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THE DEFICIENCIES AND THE ORDER OF HOLOMORPHIC MAPPINGS OF C^n INTO A COMPACT COMPLEX MANIFOLD

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1. Introduction. Edrei and Fuchs [1] proved several interesting results about the relations between the order and deficiencies of meromorphic functions of one complex variable. In particular, they showed that a meromorphic function with two deficient values is of positive lower order and that an entire function of finite order with maximal deficiency sum 2 is of positive integral order and of regular growth. Generalizations of these results were given by Toda [6], Noguchi [5] and the present author [4] for meromorphic mappings into a complex projective space $P^N C$.

In this note, we generalize results of Edrei and Fuchs concerning the relations between deficiencies and the lower order to the case of holomorphic mappings of the n-dimensional complex Euclidean space C^n into a compact complex manifold M with a positive line bundle.

2. Notation and Terminology. Let $z = (z_1, \cdots, z_n)$ be the natural coordinate system in C^n. We put $\|z\|^2 = \sum_{j=1}^n z_j \bar{z}_j$, $B(r) = \{z \in C^n | \, \|z\| < r\}$, $\partial B(r) = \{z \in C^n | \, \|z\| = r\}$, $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$, $\psi = dd^c \log \|z\|^2$, $\psi_k = \psi \wedge \cdots \wedge \psi$ (k-times) and $\sigma = d^c \log \|z\|^2 \wedge \psi_{n-1}$.

For a divisor $D(\neq 0)$ in C^n, we write

$$n(r, D) \equiv \int_{\operatorname{supp} D \cap B(t)} \psi_{n-1} \quad \text{and} \quad N(r, D) \equiv \int_0^r n(t, D)(dt/t) .$$

Let M be an m-dimensional compact complex manifold with a positive line bundle, hence a smooth projective algebraic variety by a famous embedding theorem of Kodaira. Let L be a positive line bundle over M and $\{U_\alpha\}$ be an open covering of M such that the restriction $L_{|U_\alpha}$ is trivial. Then L is determined by the 1-cocycle $\{f_{\alpha\beta}\}$ which are non-zero holomorphic functions on $U_\alpha \cap U_\beta$ and satisfies $f_{\alpha\beta} = f_{\alpha\gamma} \cdot f_{\gamma\beta}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. A holomorphic section $\phi = \{\phi_\alpha\} \in H^0(M, \mathscr{O}(L))$ of $L \to M$ is given by holomorphic functions ϕ_α on U_α which satisfy the relation $\phi_\alpha = f_{\alpha\beta}\phi_\beta$ on $U_\alpha \cap U_\beta$. A metric h in L is given by positive C^∞ functions h_α in U_α such that $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$ on $U_\alpha \cap U_\beta$. Thus, if $\phi = \{\phi_\alpha\}$ is a section of $L \to M$, then the function $|\phi|^2 = |\phi_\alpha|^2/h_\alpha$ is well defined on M and is called the norm of φ. We put $\omega = \omega_L = dd^c \log h_\alpha$, which represents the Chern class $c_1(L)$ of L, and call it the curvature form of the metric h.

Let f be a holomorphic mapping of C^n into M. We define

$$T(r, f) \equiv T_L(r, f) \equiv \int_0^r (dt/t) \int_{B(t)} f^*\omega \wedge \psi_{n-1}$$

and call it the characteristic function of f, where $f^*\omega$ denotes the pull back of the form ω by f. We note that $T(r, f)$ is independent of the choice of the metric h in L up to an $O(1)$-term (see [2], p. 182).

Let $|L|$ denote the complete linear system of effective divisors on M given by the zeros of a holomorphic section of $L \to M$.

Let $\tilde{D} \in |L|$ be an effective divisor given by the zeros of a holomorphic section $\phi \in H^0(M, \mathcal{O}(L))$ with $|\phi| \leqq 1$ on M. Assume that $\phi(f(z)) \not\equiv 0$ and let

$$m(r, \tilde{D}) \equiv \int_{\partial B(r)} (\log(1/|\phi|^2(f(z))))\sigma \quad (\geqq 0) .$$

Using Nevanlinna's first main theorem, Griffiths and King proved the following (see [2], p. 174 for the case of meromorphic functions and p. 184 for the case of divisors).

