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THE DUAL SPACE OF THE SPACE BMO FOR A STOCHASTIC POINT PROCESS

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1. Introduction. Let (Ω, F, P) be a complete probability space endowed with a non-decreasing right continuous family $(F_t)_{t\geqq 0}$ of sub σ-fields of F with $F = \bigvee_{t\geqq 0} F_t$ such that F_0 contains all null sets. Let λ be a non-negative predictable process such that $P\left(\int_0^t \lambda_s ds < \infty\right) = 1$ for all t. An F_t-adapted process $N = (N_t)$ is called a stochastic point process with the intensity λ if N has right continuous paths taking values in $Z_+ = \{0, 1, 2, \cdots\}$ with $N_0 = 0$, $\Delta N_t = N_t - N_{t-} = 0$ or 1, and if $\hat{N}_t = N_t - \int_0^t \lambda_s ds$ is a local martingale. Throughout, we assume that the stochastic point process N with the intensity λ satisfies the following conditions:

(S) $F_t = \sigma(N_s, s \leqq t)$, i.e., F_t is the completion of the σ-field generated by $(N_s, s \leqq t)$,

(B) $\hat{N}$ belongs to the space BMO.

Then we can define the finite measure μ on the σ-field Ξ of all predictable subsets A of $[0, \infty) \times \Omega$ by

$$\mu(A) = E\left[\int_0^\infty I_A \lambda_s ds\right]. \tag{1}$$

We shall adopt the following notations and definitions:

(2) $L^1(\mu)$ denotes the set of all predictable processes f with $\|f\|_{L^1(\mu)} < \infty$;

$$L^\infty(\mu) = \{f \in L^1(\mu); \|f\|_{L^\infty(\mu)} < \infty\}; \tag{3}$$

(4) Φ denotes the set of all real valued set functions ν on $([0, \infty)\times\Omega, \Xi, \mu)$ such that $\nu(A\cup B) = \nu(A) + \nu(B)$ if $A, B \in \Xi$ and $A \cap B = \emptyset$, $\|\nu\| = \sup\{\sum_{i=1}^n |\nu(A_i)|; \{A_1, \cdots, A_n\}$ is a measurable partition of $[0, \infty)\times\Omega\} < \infty$, and $\nu(A) = 0$ if $A \in \Xi$ and $\mu(A) = 0$;

$$U(f)(t) = \int_0^t f_s d\hat{N}_s \quad \text{for} \quad f \in L^1(\mu); \tag{5}$$

$$(6) \qquad V(f)(t) = \int_0^t f_s d\hat{N}_s \quad \text{for} \quad f \in L^\infty(\mu) ;$$

where the integrals in (5) and (6) are to be interpreted as Stieltjes-Lebesgue integrals.

Our aim is to prove the following theorems.

THEOREM 1. *U is an isomorphism of $L^1(\mu)$ onto H^1 such that $(\sqrt{2}\, ||\hat{N}||_{\mathrm{BMO}})^{-1} \leqq ||U|| \leqq 1$. Furthermore, V is an isomorphism of $L^\infty(\mu)$ onto* BMO *such that $1/\sqrt{2} \leqq ||V|| \leqq \sqrt{10}\, ||\hat{N}||_{\mathrm{BMO}}$.*

Now, let $\nu \in \Phi$ and define W by

$$(7) \qquad W(\nu)(Y) = \int_{[0,\infty)\times\Omega} V^{-1}(Y)d\nu , \qquad Y \in \mathrm{BMO} ,$$

where V^{-1} is the inverse of V and the integral is a Radon integral.

THEOREM 2. *The dual space* BMO* *of* BMO *is Φ. More precisely, the mapping $W: \Phi \to \mathrm{BMO}^*$ given by the formula* (7) *is isomorphic and we have $(\sqrt{10}\, ||\hat{N}||_{\mathrm{BMO}})^{-1} \leqq ||W|| \leqq \sqrt{2}$.*

The martingale representation theorem obtained in [1] plays an important role in our discussion. The reader is assumed to be familiar with the martingale theory as is given in [1] and [5].

