{aligned}
(1.16) \quad & (\nabla_X H)Y - (\nabla_Y H)X - s(X)JHY + s(Y)JHX \\
& = \frac{1}{8}\{u(X)fY - u(Y)fX
\end{aligned}
$$

$$+ u(JX)fJY - u(JY)fJX - 2g(fX, JY)JU\}$$

——Codazzi equation,

$$(1.17)\quad (\nabla_X s)(Y) - (\nabla_Y s)(X) = 2ds(X, Y)$$
$$= X\cdot s(Y) - Y\cdot s(X) - s([X, Y])$$
$$= 2g(X, JH^2Y) + \frac{1}{4}\{u(X)u(JY) - u(JX)u(Y)$$
$$+ g(X, JY) + \lambda g(fX, JY)\}$$

——Ricci equation,

$$(1.18)\quad f^2X = X - u(X)U + u(JX)JU\,,$$

$$(1.19)\quad u(fX) = -\lambda u(X)\,,$$

$$(1.20)\quad fU = -\lambda U\,,$$

$$(1.21)\quad u(U) = g(U, U) = 1 - \lambda^2\,,$$

$$(1.22)\quad (\nabla_Y f)X = h(Y, X)U + k(Y, X)JU + u(X)HY - u(JX)JHY\,,$$

$$(1.23)\quad (\nabla_Y u)X = \lambda h(Y, X) - h(Y, fX) - s(Y)u(JX)\,,$$

$$(1.24)\quad \nabla_X U = -fHX + \lambda HX + s(X)JU\,,$$

$$(1.25)\quad X\cdot\lambda = -2h(X, U) = -2u(HX)\,,$$

$$(1.26)\quad S(X, Y) = \frac{2n+1}{4}g(X, Y) - \frac{1}{4}u(X)u(Y) - \frac{1}{4}u(JX)u(JY)$$
$$- \frac{1}{4}g(fX, Y)\lambda - 2g(H^2X, Y)\,,$$

$$(1.27)\quad \rho = 2n^2 - (1 - \lambda^2) - 2\,\mathrm{trace}\,H^2\,,$$

where f; u, $\tilde{u}$; U; λ, $\tilde{\lambda}$; h, k; s; S and ρ define a symmetric linear transformation of the tangent bundle of M, two 1-forms, a vector field, two functions on M, the second fundamental tensors of the hypersurface, a normal connection form, the Ricci tensor of M and the scalar curvature of M, respectively (See [2], [3]).

If u is identically zero, then M is said to be an *invariant* hypersurface, that is, the tangent space $T_x(M)$ is invariant under $\bar{F}$. We can easily see by (1.21) that this is equivalent to $\lambda^2 = 1$.

Pick an orthonormal frame $\bar{E}_A$, $\bar{E}_{A^*} = \bar{J}\bar{E}_A$, $A = 1, \cdots, 2n$ in such a way that the first $2n-1$ $\bar{E}_A$'s satisfy $\bar{E}_a = BE_a$, and $\bar{E}_{2n} = N$[1]. Then

[1] We use the following convention on the range of indices unless otherwise stated:

$A, B, C, D = 1, \cdots, 2n$

$a, b, c, d = 1, \cdots, 2n-1$

$i, j, k, l = 1, \cdots, 2n-1, 1^*, \cdots, 2n-1^*$

because of (1.6) and (1.11) we have

$$\begin{aligned}\operatorname{trace} f &= \sum g(fE_i, E_i)\\ &= \sum \bar{g}(BfE_i, BE_i) = \sum \bar{g}(\bar{F}BE_i, BE_i)\\ &= \sum \bar{g}(\bar{F}\bar{E}_A, \bar{E}_A) + \sum \bar{g}(\bar{F}\bar{E}_{A^*}, \bar{E}_{A^*}) - \bar{g}(\bar{F}N, N) - \bar{g}(\bar{F}\bar{J}N, \bar{J}N)\\ &= \operatorname{trace} \bar{F} - 2\lambda = -2\lambda .\end{aligned}$$

From (1.21) and (1.27) we easily get

PROPOSITION 1.1. *The scalar curvature ρ of M satisfies $\rho \leqq 2n^2$. If the equality holds, then M is an invariant and totally geodesic hypersurface of $P_n(C) \times P_n(C)$.*

We will see later Theorem 2.1 that "invariant" of Proposition 1.1 automatically holds.

