Tôhoku Math. Journ.
32 (1980), 35-48.

THE RESIDUAL LIMIT POINTS OF THE FIRST KIND AND THE NEST GROUPS

TAKEHIKO SASAKI

(Received January 29, 1979, revised June 30, 1979)

1. Introduction. Let G be a finitely generated Kleinian group and denote by $\Omega(G)$ and by $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. The residual limit set $\Lambda_0(G)$ of G is a subset of $\Lambda(G)$ consisting of all the limit points which do not lie on any boundary of the component of $\Omega(G)$. Let Δ be a component of G such that $\bar{\Delta} \neq \hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$. The boundary of a component of $\bar{\Delta}^c$ is called a separator of G. A point $p \in \Lambda_0(G)$ is said to be of the first kind, if there is a sequence of separators $\{C_n\}$ of G such that C_n converges to p as n tends to ∞ and that C_n separates C_{n-1} from C_{n+1}. The set of all the residual limit points of the first kind of G is denoted by $L_1(G)$, and $L_2(G) = \Lambda_0(G) \backslash L_1(G)$ is called the residual limit set of the second kind of G. In this article we shall investigate the elements of G which have their fixed points on $L_1(G)$. Next we shall show the existence of a finitely generated Kleinian group G such that $\Lambda_0(G) = L_1(G) \neq \varnothing$. We shall call it a nest group and give a characterization of nest groups. Finally, for those groups, we shall investigate the numbers of elliptic and parabolic elements, respectively.

2. Lemmas. Throughout this section we assume that G is a finitely generated Kleinian group with $\Lambda_0(G) \neq \varnothing$. Let Δ and Δ' be distinct components of G and denote by Δ^* the component of $\bar{\Delta}^c$ containing Δ'. The auxiliary domain for Δ relative to Δ' is the complement of $\overline{\Delta^*}$ and is denoted by $D(\Delta, \Delta')$. First we shall prove the following.

LEMMA 1. *Let γ and Δ be an element and a component of G, respectively, and suppose $\gamma(\Delta) \neq \Delta$. Put $D = D(\Delta, \gamma(\Delta))$. If $\gamma(D) \supset D$, then*

i) *$\partial\gamma(D) \cap \partial D$ consists of at most one point and*

ii) *γ is either loxodromic or parabolic and the latter holds if and only if $\partial\gamma(D) \cap \partial D = \{p\}$ and $\gamma(p) = p$.*

PROOF. We shall first show that $\partial\gamma(D) \neq \partial D$. Assume the contrary. Then $\partial\gamma(D) = \partial D$ so that $\gamma(D) = D$. By the properties of auxiliary domains [7], we have

$$\emptyset = D(\Delta, \gamma(\Delta)) \cap D(\gamma(\Delta), \Delta) \supset D \cap \gamma(\Delta) = \gamma(D) \cap \gamma(\Delta) = \gamma(\Delta) ,$$

a contradiction. Hence we have $\partial\gamma(D) \neq \partial D$ so that $\gamma(D) \supsetneqq D$. Put $D^* = D(\gamma(\Delta), \Delta)$. Since both Δ and $\gamma(\Delta)$ lie in $\gamma(D)$, we see that $\partial\gamma(D) \neq \partial D^*$. Hence both $\partial\gamma(D)$ and ∂D^* are boundaries of the distinct components of $\overline{\gamma(\Delta)}^c$. Proposition 5 of [7] asserts that $\partial\gamma(D) \cap \partial D^*$ consists of at most one point. On the other hand, since $D \cap D^* = \emptyset$ and since $\partial\gamma(D) \subset \partial\gamma(\Delta) \subset \overline{D^*}$, we see that $\partial\gamma(D) \cap \partial D \subset \partial\gamma(D) \cap \partial D^*$. Thus the first assertion follows.

Next we shall show the second assertion. Since $\gamma(D) \supsetneqq D$, we have $\gamma^n(D) \supsetneqq D$ for each positive integer n. This implies that γ is not elliptic. If $\partial\gamma(D) \cap \partial D = \emptyset$, then γ is a loxodromic element with the attractive and the repelling fixed points in the exterior of $\gamma(D)$ and in the interior of D, respectively. If $\partial\gamma(D) \cap \partial D = \{p\}$ and if $\gamma(p) \neq p$, then $\partial\gamma^2(D) \cap \partial D = \emptyset$ so that γ is loxodromic. If $\partial\gamma(D) \cap \partial D = \{p\}$ and if $\gamma(p) = p$, then γ is not loxodromic, for otherwise another fixed point of γ would lie on $\partial\gamma(D) \cap \partial D$ (see [6]), which contradicts i). Thus we have our lemma.

The following two lemmas give us some sufficient conditions for an element of G to belong to a component subgroup of G.

LEMMA 2. *Let γ be an element of G. If one fixid point p of γ lies neither on the residual limit set nor on the set of separators of G, then there is a component Δ of G such that $\gamma \in G_\Delta$.*

PROOF. If $p \in \Omega(G)$, then γ is elliptic and the assertion is clear. Hence we assume $p \notin \Omega(G)$. Since p does not lie on the residual limit set of G, there is a component Δ such that $p \in \partial\Delta$. If $\gamma \notin G_\Delta$, then $\gamma(\Delta)$ is a component of G different from Δ. It is known that $\partial\Delta \cap \partial\gamma(\Delta)$ is contained in a separator of G. On the other hand, p lies on $\partial\Delta \cap \partial\gamma(\Delta)$. This contradicts our assumption. Hence we have $\gamma \in G_\Delta$.