2. **Preliminaries.** (a) The spaces H^1 and BMO. Let L denote the class of all right continuous local martingales X over (F_t) with $X_0 = 0$. Every $X \in L$ has a unique decomposition $X_t = X_t^c + X_t^d$, where X^c is the continuous part of X and X^d is the purely discontinuous part of X, orthogonal to all continuous local martingales. Let $\langle X^c, X^c\rangle$ denote the continuous increasing process such that $(X^c)^2 - \langle X^c, X^c\rangle \in L$. For every $X \in L$, $[X, X]$ denotes the process defined by $[X, X]_t = \langle X^c, X^c\rangle_t + \sum_{s\leqq t} (\Delta X_s)^2$, where the summation is taken over all points of discontinuity of X and $\Delta X_t = X_t - X_{t-}$. For $X, Y \in L$, we set $[X, Y] = \{[X + Y, X + Y] - [X - Y, X - Y]\}/4$. Let H^1 denote the Banach space of all $X \in L$ such that $||X||_{H^1} = E[[X, X]_\infty^{1/2}] < \infty$. Let us denote by $||X||_{\mathrm{BMO}}$ the smallest positive constant c such that c^2 dominates a.s., $E[[X, X]_\infty - [X, X]_{T-} | F_T]$ for every stopping time T. The space BMO is the Banach space of all $X \in L$ with $||X||_{\mathrm{BMO}} < \infty$.

LEMMA 1. *Let Y be a square integrable martingale. Then*

$$||Y||_{\mathrm{BMO}} \leqq \sqrt{5} \sup \{E[[X, Y]_\infty]; ||X||_{H^1} \leqq 1\} .$$

The proof is given in [4].

(b) The stochastic point process. Let N be a stochastic point process with the intensity λ. Then we have the following.

LEMMA 2. *Let f be a predictable process.*

(i) *If $P\left(\int_0^t |f_s|\lambda_s ds < \infty\right) = 1$ for all t, then the process $\left(\int_0^t f_s d\hat{N}_s\right)$ belongs to L.*

(ii) *If $E\left[\int_0^\infty |f_s|\lambda_s ds\right] < \infty$, then the process $\left(\int_0^t f_s d\hat{N}_s\right)$ is a uniformly integrable martingale.*

For the proof of (i), see [3, Proposition 2] or [2], and (ii) follows from (i).

We note that $[\hat{N}, \hat{N}]_T = N_T$ and $0 \leqq \varDelta\hat{N}_T = \varDelta N_T \leqq 1$ for any stopping time T. If $\hat{N}$ is a uniformly integrable martingale, we have

$$\begin{aligned}E[[\hat{N}, \hat{N}]_\infty - [\hat{N}, \hat{N}]_{T-} | F_T] &= E[N_\infty - N_{T-} | F_T] \\ &= E[\hat{N}_\infty - \hat{N}_{T-} | F_T] + E\left[\int_T^\infty \lambda_s ds \,|\, F_T\right] \\ &= \varDelta\hat{N}_T + E\left[\int_T^\infty \lambda_s ds \,|\, F_T\right] .\end{aligned}$$

Therefore, the stochastic point process N with the intensity λ satisfies the condition (B) if and only if there exists a positive constant d such that for any stopping time T,

$$E\left[\int_T^\infty \lambda_s ds \,|\, F_T\right] \leqq d^2 . \tag{8}$$

This implies that μ of (1) is a finite measure.

3. Example. Here we give an example of a stochastic point process N with the intensity λ, which satisfies the conditions (S) and (B).

Let M be a Poisson process, i.e., a stochastic point process with the intensity 1, and let $G_t = \sigma(M_s, s \leqq t)$. Define the process N by $N_t = M_{t\wedge 1}$, where $t \wedge 1 = \min(t, 1)$. Let $F_t = \sigma(N_s, s \leqq t)$ and $\lambda_t = I_{[0,1]}(t)$. Then N and λ are as required. Indeed, since

$$\hat{N}_t = N_t - \int_0^t \lambda_s ds = M_{t\wedge 1} - t \wedge 1 ,$$

it is easy to see that N is a stochastic point process with the intensity λ over (G_t). By the definition of M, the process $(M_{t\wedge 1} - t \wedge 1)$ is a G_t-martingale. Hence $\hat{N}$ is an F_t-martingale, because $F_t \subset G_t$. Thus the condition (S) is satisfied. Furthermore, for any stopping time T,

$$E\left[\int_T^\infty \lambda_s ds \,|\, F_T\right] \leqq 1 .$$

Consequently, $\hat{N}$ belongs to BMO over (F_t) by (8).