2. Integral formulas of Simons' type.

Consider the function $\varphi = 2\operatorname{trace} H^2$. We will now compute the Laplacian $\Delta\varphi$. Since M is a minimal submanifold of $P_n(C) \times P_n(C)$, the following holds ([1]):

$$\begin{aligned}\frac{1}{2}\Delta\varphi = &\sum \|\nabla^* H_\alpha\|^2 + \sum \operatorname{trace}(H_\alpha H_\beta - H_\beta H_\alpha)^2\\ &- \sum (\operatorname{trace} H_\alpha H_\beta)^2\\ &+ \sum (4\bar{g}(\bar{R}(\bar{E}_i, \bar{E}_j)\bar{E}_\beta, \bar{E}_\alpha)g(H_\alpha E_j, E_k)g(H_\beta E_i, E_k)\\ &\quad - \bar{g}(\bar{R}(\bar{E}_\beta, \bar{E}_k)\bar{E}_k, \bar{E}_\alpha)g(H_\alpha E_i, E_j)g(H_\beta E_i, E_j)\\ &\quad + 2\bar{g}(\bar{R}(\bar{E}_k, \bar{E}_j)\bar{E}_j, \bar{E}_i)g(H_\alpha E_i, E_l)g(H_\beta E_k, E_l)\\ &\quad + 2\bar{g}(\bar{R}(\bar{E}_k, \bar{E}_l)\bar{E}_j, \bar{E}_i)g(H_\alpha E_i, E_l)g(H_\beta E_j, E_k)) ,\end{aligned}$$

where Greek indices α, β have the range $\{2n, 2n^*\}$, and $H_{2n} = H$, $H_{2n}^* = JH$, and $\nabla_X^* H = \nabla_X H - s(X)JH$ ([2]). Using (1.9), (1.10), (1.11), (1.12), (1.15), (1.18), (1.21) and trace $f = -2\lambda$, the last term of the right hand side of the above equation equals to

$$\frac{n+1}{2}\varphi + \frac{1}{2}\lambda^2\varphi + 2\operatorname{trace}(fH)^2 - 3\lambda \operatorname{trace} fH^2 - 6g(H^2U, U) .$$

Moreover we have ([6])

$$\sum \operatorname{trace}(H_\alpha H_\beta - H_\beta H_\alpha)^2 = -8\operatorname{trace} H_{2n}^4 = -8\operatorname{trace} H^4 .$$

Thus we have

$$(2.1)\qquad \frac{1}{2}\Delta\varphi = \frac{n+1}{2}\varphi + \frac{1}{2}\lambda^2\varphi - \frac{1}{2}\varphi^2 + 2\,\mathrm{trace}\,(fH)^2 - 3\lambda\,\mathrm{trace}\,fH^2 - 6g(H^2U, U) - 8\,\mathrm{trace}\,H^4 + 2\|\nabla^* H\|^2 .$$

Next we want to compute $\mathrm{div}\,(fHU)$. Extend an orthonormal basis E_i's for $T_x(M)$ to vector fields in a neighborhood of x in such a way that $\nabla E_i = 0$ at x. Since $\mathrm{div}\, Z = \sum g(\nabla_{E_i}Z, E_i)$ for any vector field Z, we first have, for a vector field X,

$$\begin{aligned}\nabla_X(fHU) &= (\nabla_X f)HU + f(\nabla_X H)U + fH\nabla_X U \\ &= g(H^2U, X)U + g(JH^2U, X)JU + g(HU, U)HX - g(JHU, U)JHX \\ &\quad + f((\nabla_U H)X + s(X)JHU - s(U)JHX + \frac{1}{8}(u(X)fU - u(U)fX \\ &\quad + u(JX)fJU - u(JU)fJX - 2g(fX, JU)JU)) + fH(-fHX \\ &\quad + \lambda HX + s(X)JU) \\ &= g(H^2U, X)U + g(JH^2U, X)JU + g(HU, U)HX - g(JHU, U)JHX \\ &\quad + f(\nabla_U H)X - s(U)fJHX + \frac{1}{8}\lambda^2 u(X)U - \frac{1}{8}(1-\lambda^2)(X - u(X)U \\ &\quad + u(JX)JU) + \frac{3}{8}\lambda^2 u(JX)JU - (fH)^2X + \lambda fH^2X ,\end{aligned}$$

because of (1.16), (1.18), (1.20), (1.21), (1.22) and (1.24), from which it follows that

$$\mathrm{div}\,(fHU) = 2g(HU, HU) + \mathrm{trace}\, f\nabla_U H - \frac{n}{2}(1-\lambda^2) + \frac{1}{2}(1-\lambda^2)^2 - \mathrm{trace}\,(fH)^2 + \lambda\,\mathrm{trace}\, fH^2 .$$