THEOREM A. *Let g be a holomorphic mapping of C^n into M and let $\tilde{D} \in |L|$ be an effective divisor given by $\phi \in H^0(M, \mathcal{O}(L))$ such that the divisor (ϕ) of ϕ is equal to $\tilde{D}$, $|\phi| \leqq 1$ and $\phi(f(z)) \not\equiv 0$. Then*

$$N(r, f^*\tilde{D}) + m(r, \tilde{D}) = T(r, f) + O(1) , \tag{1}$$

where $O(1)$ depends on $\tilde{D}$ but not on r.

In the case where $f^*\tilde{D}$ passes through the origin, the definition of $N(r, f^*\tilde{D})$ must be modified by means of the Lelong numbers and, furthermore, the $O(1)$ term must be replaced by the term of $O(\log r)$. (Cf. *Griffiths and King* [2].)

For a divisor $\tilde{D} \in |L|$ on M, we define the deficiency of $\tilde{D}$ by

$$\delta(\tilde{D}, f) \equiv 1 - \limsup_{r\to\infty} N(r, f^*\tilde{D})/T(r, f) .$$

We define the order λ and the lower order μ of f as follows: $\lambda = \limsup_{r\to\infty} (\log T(r, f))/\log r$ *and* $\mu = \liminf_{r\to\infty} (\log T(r, f))/\log r$.

3. Let $f: C^n \to M$ be a holomorphic mapping of finite order λ and $f(C^n)$ is not contained in any divisor belonging to $|L|$. Let $\tilde{D}_1, \cdots, \tilde{D}_{m+1} \in |L|$ be divisors on M such that $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_{m+1} \in |L^{m+1}|$ has normal crossings and let $\phi^j = \{\phi^j_\alpha\} \in H^0(M, \mathcal{O}(L))$ be holomorphic sections such that the divisor (ϕ^j) of ϕ^j is equal to $\tilde{D}_j$ and $|\phi^j| \leqq 1$ on M for $j=1, \cdots, m+1$. Set $D_j = f^*(\tilde{D}_j)$. Then the system $\{\phi^1, \cdots, \phi^{m+1}\}$ has no common zeros, since $\tilde{D}$ has normal crossings. Thus the function $h = \{h_\alpha\} \equiv \sum_{j=1}^{m+1} |\phi_\alpha|^2$ is a positive C^∞ function on M satisfying $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$ on $U_\alpha \cap U_\beta$. Hence we may take h as a metric on L. Note that, if η^1 and η^2 are two holomorphic sections of $L \to M$, then its ratio η^1/η^2 is a global meromorphic function on M. Since $\tilde{D}$ has normal crossings, there is an i (say $i=1$) such that $f^*(\tilde{D}_i)$ does not contain the origin. By Theorem A for the case of divisors, we have

$$\begin{aligned} T(r, f) &= N(r, f^*(\tilde{D}_1)) + m(r, \tilde{D}_1) + O(1) \\ &= N(r, D_1) + \int_{\partial B(r)} (\log(h_\alpha(f(z))/|\phi_\alpha^1|^2(f(z))))\sigma(z) + O(1) \\ &= N(r, D_1) + \int_{\partial B(r)} (\log(\sum_{j=1}^{m+1} |\phi_\alpha^j(f(z))/\phi_\alpha^1(f(z))|^2))\sigma(z) + O(1) , \end{aligned}$$

or

$$(2) \qquad T(r, f) = N(r, D_1) + \int_{\partial B(r)} (\max_{1 \leqq j \leqq m+1} \log |\phi_\alpha^j(f)/\phi_\alpha^1(f)|^2)\sigma + O(1) .$$

Note that $T(r, f) \geqq n(r/2, D_1) \cdot \log 2$. Hence $n(r, D_1)$ has the order not exceeding the order $\lambda(<\infty)$ of f. Hence there exists an integer q with $\int^\infty t^{-q-1} dn(t, D_1) < \infty$, and an entire function $F(z)$ with the divisor $(F) = D_1(\not\ni 0)$ and of order at most $\max(q, \text{ord}.D_1)$ by Lelong [3].