4. **Proof of Theorem 1.** First, we show that U is a continuous linear mapping of $L^1(\mu)$ into H^1. For this purpose, let $f \in L^1(\mu)$. By Lemma 2, the process $\left(\int_0^t |f_s| d\hat{N}_s\right)$ is a uniformly integrable martingale. Since $\Delta U(f)(t) = f_t \Delta N_t$, we have

$$\begin{aligned} \|U(f)\|_{H^1} &= E[[U(f), U(f)]_\infty^{1/2}] = E\left[\left(\sum_s (f_s \Delta N_s)^2\right)^{1/2}\right] \\ &\leqq E\left[\sum_s |f_s| \Delta N_s\right] = E\left[\int_0^\infty |f_s| dN_s\right] \\ &= E\left[\int_0^\infty |f_s| \lambda_s ds\right] = \|f\|_{L^1(\mu)} . \end{aligned}$$

Thus $\|U\| \leqq 1$, and the linearity of U is obvious. On the other hand, by the martingale representation theorem [1, Theorem 3.4], every $X \in H^1$ has a representation $X_t = \int_0^t f_s d\hat{N}_s$, where f is a predictable process such that $P\left(\int_0^t |f_s| \lambda_s ds < \infty\right) = 1$ for all t. Define $X' \in L$ by $X'_t = \int_0^t |f_s| d\hat{N}_s$. Then we have $[X', X']_t = \sum_{s \leqq t} (|f_s| \Delta N_s)^2 = [X, X]_t$, which implies that $\|X\|_{H^1} = \|X'\|_{H^1}$ and $X' \in H^1$. By Fefferman's inequality [5], we have

$$\begin{aligned} (9) \qquad \|f\|_{L^1(\mu)} &= E\left[\int_0^\infty |f_s| dN_s\right] = E\left[\sum_s (|f_s| \Delta N_s)(\Delta N_s)\right] \\ &= E[[X', \hat{N}]_\infty] \leqq \sqrt{2}\, \|X\|_{H^1} \|\hat{N}\|_{\mathrm{BMO}} , \end{aligned}$$

so that $f \in L^1(\mu)$ and $X = U(f)$. Namely, $U: L^1(\mu) \to H^1$ is isomorphic, and $(\sqrt{2}\, \|\hat{N}\|_{\mathrm{BMO}})^{-1} \leqq \|U\|$.

Next, we show the latter part of the theorem. Let $g \in L^\infty(\mu)$ and $V(g)(t) = \int_0^t g_s d\hat{N}_s$. Let $X \in H^1$ and $f = U^{-1}(X) \in L^1(\mu)$, where U^{-1} denotes the inverse of U. Then we have

$$\begin{aligned} E[[X, V(g)]_\infty] &\leqq E[|[X, V(g)]_\infty|] = E\left[\left|\int_0^\infty f_s g_s dN_s\right|\right] \\ &\leqq E\left[\int_0^\infty |f_s g_s| dN_s\right] = E\left[\int_0^\infty |f_s g_s| \lambda_s ds\right] \\ &\leqq \|f\|_{L^1(\mu)} \|g\|_{L^\infty(\mu)} = \|U^{-1}(X)\|_{L^1(\mu)} \|g\|_{L^\infty(\mu)} \\ &\leqq \sqrt{2}\, \|\hat{N}\|_{\mathrm{BMO}} \|X\|_{H^1} \|g\|_{L^\infty(\mu)} , \end{aligned}$$

which follows from (9). Since $E[[V(g), V(g)]_\infty] = E\left[\int_0^\infty g_s^2 \lambda_s ds\right] < \infty$, $V(g)$ is a square integrable martingale. It follows from Lemma 1 that

$$\|V(g)\|_{\mathrm{BMO}} \leqq \sqrt{5} \sup \{E[[X, V(g)]_\infty]; \|X\|_{H^1} \leqq 1\} \leqq \sqrt{10}\, \|\hat{N}\|_{\mathrm{BMO}} \|g\|_{L^\infty(\mu)} .$$