On the other hand, we have

$$\begin{aligned}\mathrm{trace}\, fH &= \sum \{g(fHE_a, E_a) + g(fHJE_a, JE_a)\} \\ &= \sum \{g(JfHE_a, JE_a) + g(fHJE_a, JE_a)\} \\ &= \sum \{- g(fHJE_a, JE_a) + g(fHJE_a, JE_a)\} = 0 ,\end{aligned}$$

from which we obtain

$$\begin{aligned}0 &= \nabla_X(\mathrm{trace}\, fH) \\ &= \sum \nabla_X(g(fHE_i, E_i)) \\ &= \sum \{g((\nabla_X f)HE_i, E_i) + g(f(\nabla_X H)E_i, E_i)\} \\ &= \sum \{g(H^2X, E_i)g(U, E_i) + g(HJHX, E_i)g(JU, E_i)\end{aligned}$$

$$+ g(HU, E_i)g(HX, E_i) + g(HJU, E_i)g(JHX, E_i)\}$$
$$+ \operatorname{trace} f\nabla_X H$$
$$= \operatorname{trace} f\nabla_X H ,$$

because of (1.22), from which it follows that

$$\operatorname{div}(fHU) = 2g(HU, HU) - \operatorname{trace}(fH)^2 + \lambda \operatorname{trace} fH^2 - \frac{n}{2}(1 - \lambda^2) + \frac{1}{2}(1 - \lambda^2)^2 . \tag{2.2}$$

Now we compute $\operatorname{div}(\lambda HU)$. From (1.16), (1.24) and (1.25), we have

$$\begin{aligned} \nabla_X(\lambda HU) &= (X\cdot\lambda)HU + \lambda(\nabla_X H)U + \lambda H\nabla_X U \\ &= -2u(HX)HU + \lambda(\nabla_U H)X - \lambda s(U)JHX \\ &\quad + \frac{\lambda}{8}\{u(X)fU - u(U)fX + u(JX)fJU - 2g(fX, JU)JU\} \\ &\quad - \lambda HfHX + \lambda^2H^2X . \end{aligned}$$

Therefore

$$\operatorname{div}(\lambda HU) = -2g(HU, HU) + \frac{1}{2}\lambda^2(1 - \lambda^2) - \lambda \operatorname{trace} fH^2 + \frac{1}{2}\lambda^2\varphi . \tag{2.3}$$

Thus we have

$$\begin{aligned} &\frac{1}{2}\Delta\varphi + 2\operatorname{div}(fHU) - \operatorname{div}(\lambda HU) \\ &= \frac{n+1}{2}\varphi - \frac{1}{2}\varphi^2 - 8\operatorname{trace} H^4 - \frac{1}{2}(2n - 2 + 3\lambda^2)(1 - \lambda^2) \\ &\quad + 2\|\nabla^*H\|^2 . \end{aligned} \tag{2.4}$$

From (2.2), (2.3) or (2.4) we easily get

THEOREM 2.1. *A totally geodesic hypersurface of $P_n(C) \times P_n(C)$ is an invariant hypersurface.*

Assume that the hypersurface M is compact. Integrating the above equation over M, we get, because of Green-Stokes' theorem,

$$\int_M\left\{\frac{n+1}{2}\varphi - \frac{1}{2}\varphi^2 - 8\operatorname{trace} H^4 - \frac{1}{2}(2n - 2 + 3\lambda^2)(1 - \lambda^2) + 2\|\nabla^*H\|^2\right\}dM = 0 . \tag{2.5}$$

Applying $(1/(2n-1))\varphi^2 \leqq 8\,\mathrm{trace}\, H^4$ ([5]) to (2.5), we have

THEOREM 2.2. *A compact Kaehler hypersurface of* $P_n(C) \times P_n(C)$ *satisfying*

$$\int_M \left(\frac{2n+1}{2n-1}\varphi^2 - (n+1)\varphi\right) dM \geqq 4\int_M \|\nabla^* H\|^2 dM \tag{2.6}$$

is an invariant hypersurface.