Now, let $\gamma_\rho(z, z_0)$ be an automorphism of $B(\rho)$ such that $\gamma_\rho(z, z_0) = 0$ for $z_0 \in B(\rho)$. We write $\psi_\rho(z, z_0) = \psi \circ \gamma_\rho(z, z_0)$ and $\sigma_\rho(z, z_0) = \sigma \circ \gamma_\rho(z, z_0)$. If $z_0 = (r, 0, \cdots, 0)$, $\zeta = (\zeta_1, \cdots, \zeta_n)$ and if

$$\gamma_\rho(\zeta, z_0) = \frac{\rho}{\rho - (r/\rho)\zeta_1}(\zeta_1 - r, (1-(r/\rho)^2)^{1/2}\zeta_2, \cdots, (1-(r/\rho)^2)^{1/2}\zeta_n) ,$$

then we have

$$(3) \qquad \frac{(1-(r/\rho)^2)^n}{(1+(r/\rho))^{2n}}\sigma(\zeta) \leqq \sigma_\rho(\zeta, z) \leqq \frac{(1-(r/\rho)^2)^n}{(1-(r/\rho))^{2n}}\sigma(\zeta)$$

for $\zeta \in \partial B(\rho)$. Hence we see that $\sigma_\rho(\zeta, z) \equiv (1+Q)\sigma(\zeta)$, where

$$Q \leqq \{(\tau_\rho + 1)^n - (\tau_\rho - 1)^n\}/(\tau_\rho - 1)^n \quad \text{for} \quad \tau_\rho = \rho/r > 1 .$$

4. We now prove the following theorem which yields a relation between the lower order and the deficiencies.

THEOREM 1. *Let $f: C^n \to M$ be a holomorphic mapping of finite order λ such that the image $f(C^n)$ is not contained in any divisor in $|L|$ and let $\tilde{D}_1, \cdots, \tilde{D}_{m+1} \in |L|$ be divisors on M such that $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_{m+1}$*

has normal crossings. If $\tau > \tau_0$, then

(4) $T(r, f) \leqq (5n/2\tau)T(\tau r, f) + \max_{1\leqq j\leqq m+1} N(\tau r, f^*\tilde{D}_j) + O(\log r) \quad (r \to \infty)$,

where $\tau_0 \in R$ is the maximal real number of τ_0 such that

$$\{(\tau_0+1)^n - (\tau_0-1)^n\}(\tau_0-1)^{-n} = 5n\cdot(2\tau_0)^{-1} .$$

PROOF. Since ϕ^j/ϕ^1 is a meromorphic function on M and since F is an entire function on C^n with $(F) = D_1$, we see that $(\phi^j(f(z))/\phi^1(f(z)))\cdot F(z)$ is an entire function without poles on C^n. Hence the function $\log(|\phi^j(f)/\phi^1(f)|\,|F|)$ is a subharmonic function on C^n. Thus, for $\|z\| = r < R$, (2) implies

$$\begin{aligned}
\log|\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2 &\leqq \int_{\partial B(R)} (\log|\{\phi^j(f(\zeta))/\phi^1(f(\zeta))\}F(\zeta)|^2)\sigma(\zeta, z) \\
&= \int_{\partial B(R)} (\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2)\sigma(\zeta, z) + \int_{\partial B(R)} (\log|F(\zeta)|^2)\sigma(\zeta, z) \\
&\leqq \int_{\partial B(R)} (\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2)\sigma(\zeta) \\
&\quad + (5n/2\tau)\int_{\partial B(R)} (|\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2|)\sigma(\zeta) + \int_{\partial B(R)} (\log|F(\zeta)|^2)\sigma(\zeta, z) \\
&= N(R, D_j) - N(R, D_1) + (5n/2\tau)\int_{\partial B(R)} (|\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2|)\sigma(\zeta) \\
&\quad + \int_{\partial B(R)} (\log|F(\zeta)|^2)\sigma(\zeta, z) + O(\log R) \quad (j = 1, \cdots, m+1) ,
\end{aligned}$$

hence

$$\begin{aligned}
(5)\qquad &\max_{1\leqq j\leqq m+1} \log|\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2 \\
&\leqq \max_{1\leqq j\leqq m+1} N(R, D_j) - N(R, D_1) + (5n/2\tau)\int_{\partial B(R)} (\max_{1\leqq j\leqq m+1} |\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2|)\sigma(\zeta) \\
&\qquad + \int_{\partial B(R)} (\log|F(\zeta)|^2)\sigma(\zeta, z) + O(\log R)
\end{aligned}$$