Let Δ be a component of G and let p be a point not lying on the closure of Δ. Here we shall consider the domain $D(\Delta, p)$ which is of the same kind as an auxiliary domain, that is, let Δ^* be the component of $\bar{\Delta}^c$ containing p. We then denote the complement of the closure of Δ^* by $D(\Delta, p)$ and call it the auxiliary domain for Δ relative to p. We shall also use the following terminology. Let p and q be distinct points and let Δ be a component of G. Then we say that Δ separates p from q if they lie in the distinct complementary components of $\bar{\Delta}$.

LEMMA 3. *Let γ be an elliptic element of G and let p and q be the fixed points of γ. If there is a component Δ of G which separates p from q, then $\gamma \in G_\Delta$.*

PROOF. Assume $\gamma(\Delta) \neq \Delta$. Put $D = D(\Delta, p)$. Since ∂D separates p from q, we see $q \in D$. Hence either $\gamma(D) \supset D$ or $\gamma^{-1}(D) \supset D$. The former and the latter cases imply $D = D(\Delta, \gamma(\Delta))$ and $D = D(\Delta, \gamma^{-1}(\Delta))$, respectively. This contradicts Lemma 1. Therefore, we have $\gamma(\Delta) = \Delta$ so that $\gamma \in G_\Delta$.

Later we need the following.

LEMMA 4. *Let γ and Δ be a parabolic element and a component of G, respectively. If the fixed point p of γ does not lie on $\partial\Delta$, then $\gamma(D(\Delta, p)) \cap D(\Delta, p) = \emptyset$.*

PROOF. Put $D = D(\Delta, p)$. Assume $\gamma(D) \cap D \neq \emptyset$. Then either $\partial\gamma(D) \subset \bar{D}$ or $\partial D \subset \overline{\gamma(D)}$ holds. Since $p \notin D \cup \gamma(D)$, we see that either $\gamma(D) \subsetneqq D$ or $\gamma^{-1}(D) \subsetneqq D$. Since $p \notin \partial\Delta$, we see by Lemma 1 that γ is loxodromic, a contradiction. Hence $\gamma(D) \cap D = \emptyset$.

Next lemma gives us a characterization of the points of $L_1(G)$.

LEMMA 5. *Let $p \in L_1(G)$ and let q be a point different from p. Then there is a sequence of couples (Δ_n, δ_n) of components and elements of G with the following properties:*

i) $D_n \supset \bar{D}_{n+1}$,

ii) $\delta_n(D_1) = D_n$ *and*

iii) D_n *converges to p as n tends to ∞, where $D_n = D(\Delta_n, q)$.*

PROOF. Let $\{C_k\}$ be a sequence of separators converging to p such that C_k separates C_{k-1} from C_{k+1}. Delete, if necessary, a finite number of terms from $\{C_k\}$ which do not separate p from q, and denote the new sequence also by $\{C_k\}$. Now there is a sequence $\{\Delta_k\}$ of components of G such that $C_k \subset \partial\Delta_k$. Since C_{k-1}, C_k and C_{k+1} are included in $\Lambda(G)$, we see that $\{\Delta_{2k}\}$ is a sequence of distinct components of G such that Δ_{2k} converges to p and separates p from q. By the Ahlfors finiteness theorem, we can choose an infinite subsequence of $\{\Delta_{2k}\}$ such that this subsequence is a subset of an equivalence class of components of G. We denote by $\{\Delta_m\}$ this subsequence. Let γ_m be an element of G such that $\gamma_m(\Delta_1) = \Delta_m$. The set $\{\gamma_m^{-1}(\partial D(\Delta_m, q))\}$ is a set of separators of G lying on $\partial\Delta_1$. Now the Ahlfors finiteness theorem implies that there is a finite number of separators on $\partial\Delta_1$ which are not equivalent to each other under G_{Δ_1}. Hence, choosing a subsequence, we have an infinite subsequence $\{\gamma_{m_j}^{-1}(\partial D(\Delta_{m_j}, q))\}_{j=2}^{\infty}$ of $\{\gamma_m^{-1}(\partial D(\Delta_m, q))\}$ such that, for each $j > 1$,

$$\gamma_{m_j}\varepsilon_{m_j}\gamma_{m_2}^{-1}(\partial D(\Delta_{m_2}, q)) = \partial D(\Delta_{m_j}, q) ,$$

where $\varepsilon_{m_j} \in G_{\Delta_1}$. (See Figure 1). Put $\delta_{m_j} = \gamma_{m_j}\varepsilon_{m_j}\gamma_{m_2}^{-1}$. Now it is easy to see that, if we change m_j by m_{j-1}, the couples $(\Delta_{m_j}, \delta_{m_j})$ satisfy the properties i), ii) and iii).

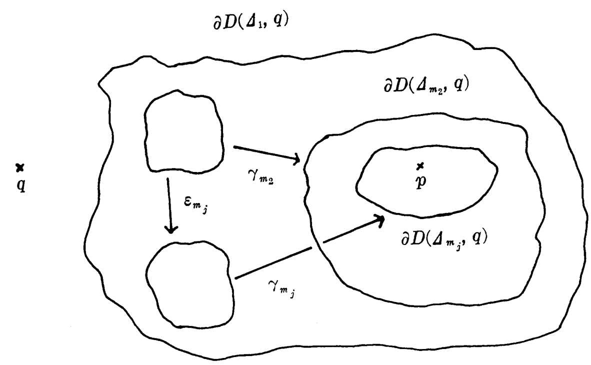


FIGURE 1

3. **Fixed points on $L_1(G)$.** Now we shall prove the following.