REMARK. From (2.5) and (2.6), we easily see that a compact Kaehler hypersurface with parallel second fundamental tensor of $P_n(C) \times P_n(C)$ satisfying $\varphi \geqq (2n-1)(n+1)/(2n+1)$ is an invariant hypersurface and $\varphi \equiv (2n-1)(n+1)/(2n+1)$. However, we will see later Theorem 3.7 that there exist no such invariant hypersurfaces.

3. Invariant hypersurfaces of $P_n(C) \times P_n(C)$. In this section we assume that the hypersurface M is invariant, i.e., (1.11) can be written as

$$\bar{F}BX = BfX .$$

Since the 1-form u and the vector field U vanish identically, we have

$$f^2X = X , \tag{3.1}$$

$$1 - \lambda^2 = 0 , \tag{3.2}$$

$$\nabla_X f = 0 , \tag{3.3}$$

$$X \cdot \lambda = 0 . \tag{3.4}$$

We may assume that $\lambda = 1$ in the following discussions. Then the formula (2.5) becomes

$$\int_M \left\{\frac{n+1}{2}\varphi - \frac{1}{2}\varphi^2 - 8\,\mathrm{trace}\, H^4 + 2\|\nabla^* H\|^2\right\} dM = 0 . \tag{3.5}$$

Thus noting that $8\,\mathrm{trace}\, H^4 \leqq \varphi^2$ ([5], [9]), we get

THEOREM 3.1. *Let M be a compact invariant Kaehler hypersurface of $P_n(C) \times P_n(C)$. Then either M is the totally geodesic hypersurface, $\varphi \equiv (n+1)/3$, or $\varphi(x) > (n+1)/3$ at some $x \in M$.*

COROLLARY 3.2. *Let M be a compact invariant Kaehler hypersurface of $P_n(C) \times P_n(C)$. If $\varphi < (n+1)/3$, then M is a totally geodesic hypersurface.*

Now let

$$T_1(x) = \{X \in T_x(M);\, fX = X\} ,$$
$$T_{-1}(x) = \{X \in T_x(M);\, fX = -X\} .$$

Then $x \longrightarrow T_1(x)$ and $x \longrightarrow T_{-1}(x)$ define $(n-1)$-dimensional and n-dimensional distributions respectively, since trace $f = -2\lambda = -2$. By virtue of (3.3) it follows that both distributions are involutive. We easily see that if $X \in T_1$ and $Y \in T_{-1}$, then $\nabla_Y X \in T_1$ and $\nabla_X Y \in T_{-1}$. Hence both distributions are parallel. Moreover, for the vector fields X and Y chosen in the above way, we have $g(\nabla_Z X, Y) = 0$ and $g(\nabla_W Y, X) = 0$, where $Z \in T_1$ and $W \in T_{-1}$. Thus the maximal integral manifolds through each $x \in M$ of T_1 and T_{-1} are both totally geodesic in M. By standard arguments (See [3]) we know that M is a product of the maximal integral manifolds of the distributions T_1 and T_{-1}. In the next step we want to show that the maximal integral manifold of T_{-1} is $P_n(C)$.

Let $X \in T_{-1}$. Then by virtue of (1.1) and (1.2) it follows that

$$\bar{P}BX = \frac{1}{2}(IBX + \bar{F}BX) = \frac{1}{2}(BX + BfX) = 0 .$$

Thus BX belongs to the tangent space $T(P_n(C))$ which is defined by $V_Q = \{\bar{X}; \bar{Q}\bar{X} = \bar{X}\}$. Conversely, if we take a vector field $\bar{X}$ belonging to V_Q, $\bar{X}$ can be written as a sum of the tangential components and the normal components. So we put

$$\bar{X} = BX + \alpha N + \tilde{\alpha}\bar{J}N .$$

Applying $\bar{P}$ to the above equation, we have

$$\begin{aligned} 0 = \bar{P}\bar{X} &= \bar{P}BX + \alpha\bar{P}N + \tilde{\alpha}\bar{P}\bar{J}N \\ &= \frac{1}{2}\{(IBX + \bar{F}BX) + \alpha(IN + \bar{F}N) + \tilde{\alpha}(I\bar{J}N + \bar{F}\bar{J}N)\} \\ &= \frac{1}{2}\{BX + BfX + 2\alpha N + 2\tilde{\alpha}\bar{J}N\} , \end{aligned}$$