by (1) for the case of meromorphic functions. Since $\max_{1\leqq j\leqq m+1}|\phi^j(f)/\phi^1(f)| \geqq 1$, we see $\max_{1\leqq j\leqq m+1}|\log|\phi^j(f)/\phi^1(f)|| = \max_{1\leqq j\leqq m+1}\log|\phi^j(f)/\phi^1(f)|$ and we have

$$\begin{aligned}
&\int_{\partial B(R)} (\max_{1\leqq j\leqq m+1} |\log|\phi^j(f(\zeta))/\phi^1(f(\zeta))|^2|)\sigma(\zeta) \\
&\qquad = \int_{\partial B(R)} (\max_{1\leqq j\leqq m+1} \log|\phi^j(f(\zeta))/\phi^1(f(\zeta))^2|)\sigma(\zeta) \leqq T(R, f) + O(1) .
\end{aligned}$$

Therefore, integrating both sides of (5) on $\partial B(r)$, we obtain

$$\int_{\partial B(r)} (\max_{1\leq j\leq m+1} \log |\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2)\sigma(z) \leqq \max_{1\leq j\leq m+1} N(R, D_j) - N(R, D_1) + (5n/2\tau)T(R, f) + N(R, D_1) + O(\log R) ,$$

since

$$\int_{\partial B(r)} \left\{\int_{\partial B(R)} (\log |F(\zeta)|^2)\sigma(\zeta, z)\right\}\sigma(z) = \int_{\partial B(R)} (\log |F(\zeta)|^2)\sigma(\zeta) = N(R, (F)) = N(R, D_1) .$$

Thus we have

$$\begin{aligned} T(r, f) &= \int_{\partial B(r)} (\max_{1\leq j\leq m+1} \log |\phi^j(f(z))/\phi^1(f(z))|^2)\sigma(z) + N(r, D_1) + O(1) \\ &= \int_{\partial B(r)} (\max_{1\leq j\leq m+1} \log |\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2)\sigma(z) + O(1) \\ &= \max_{1\leq j\leq m+1} N(R, D_j) + (5n/2\tau)T(R, f) + O(\log R) \quad (R \to \infty) . \end{aligned}$$

This completes the proof of Theorem 1 with $R = \tau r$.

COROLLARY 1. *Under the same assumption as in Theorem* 1, *if f is transcendental and is of lower order μ and if* $\gamma \equiv \max_{1\leq j\leq m+1} (1-\delta(\tilde{D}_j, f)\} < 1$, *then*

$$\mu \geqq \frac{\log (1/\gamma(2-\gamma))}{\log \tau} \quad \textit{for} \quad \gamma \neq 0 ,$$

$$\mu \geqq 1 \qquad \textit{for} \quad \gamma = 0 ,$$

where $\tau = \max (\tau_0, 5n/\gamma(1-\gamma))$.

PROOF. Using a method similar to that of Edrei and Fuchs [1], we deduce these inequalities from (4).

COROLLARY 2. *Under the same assumption as in Theorem* 1, *if f is transcendental and if there are $m+1$ divisors $\tilde{D}_j \in |L|$ $(j = 1, \cdots, m+1)$ on M such that $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_{m+1} \in |L^{m+1}|$ has normal crossings and $\delta(\tilde{D}_j, f) > 0$ for $j = 1, \cdots, m+1$, then the lower order μ of f is positive.*

We next give an estimate for $K(f) \equiv \limsup_{r\to\infty} \sum_{j=1}^{m+1} N(r, D_j)/T(r,f)$. A similar estimate for the case of meromorphic mappings into $\boldsymbol{P}^N\boldsymbol{C}$ was given by Noguchi [5].

THEOREM 2. *Let $f\colon \boldsymbol{C}^n \to M$ be a holomorphic mapping of finite order λ which is not a positive integer. Assume that the image $f(\boldsymbol{C}^n)$ is not contained in any divisor belonging to $|L|$. Then, for any $m+1$ divisors $\tilde{D}_j \in |L|$, $j = 1, \cdots, m+1$ on M such that $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_{m+1}$ has normal* *crossings and* $f^*\tilde{D}_j \not\ni 0$ $(j=1,\cdots,m+1)$, *we have*

$$K(f) \equiv \limsup_{r\to\infty} \sum_{j=1}^{m+1} N(r, f^*\tilde{D}_j)/T(r, f) \geqq k(\lambda)\ ,$$

where $k(\lambda)$ *is a positive constant depending only on* λ *and is not less than* $\{2\Gamma^4(3/4)|\sin\pi\lambda|\}/\{\pi^2\lambda + \Gamma^4(3/4)|\sin\pi\lambda|\}$. *In particular, if* $0 \leqq \lambda < 1$, *then* $k(\lambda)$ *satisfies* $k(\lambda) \geqq 1-\lambda$, *where* $\Gamma(\cdot)$ *denotes the gamma-function.*