THEOREM 1. *Let G be a finitely generated Kleinian group with $L_1(G) \neq \emptyset$ and let γ ($\neq$id) be an element of G. If γ has a fixed point on $L_1(G)$, then γ is either loxodromic or elliptic. In the latter case there exists a loxodromic element of G having the same fixed points as γ.*

PROOF. Let p be the fixed point of γ lying on $L_1(G)$. Assume that γ is parabolic. Without loss of generality we may assume that $p = \infty$ and that $\gamma(z) = z + 1$. Let $\{C_n\}$ be a nested sequence of separators converging to ∞. Let Δ_n be a component of G with $C_n \subset \partial \Delta_n$. Put $D_n = D(\Delta_n, \infty)$. Then, by Lemma 4, we see $\gamma(D_n) \cap D_n = \emptyset$ for each n. Since D_1 is bounded, $\gamma(D_1)$ is also bounded. Since $\{C_n\}$ is a nested sequence converging to ∞ and since $\gamma(D_1) \cup D_1$ is bounded, there is an n_0 such that $D_{n_0} \supset \gamma(D_1) \cup D_1$. Then we have

$$\gamma(D_{n_0}) \cap D_{n_0} \supset \gamma(\gamma(D_1) \cup D_1) \cap (\gamma(D_1) \cup D_1) \supset \gamma(D_1) .$$

This contradicts the fact $\gamma(D_{n_0}) \cap D_{n_0} = \emptyset$. Therefore, γ is not parabolic.

Next assume that γ is elliptic. Let q be another fixed point of γ. We shall show that there is a loxodromic element of G having p and q as the fixed points. Let $\{(\Delta_n, \delta_n)\}$ be a sequence of couples obtained in Lemma 5. We note that Δ_n separates p from q for each n. By Lemma 3, we see $\gamma \in G_{\Delta_n}$ for each n so that $\delta_n^{-1}\gamma\delta_n \in G_{\Delta_1}$. Since $\delta_n(\partial D_1) = \partial D_n$, we see that $\delta_n^{-1}\gamma\delta_n$ is an element of the component subgroup $G_{\Delta_1^*}$ of G_{Δ_1}, where Δ_1^* is the component of G_{Δ_1} containing q. Since there is only a finite number of non-conjugate elliptic elements in $G_{\Delta_1^*}$, there are numbers m and k ($>m$) such that $\delta_m^{-1}\gamma\delta_m$ is conjugate to $\delta_k^{-1}\gamma\delta_k$ in $G_{\Delta_1^*}$. Let ε be an element of $G_{\Delta_1^*}$ such that $\delta_k^{-1}\gamma\delta_k = \varepsilon\delta_m^{-1}\gamma\delta_m\varepsilon^{-1}$. Putting $\gamma^* = \delta_k\varepsilon\delta_m^{-1}$, we see $\gamma\gamma^* = \gamma^*\gamma$. Since $\gamma^*(D(\Delta_m, q)) = D(\Delta_k, q)$ and $D(\Delta_m, q) \supsetneqq D(\Delta_k, q)$, we see that γ^* is a desired loxodromic element of G with the same fixed points as γ.

For the fixed points of elliptic elements of G we have the following.

COROLLARY. *Let G be a finitely generated Kleinian group and let γ be an elliptic element of G. If one of the fixed points of γ lies on $L_1(G)$, then both of them lie on $L_1(G)$.*

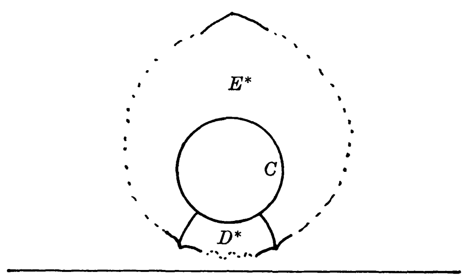
PROOF. Let p and q be the fixed points of γ and let $p \in L_1(G)$. By Theorem 1, we see that there is a loxodromic element of G whose fixed points coincide with p and q. Since $p \in L_1(G)$, there is a separator separating p from q. Hence we see $q \in L_1(G)$.

REMARK. The existence of a finitely generated Kleinian group G containing elliptic elements whose fixed points lie on $L_1(G)$ can be shown by means of the following combination theorem of Maskit.

THEOREM ([4]). *Let G_1 and G_2 be Kleinian groups and let H be a common subgroup of G_1 and G_2. Let D_1, D_2 and Δ be partial fundamental sets of G_1, G_2 and H, respectively. For $i = 1, 2$, set $E_i = \bigcup_{h \in H} h(D_i)$. Assume that $E_1 \cup E_2 \supset R(G_1) \cup R(G_2)$ and $D' = \mathrm{int}(D) \neq \emptyset$, where $D = E_1 \cap E_2 \cap \Delta$ and $R(G_i) = \Omega(G_i) \setminus \{$fixed points of elliptic elements of $G_i\}$ $(i = 1, 2)$. Then $G = \langle G_1, G_2 \rangle$ is Kleinian, no two points of D are equivalent under G, D' is a partial fundamental set for G, and G is the free product of G_1 and G_2 with the amalgamated subgroup H.*

Now let G_1 be a finitely generated Fuchsian group of the first kind such that $\Lambda(G_1) = R \cup \{\infty\}$ and that G_1 contains the elliptic element h with the fixed points i and $-i$. Put $U_\varepsilon = \{z \mid 0 < |z - i| < \varepsilon\} \cup \{z \mid 0 < |z + i| < \varepsilon\}$. Then, for a small ε, there is a fundamental set D_1 of G_1 such that $U_\varepsilon \subset \bigcup_{h \in H} h(D_1)$, where $H = \langle h \rangle$. Let g be a loxodromic transformation with the fixed points i and $-i$ such that both isometric circles of g and g^{-1} lie in U_ε. Let $G_2 = \langle g, H \rangle$ and let Δ be a fundamental set of H bounded by two circular arcs with the same endpoints i and $-i$. Finally, let D_2 be the fundamental set of G_2 bounded by the isometric circles of g and g^{-1} and by the boundary of Δ.