from which we have

$$fX = -X , \quad \alpha = 0 , \quad \tilde{\alpha} = 0 .$$

This means that $\bar{X} = BX$, and consequently $V_Q = BT_{-1}$. Thus, if M is complete, the maximal integral manifold of T_{-1} must be $P_n(C)$. If $X \in T_1$, then the same discussion as above shows that $BX \in V_P = \{\bar{X}; \bar{P}\bar{X} = \bar{X}\}$. Since the integral submanifold of V_P is another $P_n(C)$, the maximal integral manifold of T_1 is a hypersurface of $P_n(C)$. Thus we have

THEOREM 3.3. *A complete invariant Kaehler hypersurface of $P_n(C) \times P_n(C)$ is a product manifold $M' \times P_n(C)$, where M' is a Kaehler hypersurface of $P_n(C)$.*

In order to get further results, we prove

LEMMA 3.4. *Let P and Q be the projection of $T(M)$ into $T(M')$ and $T(P_n(C))$ respectively. Then we have*

$$HQ = 0 . \tag{3.6}$$

PROOF. By the definitions of $\bar{F}$, P and Q, we have

$$\bar{F}BQX = (\bar{P} - \bar{Q})BQX = (\bar{P} - \bar{Q})\bar{Q}BX = -\bar{Q}BX = -BQX ,$$

since $V_Q = BT_{-1}$. Hence

$$\begin{aligned} \bar{\nabla}_{BY}(\bar{F}BQX) &= -\bar{\nabla}_{BY}(BQX) \\ &= -B\nabla_Y(QX) - h(Y, QX)N - k(Y, QX)\bar{J}N . \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} &\bar{\nabla}_{BY}(\bar{F}BQX) \\ &= \bar{F}(B\nabla_Y(QX) + h(Y, QX)N + k(Y, QX)\bar{J}N) \\ &= -B\nabla_Y(QX) + h(Y, QX)\bar{F}N + k(Y, QX)\bar{F}\bar{J}N \\ &= -B\nabla_Y(QX) + h(Y, QX)N + k(Y, QX)\bar{J}N , \end{aligned} \tag{3.8}$$

because of the fact that $\nabla_Y(QX) \in V_Q$, $\bar{F}N = N$ and $\bar{F}\bar{J}N = \bar{J}N$.

Comparing (3.7) and (3.8), we have $h(Y, QX) = k(Y, QX) = 0$, from which (3.6) follows.

We consider the immersion $i' \colon M' \to M' \times P_n(C) = M$, and denote the differential of i' by B'. Then we have

$$\begin{aligned} \bar{\nabla}_{BB'Y'}BB'X' &= BB'\nabla'_{Y'}X' \\ &\quad + \sum_{A=1}^{n+1} h'_A(X', Y')N'_A + \sum_{A=1}^{n+1} k'_A(X', Y')\bar{J}N'_A , \end{aligned} \tag{3.9}$$

where X' and $Y' \in T(M')$, and h'_A and k'_A's are the second fundamental tensors with respect to the normals N'_A and $\bar{J}N'_A$ respectively. Now we choose the last normal N'_{n+1} in such a way that N'_{n+1} is the unit normal to M' in $P_n(C)$.

On the other hand, we have

$$\begin{aligned} &\bar{\nabla}_{BB'Y'}BB'X' \\ &\quad = B\nabla_{B'Y'}B'X' + h(B'X', B'Y')N + k(B'X', B'Y')\bar{J}N , \end{aligned}$$

from which it follows that

$$\begin{aligned} \bar{\nabla}_{BB'Y'}BB'X' &= BB'\nabla'_{Y'}X' \\ &\quad + \sum_{\alpha=1}^{n} h_\alpha(X', Y')BN_\alpha + \sum_{\alpha=1}^{n} k_\alpha(X', Y')BJN_\alpha \\ &\quad + h(B'X', B'Y')N + k(B'X', B'Y')\bar{J}N . \end{aligned} \tag{3.10}$$