PROOF. Since $f^*\tilde{D}_1 \equiv D_1 \not\ni 0$, there is Lelong's canonical function F with $(F) = D_1$ and of order at most $\max(q, \mathrm{ord}.D_1)$, where q is the least integer satisfying $\int^\infty t^{-q-1}dn(t, D_1) < \infty$. Thus, by (2) we have

$$T(r, f) = N(r, D_1) + \int_{\partial B(r)} \Big(\log \sum_{j=1}^{m+1} |\phi^j(f(z))/\phi^1(f(z))|^2\Big)\sigma(z) + O(1)$$
$$= \int_{\partial B(r)} (\max_{1\leqq j\leqq m+1} \log |\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2)\sigma(z) + O(1)$$

or

$$T(r, f) \leqq \sum_{j=1}^{m+1}\int_{\partial B(r)} (\log^+ |\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2)\sigma(z) + O(1)\ .$$

Now we can write $\{\phi^j(f(z))/\phi^1(f(z))\}F(z) \equiv G_j(z)\cdot\exp(P_j(z))$, where G_j is the canonical function associated with the divisor $f^*\tilde{D}_j \equiv D_j$ and P_j is a polynomial of degree not greater than the order of $\{\phi^j(f)/\phi^1(f)\}F$. We also see that

$$T(r, f) \geqq \int_{\partial B(r)} (\log^+ |\phi^j(f(z))/\phi^1(f(z))|^2)\sigma(z) - O(1)\ ,$$

hence

$$\int_{\partial B(r)} (\log^+ |\exp(P_j(z))|^2)\sigma(z)$$
$$\leqq \int_{\partial B(r)} (\log^+ |G_j(z)\exp(P_j(z))|^2)\sigma(z) + \int_{\partial B(r)} (\log^+ |G_j(z)|^{-2})\sigma(z)$$
$$= \int_{\partial B(r)} (\log^+ |\{\phi^j(f(z))/\phi^1(f(z))\}F(z)|^2)\sigma(z) + \int_{\partial B(r)} (\log^+ |G_j(z)|^{-2})\sigma(z)$$
$$\leqq \int_{\partial B(r)} (\log^+ |\phi^j(f)/\phi^1(f)|^2)\sigma(z) + \int_{\partial B(r)} (\log^+ |F|^2)\sigma(z)$$
$$+ \int_{\partial B(r)} (\log^+ |G_j|^{-2})\sigma(z) \leqq T(r, f) + T_1(r, F) + T_1(r, G_j) + O(1)\ .$$

Here $T_1(r, F)$ and $T_1(r, G_j)$ are characteristic functions of F and G_j, respectively. Hence the order of $\exp(P_j)$ is not greater than the order λ of f, since F and G_j are of order at most λ. Thus

$$T(r, f) \leqq \sum_{j=1}^{m+1} \int_{\partial B(r)} (\log^+ |G_j|^2) \sigma(z) + O(r^q) ,$$

where q is the largest integer not greater than $[\lambda]$ $(<\lambda)$. Therefore, putting $n(t) = \sum_{j=1}^{m+1} n(t, D_j)$ and using a method similar to that of Noguchi [5], we have the conclusion of Theorem 2.

COROLLARY 3. *Let* $f: C^n \to M$ *be a holomorphic mapping such that the image* $f(C^n)$ *is not contained in a divisor in* $|L|$. *If there are* $m + 1$ *divisors* $\tilde{D}_j \in |L|$ $(j = 1, \cdots, m + 1)$ *on* M *such that* $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_{m+1}$ *has normal crossings and* $\delta(\tilde{D}_j, f) = 1$ *for* $j = 1, \cdots, m + 1$, *then* f *is of positive integral order or of infinite order.*

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