Then it is easy to see that fundamental sets D_1, D_2 and Δ constructed just above satisfy the assumption of Maskit's theorem stated above. Hence $G = \langle G_1, G_2 \rangle$ is a finitely generated Kleinian group. By the construction, we see that D is a fundamental set of G. Let D^* and D_1^* be components of D and D_1 lying in the upper half plane, respectively. Put $E^* = \bigcup_{h \in H} h(D^*)$ and $E_1^* = \bigcup_{h \in H} h(D_1^*)$. Then E^* is an annulus obtained from E_1^* by deleting a subset of E_1^*. (See Figure 2). Hence we see that the component of G containing D^* has $R \cup \{\infty\}$ on its boundary. It is clear that $R \cup \{\infty\}$ is a separator of G so that h is an elliptic element of G with the fixed points on $L_1(G)$.



C: isometric circle of g

FIGURE 2

4. **Nest groups.** First we shall show the existence of a finitely generated Kleinian group G with $\Lambda_0(G) = L_1(G) \neq \emptyset$. We shall call such a group a nest group. The group which we consider in the following is the one which is constructed by Abikoff in order to prove the existence of the residual limit points [1].

Let Γ be a finitely generated Fuchsian group of the first kind such that $\gamma(U) = U$ for each $\gamma \in \Gamma$, where $U = \{z \mid |z| < 1\}$. Taking the conjugate of Γ with respect to a linear transformation, we may assume that 0 and ∞ are not fixed points of any elliptic element of Γ. Then each $\gamma(\neq \mathrm{id}) \in \Gamma$ has the form $z \mapsto (az + b)/(\bar{b}z + \bar{a})$, $|a|^2 - |b|^2 = 1$ and $b \neq 0$. It is well known that there is a positive number $\tilde{b}$ such that $|b| \geqq \tilde{b}$ for each $\gamma \in \Gamma$ except for the identity. The isometric circle of γ is the set $\{z \mid |z + \overline{a/b}| = 1/|b|\}$ with the center $-\overline{a/b}$ and the radius $1/|b|$. The Ford fundamental region of Γ consists of two components R_1 and R_2, the former bounded and containing 0 and the latter unbounded and containing ∞. It is easy to see that there is a positive number C such that $\{z \mid |z| < C\} \subset R_1$. Choose a point $p \in R_2$ on the positive real axis. Let δ be a hyperbolic transformation such that the repelling and the attractive fixed points of δ are 0 and p, respectively, and that the isometric circles I_1 of δ and I_2 of δ^{-1} lie in R_1 and in R_2, respectively.

Then, by Klein's combination theorem, we see that $G = \langle \Gamma, \delta \rangle$ is a finitely generated Kleinian group with the fundamental set $(R_1 \cap \overline{\mathrm{Ext}\, I_1}) \cup (R_2 \cap \mathrm{Ext}\, I_2)$. By the construction, we see that all the components of G are equivalent to each other. Let Δ be the component of G containing $R_1 \cap \overline{\mathrm{Ext}\, I_1}$. Then $(R_1 \cap \overline{\mathrm{Ext}\, I_1}) \cup \delta^{-1}(R_2 \cap \mathrm{Ext}\, I_2)$ is the fundamental set of G_Δ

in Δ, and $\{z \mid |z| = 1\}$ and $\delta^{-1}(\{z \mid |z| = 1\})$ are contained in $\partial\Delta$.

We shall show that we can choose p and the multiplier of δ so large that the isometric circles of Γ do not intersect the isometric circles of $\delta^{-1}\Gamma\delta$. Choose the multiplier κ of δ such that $\kappa = p > 1$. Then the isometric circle of $\delta^{-1}\gamma\delta$ is the set $\{z \mid |Az + B| = 1\}$, where

$$A = -(a(\kappa - 1) + b(\kappa - 2 + 1/\kappa)/p)/p + \bar{b}\kappa + \bar{a}(\kappa - 1)/p = \bar{b}p + O(1)$$

and

$$B = -b(1 - 1/\kappa)/p + \bar{a} = \bar{a} + o(1)\ ,$$

as p tends to ∞. Since $|b| \geqq \tilde{b} > 0$, we see that, for a sufficiently large p,

$$|B/A| + 1/|A| < C\ ,$$

so that each isometric circle of $\delta^{-1}\Gamma\delta$ lies in R_1. Hence the isometric circles of Γ do not intersect those of $\delta^{-1}\Gamma\delta$. Put $G^* = \langle \Gamma, \delta^{-1}\Gamma\delta \rangle$. Then we easily see that G^* is a free product of Γ and $\delta^{-1}\Gamma\delta$ and that $G^* = G_\Delta$. This implies that each boundary of non-invariant components of G_Δ is equivalent to either $\{z \mid |z| = 1\}$ or $\delta^{-1}(\{z \mid |z| = 1\})$.