Therefore, V is a continuous linear mapping of $L^\infty(\mu)$ into BMO such that $\|V\| \leqq \sqrt{10}\,\|\hat{N}\|_{\mathrm{BMO}}$. To see that $L^\infty(\mu)$ and BMO are isomorphic under the mapping V, let $Y \in \mathrm{BMO}$ and set $C(f) = E[[U(f), Y]_\infty]$ for every $f \in L^1(\mu)$. Then by Fefferman's inequality and the former part,

$$|C(f)| \leqq \sqrt{2}\,\|U(f)\|_{H^1}\|Y\|_{\mathrm{BMO}} \leqq \sqrt{2}\,\|f\|_{L^1(\mu)}\|Y\|_{\mathrm{BMO}}\ ,$$

from which it follows that C is a bounded linear functional on $L^1(\mu)$. Hence, there exists $g \in L^\infty(\mu)$ such that $C(f) = \int fgd\mu$ for all $f \in L^1(\mu)$. Namely,

$$C(f) = E\left[\int_0^\infty f_s g_s \lambda_s ds\right] = E\left[\int_0^\infty f_s g_s dN_s\right] = E[[U(f),\ V(g)]_\infty]\ ,\quad f \in L^1(\mu)\ ,$$

and by the definition of $C(\cdot)$ we have $E[[X, V(g)]_\infty] = E[[X, Y]_\infty]$ for all $X \in H^1$. Therefore, $Y = V(g)$. Furthermore, if $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, then, as $\left|\int fgd\mu\right| = |E[[U(f), V(g)]_\infty]| \leqq \sqrt{2}\,\|f\|_{L^1(\mu)}\|V(g)\|_{\mathrm{BMO}}$, we have

$$\|g\|_{L^\infty(\mu)} = \sup\left\{\left|\int fgd\mu\right|;\ \|f\|_{L^1(\mu)} \leqq 1\right\} \leqq \sqrt{2}\,\|V(g)\|_{\mathrm{BMO}}\ . \tag{10}$$

This implies that $1/\sqrt{2} \leqq \|V\|$. Thus the theorem is proved.

5. **Proof of Theorem 2.** Let $\nu \in \Phi$ and $Y \in \mathrm{BMO}$. Then it is clear that $W(\nu)$ is linear, and by (10) we have

$$|W(\nu)(Y)| \leqq \|V^{-1}(Y)\|_{L^\infty(\mu)}\|\nu\| \leqq \sqrt{2}\,\|Y\|_{\mathrm{BMO}}\|\nu\|\ ,$$

which implies that $W(\nu) \in \mathrm{BMO}^*$ and $\|W\| \leqq \sqrt{2}$. On the other hand, if $B \in \mathrm{BMO}^*$, then from Theorem 1 it follows that for every $g \in L^\infty(\mu)$,

$$|B \cdot V(g)| \leqq \|B\|\,\|V(g)\|_{\mathrm{BMO}} \leqq \|B\|(\sqrt{10}\,\|\hat{N}\|_{\mathrm{BMO}}\|g\|_{L^\infty(\mu)})\ .$$

Namely, $B \cdot V$ is a bounded linear functional on $L^\infty(\mu)$. Then there exists $\nu \in \Phi$ such that

$$B \cdot V(g) = \int_{[0,\infty)\times\Omega} gd\nu = W(\nu)\cdot V(g)\quad \text{for all}\quad g \in L^\infty(\mu)\ .$$

Therefore, $B = W(\nu)$ on BMO by Theorem 1. Furthermore, we have by Theorem 1,

$$\|\nu\| = \|W(\nu)\cdot V\| \leqq \|W(\nu)\|(\sqrt{10}\,\|\hat{N}\|_{\mathrm{BMO}})\ ,$$

so that $(\sqrt{10}\,\|\hat{N}\|_{\mathrm{BMO}})^{-1} \leqq \|W\|$. Thus the theorem is established.

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