Comparing (3.9) and (3.10), we get

$$h_\alpha(X', Y') = h'_\alpha(X', Y')\,, \quad k_\alpha(X', Y') = k'_\alpha(X', Y')\,, \qquad \text{for} \quad \alpha = 1, \cdots, n\,,$$

$$h(B'X', B'Y') = h'_{n+1}(X', Y')\,,$$

$$k(B'X', B'Y') = k'_{n+1}(X', Y')\,.$$

Since M' is a totally geodesic submanifold in $M' \times P_n(C)$, it follows that $h_\alpha(X', Y') = k_\alpha(X', Y') = 0$ for $\alpha = 1, \cdots, n$. Also, for any positive integer p,

$$\begin{aligned}\operatorname{trace} H^p &= \sum g(H^p E_a, E_a) + \sum g(H^p J E_a, J E_a) \\ &= \sum_{A=1}^{n-1} g(H^p B' E_A, B' E_A) + \sum_{t=1}^{n} g(H^p N'_t, N'_t) \\ &\quad + \sum_{A=1}^{n-1} g(H^p J B' E_A, J B' E_A) + \sum_{t=1}^{n} g(H^p J N'_t, J N'_t)\,,\end{aligned}$$

where N'_t, $t = 1, \cdots, n$ are unit normals to M' in $M' \times P_n(C)$. Since there exist X_t in $T(M)$ such that $N'_t = Q X_t$, we have $H^p N'_t = 0$, because of Lemma 3.4. Thus we get

$$\begin{aligned}\operatorname{trace} H^p &= \sum_{A=1}^{n-1} g(H^p B' E_A, B' E_A) + \sum_{A=1}^{n-1} g(H^p J B' E_A, J B' E_A) \\ &= \sum_{A=1}^{n-1} g(H'^p_{n+1} E_A, E_A) + \sum_{A=1}^{n-1} g(H'^p_{n+1} J'' E_A, J'' E_A) \\ &= \operatorname{trace} H'^p_{n+1}\,,\end{aligned}$$

where J'' is the complex structure of M'. This shows that, once we fix a choice of normals in the above way, trace H^p is a function on M'. The immersion $\underline{i}\colon M \to P_n(C) \times P_n(C)$ being $\underline{i}' \times id\colon M' \times P_n(C) \to P_n(C) \times P_n(C)$, we have that the second fundamental tensor H'_{n+1} is identical with that of M' in $P_n(C)$. Thus, denoting the second fundamental tensor of M' in $P_n(C)$ by H' we can easily see that if $\rho = 2n^2 - \varphi =$ constant, then $\rho' = n(n-1) - 2\operatorname{trace} H'^2 = n(n-1) - \varphi' =$ constant, where ρ' is the scalar curvature of M'.

If $\varphi = 0$, it follows that $\varphi' = 0$ and consequently M' is totally geodesic in $P_n(C)$. Thus we have $M = P_{n-1}(C) \times P_n(C)$.

If $\varphi = (n+1)/3$, then $\varphi' = (n+1)/3$. Hence $n = 2$ and M' is imbedded as a complex quadric $Q_1(C)$ in $P_2(C)$ ([9]). Thus $M = Q_1(C) \times P_2(C)$.

If $\varphi = n - 1$, then $\varphi' = n - 1$. Thus $\rho' = (n-1)^2$.

From the above fact, we have

THEOREM 3.5. *If $\varphi \leqq (n+1)/3$, then $M = P_{n-1}(C) \times P_n(C)$ or $n = 2$*

and $M = Q_1(C) \times P_2(C)$.

Moreover, combining Theorem 2.1, we get

THEOREM 3.6. *$P_{n-1}(C) \times P_n(C)$ is the only totally geodesic hypersurface of $P_n(C) \times P_n(C)$.*

Applying Kon's theorem (See [4], Theorem 1) and combining Theorem 3.3, we have

THEOREM 3.7. *$P_{n-1}(C) \times P_n(C)$ and $Q_{n-1}(C) \times P_n(C)$ are the only compact invariant Kaehler hypersurfaces of $P_n(C) \times P_n(C)$ with constant scalar curvature, where $Q_{n-1}(C)$ is the complex quadric.*

COROLLARY 3.8. *There exist no compact invariant Einstein Kaehler hypersurfaces of $P_n(C) \times P_n(C)$.*

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