Now we can prove the non-existence of the residual limit point of the second kind of G. Let $p \in \Lambda_0(G)$ and let Δ^* be the component of G_Δ containing p. Since Δ^* is equivalent to either $\{z \mid |z| > 1\}$ or $\delta^{-1}(\{z \mid |z| < 1\})$, there is an element $g \in G_\Delta$ such that $g^{-1}(\Delta^*) = \{z \mid |z| > 1\}$ or $= \delta^{-1}(\{z \mid |z| < 1\})$. Then $\Delta_1 = g\delta(\Delta)$ (or $= g\delta^{-1}(\Delta)$) lies in Δ^* and $\partial\Delta_1 \cap \partial\Delta^* = \partial\Delta^*$. Let Δ_1^* be the component of G_{Δ_1} containing p. In the same way as above, we can find a component Δ_2 of G such that $\Delta_2 \subset \Delta_1^*$ and that $\partial\Delta_2 \cap \partial\Delta_1^* = \partial\Delta_1^*$, and so on. Then the sequence of separators $\{\partial\Delta_i^*\}$ converges to p and $\partial\Delta_i^*$ separates $\partial\Delta_{i+1}^*$ from $\partial\Delta_{i-1}^*$. This implies $p \in L_1(G)$. It is clear that $\Lambda_0(G) \neq \emptyset$. Thus we have shown the following.

THEOREM 2. *There exists a nest group, that is, a finitely generated Kleinian group G with $\Lambda_0(G) = L_1(G) \neq \emptyset$.*

REMARK. The group G constructed just above has a set of generators $\{\delta, \gamma_1, \cdots, \gamma_n\}$, where $\{\gamma_1, \cdots, \gamma_n\}$ is a set of generators of Γ. The component subgroup G_Δ is the free product of Γ and $\delta^{-1}\Gamma\delta$ so that, for each integer $m \neq 0$, we have $\delta^m \notin G_\Delta$. Since each component of G is equivalent to Δ, we see that δ^m is not an element of any component subgroup of G for each integer $m \neq 0$. This implies that G is not generated by any collection of component subgroups of G.

In connection with Theorem 1.1 in [9], Kuroda pointed out the following.

THEOREM 3. *There exists a finitely generated Kleinian group G such that any set of generators of G contains at least one loxodromic element having the fixed points on $\Lambda_0(G)$.*

PROOF. For the group G constructed above, we can choose Γ to be purely loxodromic. Since G is a free product, G contains no elliptic element. Assume that there are non-elliptic element g of G and a component Δ of G such that the fixed points of g lie on $\partial\Delta$ and that $g(\Delta) \neq \Delta$. Without loss of generality we may assume that Δ is the component in the proof of Theorem 2 and that the fixed points of g lie on the unit circle. Then $\delta^{-1}g(\Delta) = \Delta$ so that all the non-invariant components of G_Δ are conjugate to each other, a contradiction. Thus the theorem follows.

Now we shall give a characterization of nest groups.

THEOREM 4. *Let G be a finitely generated Kleinian group with $\Lambda_0(G) \neq \emptyset$. Then $L_2(G) = \emptyset$ if and only if each separator of G is a common boundary of two components of G.*

PROOF. Without loss of generality we may assume $\infty \in \Omega(G)$. First we shall show the sufficiency. Let $p \in \Lambda_0(G)$. Denote by Δ_1 the component containing ∞ and put $D_1 = D(\Delta_1, p)$. Then p lies in the interior of ∂D_1. Since ∂D_1 is a separator, there is a component Δ_2 with $\partial D(\Delta_2, \Delta_1) = \partial D_1$. Since any two separators lying on $\partial\Delta_1$ have at most one common point, we see $D(\Delta_1, \Delta_2) = D_1$. Hence $p \in D(\Delta_2, \Delta_1)$. Put $D_2 = D(\Delta_2, p)$. Then $\partial D_2 \neq \partial D_1$ and p lies in the interior of ∂D_2. Repeating the same procedure, we have an infinite sequence of separators $\{\partial D_i\}$ such that p lies in the interior of ∂D_i and that ∂D_{i+1} lies in the interior of ∂D_i ($i = 1, 2, \cdots$). Hence $p \in L_1(G)$.

In order to show the necessity we assme that there is a separator C lying only on the boundary of one component Δ. Denote by D the auxiliary domain for Δ with $\partial D = C$. Let Δ_1 be a component of G lying in the exterior of D. Then $\partial D(\Delta_1, \Delta) \neq \partial D$. Let $\{\Delta_i\}$ be the set of all components such that $D(\Delta_i, \Delta) \cap D = \emptyset$ and that $D(\Delta_i, \Delta) \supset D(\Delta_1, \Delta)$. Since each $\partial D(\Delta_i, \Delta)$ is a separator, there is a component Δ^* in $\{\Delta_i\}$ such that $D(\Delta^*, \Delta) \supset D(\Delta_i, \Delta)$ for each Δ_i in $\{\Delta_i\}$. Put $D^* = D(\Delta^*, \Delta)$. Then $D \cap D^* = \emptyset$ and $\partial D \neq \partial D^*$. We assert that there are loxodromic elements $\gamma \in G_\Delta$ and $\delta \in G_{\Delta^*}$ such that fixed points of γ (or δ) lie on ∂D (or ∂D^*) and that there is no component of G on whose boundary all four fixed points of γ and δ lie. Let γ (or δ) be a loxodromic element of G_Δ (or G_{Δ^*}) with the fixed points ξ, ξ' (or η, η') on $\partial D \backslash \partial D^*$ (or $\partial D^* \backslash \partial D$). If there is a component Δ' on whose boundary ξ, ξ', η and η' lie, then, putting $D' =$

$D(\Delta', \Delta)$, we see $D' = D(\Delta', \Delta^*)$ and $D \cap D' = D^* \cap D' = \emptyset$. Choose a loxodromic element $\varepsilon \in G_\Delta$ whose fixed points lie on $\partial D \backslash \partial D'$ and separate ξ from ξ'. Here all four fixed points of γ and ε are considered as points on the Jordan curve ∂D. Then we see easily that there is no component on whose boundary all four fixed points of ε and δ lie. Thus we have our assertion. By Lemma 4.2 in [9], we see that there is an integer m such that $\gamma\delta^m$ is a loxodromic element of G with the fixed points on $\Lambda_0(G)$. Let p and q be the fixed points of $\gamma\delta^m$. We assert $p \in L_2(G)$.

Otherwise we have $p \in L_1(G)$. Let $\{\Delta_j\}$ be the set of components such that $D(\Delta_j, \Delta) = D(\Delta_j, \Delta^*)$ and that $p \in D(\Delta_j, \Delta)$. Put $D_j = D(\Delta_j, \Delta)$. Then there is a component Δ_0 in $\{\Delta_j\}$ such that $D(\Delta_0, \Delta) \supset D(\Delta_j, \Delta)$ for each Δ_j in $\{\Delta_j\}$. Putting $D_0 = D(\Delta_0, \Delta)$, we see either $\delta^m(D_0) = D_0$ or $\delta^m(D_0) \cap D_0 = \emptyset$. (See Figure 3). If $\delta^m(D_0) = D_0$, then $\gamma\delta^m(D_0) \cap D_0 = \emptyset$, which contradicts $p \in D_0$. Hence $\delta^m(D_0) \cap D_0 = \emptyset$. By the choice of Δ^*, there is no separator other than ∂D and ∂D^* which separates D from D^*. Hence we see $\delta^m(D_0) = D$ or $\delta^m(D_0) \cap D = \emptyset$. If $\delta^m(D_0) = D$, then $\gamma\delta^m(D_0) = D$, which contradicts $p \in D_0$. Hence $\gamma\delta^m(D_0) \cap D = \emptyset$. Taking $(\gamma\delta^m)^{-1}$, we have also $(\gamma\delta^m)^{-1}(D_0) \cap D^* = \emptyset$. Since p is a point of $\gamma\delta^m(D_0) \cap D_0$, we see either $\gamma\delta^m(D_0) \supset D_0$ or $(\gamma\delta^m)^{-1}(D_0) \supset D_0$. We shall only consider the first case, because the second case can be treated in the same manner. If $\gamma\delta^m(D_0) = D_0$, then $\gamma\delta^m$ has the fixed points on a separator, a contradiction. If $\gamma\delta^m(D_0) \supsetneqq D_0$, then it contradicts the choice of Δ_0. In any case we have a contradiction. Therefore, $p \in L_2(G)$ and we have completed the proof of our theorem.

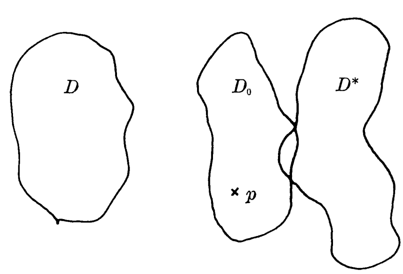


FIGURE 3

5. A web group. In this section we shall show that the method of the proof of Theorem 4 gives us a sufficient condition for a group to be a web group. Before stating the theorem, we recall some results in [2].

Let G be a finitely generated Kleinian group with $L_2(G) \neq \emptyset$ and let p be a point on $L_2(G)$. The maximal separator for p relative to a point $z \in \Omega(G)$ is the separator which separates p from z and which is not separated by any separator from p. The set of all maximal separators for p is denoted by $M(p)$. The web of p, which is denoted by $\Phi(p)$, is the closure of $M(p)$. The web subgroup of p, denoted by $H(p)$, is the maximal subgroup of G which keeps $\Phi(p)$ invariant. Then $\Lambda(H(p)) = \Phi(p)$; each component of $\Omega(H(p))$ is simply connected and bounded by a quasi-circle in $M(p)$; for each maximal separator C in $\Phi(p)$, the maximal subgroup of G which keeps invariant the complementary component ($\not\ni p$) of C is a subgroup of $H(p)$.

THEOREM 5. *Let G be a finitely generated Kleinian group with $\Lambda_0(G) \neq \emptyset$. If there is a collection of components $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ of G with the following properties, then G is a web group.*

i) *each G_{Δ_i} is quasi-Fuchsian $(i = 1, 2, \cdots, n)$,*

ii) *there is no separator separating $\partial\Delta_i$ from $\partial\Delta_j$ $(i, j = 1, 2, \cdots, n)$ and*

iii) $G = \langle G_{\Delta_1}, G_{\Delta_2}, \cdots, G_{\Delta_n}\rangle$.

PROOF. Since $\Lambda_0(G) \neq \emptyset$, we note $n > 1$. By i) and by $\Lambda_0(G) \neq \emptyset$, we see $\partial\Delta_1 \neq \partial\Delta_i$ $(i = 2, 3, \cdots, n)$. If there is a component Δ of G with $\partial\Delta_1 \subsetneqq \partial\Delta$, then each Δ_i lies in a complementary component of Δ $(i = 2, \cdots, n)$. By ii), we see $G = G_\Delta$. This contradicts the assumption $\Lambda_0(G) \neq \emptyset$. Hence $\partial\Delta_1$ is a separator which is not the common boundary of two components of G. By ii), we can choose Δ_1 and Δ_2 as Δ and Δ^* in the proof of Theorem 4, respectively. Then there are loxodromic elements $\gamma \in G_{\Delta_1}$, $\delta \in G_{\Delta_2}$ and an integer m so that $\gamma\delta^m$ is a loxodromic element of G with the fixed points on $L_2(G)$. Let p and q be the fixed points of $\gamma\delta^m$.

We assert that there is no separator separating p from $\partial\Delta_i$ $(i = 1, 2, \cdots, n)$. First we shall show this for $i = 1$. Suppose on the contrary that there is a separator C separating p from $\partial\Delta_1$. Without loss of generality we may assume that there is no separator separating C from $\partial\Delta_1$. Let Δ be the component with $C \subset \partial\Delta$. Put $D = D(\Delta, \Delta_1)$. Then $\partial D = C$. By ii), we see $D \cap \Delta_2 = \emptyset$. If $\delta^m(\partial D) = \partial D$, then $\delta^m(D) = D$ and the fixed points of γ do not lie on ∂D. Hence we see $\gamma\delta^m(D) \cap D = \emptyset$, which contradicts $p \in D$. If $\delta^m(\partial D) \subset \bar{D}$, then $\delta^m(D) \subsetneqq D$ so that the attractive fixed point of δ lie in D, a contradiction. If $\delta^m(\partial D)$ lies in the complement of D, then we see either $\delta^m(D) = \Delta_1$ or $\delta^m(D) \cap \Delta_1 = \emptyset$ and $D \cap \delta^m(D) = \emptyset$. In the former case we have $\gamma\delta^m(D) \cap D = \emptyset$, a contradiction. In the latter case, $\gamma\delta^m(\partial D)$ lies in $\bar{D}$ or in D^c. If $\gamma\delta^m(\partial D)$ lies in $\bar{D}$ (or in D^c), then $\gamma\delta^m(D) \subset D$ (or $\supset D$). This implies that the repelling (or the attractive) fixed point of $\gamma\delta^m$ lies outside D, which contradicts $p, q \in D$. Thus, in any case, we have a contradiction. Therefore, we have shown that there is no separator separating p from $\partial\Delta_1$. Next we shall show that there is no separator separating p from $\partial\Delta_i (i>1)$. Suppose on the contrary that there is a separator C separating p from $\partial\Delta_i$. Since C does not separate $\partial\Delta_1$ from $\partial\Delta_i$, we see that $\partial\Delta_1$ lies in the complementary component of C containing $\partial\Delta_i$. This implies C separates p from $\partial\Delta_1$, a contradiction. Thus we have shown our assertion.

Our assertion just proved implies $\partial\Delta_i \in M(p)$ $(i = 1, 2, \cdots, n)$. Hence, for each i $(i = 1, 2, \cdots, n)$, $G_{\Delta_i} \subset H(p)$. By iii), we see $G = H(p)$. Since G is finitely generated, we conclude G is a web group.

As an application of Theorem 5 we shall show that the group G' in [2] is a web group. The Kleinian group G' is constructed as follows: Let $C_j = \{z \mid |z - e^{(j-1)\pi i/2}| = 1/\sqrt{2}\}$ and let Γ_j be a Fuchsian group of the first kind generated by four parabolic generators, two of which have the fixed points on $C_j \cap C_{j-1}$ and $C_j \cap C_{j+1}$. Then $G' = \langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \rangle$. It is easy to see that the interiors of four circles C_1, C_2, C_3 and C_4 satisfy the properties i), ii) and iii) in Theorem 5. Hence G' is a web group.

6. Numbers of elliptic and parabolic elements. In this section we shall investigate the numbers of elliptic and parabolic elements of nest groups. In the decomposition of finitely generated function groups, Maskit showed the following.

THEOREM ([5]). *Let G be a finitely generated function group. Then the following hold.*

i) *There is a finite set of elliptic elements of G such that each elliptic element of G is conjugate to an element of the set.*

ii) *There is a finite set of parabolic cyclic and doubly periodic subgroups of G such that each parabolic element of G is conjugate to an element of a subgroup in the set.*

We shall generalize this as follows.

THEOREM 6. *Let G be a finitely generated Kleinian group with $L_2(G) = \emptyset$. Then the following hold.*

i) *There is a finite set of elliptic elements of G such that each elliptic element of G is conjugate to an element of the set.*

ii) *There is a finite set of parabolic cyclic and doubly periodic subgroups of G such that each parabolic element of G is conjugate to an element of a subgroup in the set.*

PROOF. By the theorem of Maskit and by the Ahlfors finiteness theorem, we may only consider the elements of G which do not belong to any component subgroup of G. By Theorem 1 and by Lemmas 2 and 3, we may only consider the elements whose fixed points lie on the set of separators.

First we consider the parabolic elements of G. Let γ and $\varDelta$ be a parabolic element and a component of G, respectively, such that $\gamma \notin G_\varDelta$ and that the fixed point of γ lies on $\partial\varDelta$. Then it is shown in [8] that there is a parabolic element of $G_\varDelta$ with the same fixed point as γ. Hence γ is conjugate to an element of a doubly periodic subgroup. Since there is a finite number of maximal non-conjugate parabolic doubly periodic subgroups in G, the assertion follows.

Next we consider the elliptic elements. Let γ be an elliptic element of G which does not belong to any component subgroup of G and whose fixed points lie on the set of separators. Let ξ_1 and ξ_2 be the fixed points of γ. We shall show that there is a separator on which both ξ_1 and ξ_2 lie. Suppose on the contrary that there is no separator on which both ξ_1 and ξ_2 lie. Let C_1 and C_2 be separators on which ξ_1 and ξ_2 lie, respectively. By Theorem 4, we see that there are components $\varDelta_1$, $\varDelta_2$, $\varDelta_3$ and $\varDelta_4$ such that $\partial\varDelta_1 \cap \partial\varDelta_2 = C_1$ and $\partial\varDelta_3 \cap \partial\varDelta_4 = C_2$. Without loss of generality we may assume $D(\varDelta_1, \varDelta_2) \subset D(\varDelta_3, \varDelta_4)$ and $D(\varDelta_4, \varDelta_3) \subset D(\varDelta_3, \varDelta_1)$. (See Figure 4). If $\varDelta_2 = \varDelta_3$, then $\varDelta_1$ and $\varDelta_4$ lie in the distinct components of $\bar{\varDelta}_2^c$ so that $D(\varDelta_1, \varDelta_2) \cap D(\varDelta_4, \varDelta_2) = \varnothing$. Since ξ_1 (or ξ_2) lies on C_1 (or C_2), we see $\gamma(\varDelta_2) \cap D(\varDelta_1, \varDelta_2) \neq \varnothing$ (or $\gamma(\varDelta_2) \cap D(\varDelta_2, \varDelta_4) \neq \varnothing$). This contradicts the connectivity of $\gamma(\varDelta_2)$. Hence $\varDelta_2 \neq \varDelta_3$. By the assumption, we see that if $\xi_2 \in \partial D(\varDelta_2, \varDelta_3)$, then $\xi_1 \notin \partial D(\varDelta_2, \varDelta_3)$. Hence, if $\xi_2 \in \partial D(\varDelta_2, \varDelta_3)$, then $\xi_1 \in D(\varDelta_2, \varDelta_3)$ so that $D(\varDelta_2, \varDelta_3)$ must be invariant under γ. Hence $\gamma \in G_{\varDelta_2}$, a contradiction. If ξ_2 is not in the closure of $D(\varDelta_2, \varDelta_3)$, then $\xi_2 \in D(\varDelta_2, \varDelta_1)$ so that $D(\varDelta_2, \varDelta_1)$ must be invariant under γ. Hence $\gamma \in G_{\varDelta_2}$, a contradiction. Thus, in any case, we have a contradiction so that we have shown the existence of a separator on which both ξ_1 and ξ_2 lie.

We shall next show that there is only a finite number of non-conjugate

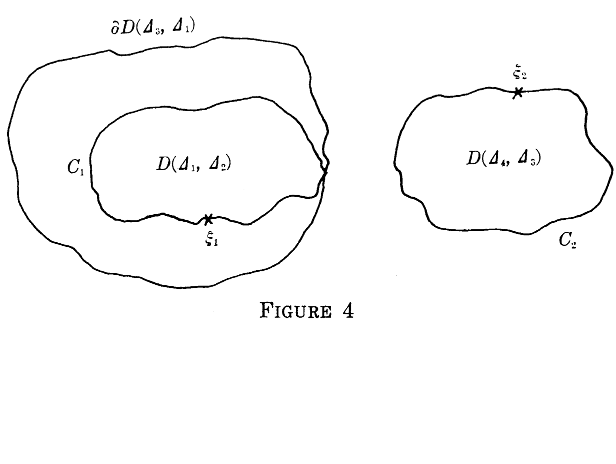


FIGURE 4

elliptic elements whose fixed points lie on a single separator. Let C be a separator and let $\{\gamma_i\}$ be a set of non-conjugate elliptic elements of G whose fixed points lie on C. Let Δ be a component with $C \subset \partial\Delta$ and let Δ' be the complementary component of Δ with $\partial\Delta' = C$. We assert that there is only a finite number of elements in $\{\gamma_i\}$ which are non-conjugate under the component subgroup $G_{\Delta'}$ of G_Δ. Since $G_{\Delta'}$ is a finitely generated quasi-Fuchsian group of the first kind, there are a finitely generated Fuchsian group F of the first kind with $\Lambda(F) = \{z \mid |z| = 1\}$ and a quasi-conformal mapping w of the unit disc onto Δ' such that $wFw^{-1} = G_{\Delta'}$. Without loss of generality we may assume that 0 and ∞ are not the fixed points of any elliptic element of F. Denote by D the bounded component of the Ford fundamental region for F. Put $g_i = w^{-1}\gamma_i w$ and denote by l_i the non-Euclidean line connecting the fixed points of g_i. Taking the conjugate of g_i with respect to an element of F, we may assume $l_i \cap D \neq \emptyset$. Put $D_r = D \cap \{z \mid |z| < r\}$. If $D_r = D$ for some $r < 1$, then each l_i passes through $\{z \mid |z| < r\}$ so that the Euclidean distances between the endpoints of l_i are bounded below by a positive constant. If $D_r \neq D$ for any $r < 1$, then there is a finite number of cusped regions of D. Denote by $\{D_j\}_{j=1}^k$ the cusped regions of D and by f_j the corresponding parabolic element of F to D_j. Let $r_0 < 1$ be a positive number such that $D \backslash D_{r_0} \subset \bigcup_{j=1}^k D_j$. If $l_i \cap D_{r_0} = \emptyset$, then there is a D_j with $l_i \cap D_j \neq \emptyset$. The conjugation of g_i by f_j^m with an integer m implies that the non-Euclidean line connecting the fixed points of $f_j^{-m} g_i f_j^m$ passes through D_{r_0}. Taking such an elliptic element for g_i, we may assume $l_i \cap D_{r_0} \neq \emptyset$. Hence the Euclidean distances between the endpoints of l_i are bounded below by a positive constant. Then we see that there is a subsequence $\{l_{i_n}\}$ of $\{l_i\}$ converging to a non-Euclidean line l with the distinct endpoints. Since the endpoints of $w(l)$ are distinct and are the cluster points of fixed points of $\{\gamma_{i_n}\}$, we conclude G is not Kleinian, a contradiction. Thus we have shown that there is only a finite number of elements in $\{\gamma_i\}$ which are non-conjugate under $G_{\Delta'}$, hence under G. Since there is only a finite number of non-equivalent separators, there is only a finite number of non-conjugate elliptic elements in G and the proof of the theorem is completed.

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
YAMAGATA UNIVERSITY
YAMAGATA, 990 JAPAN