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# ON A GENERAL SOLUTION OF A NONLINEAR 2-SYSTEM OF THE FORM $x^2dw/dx = \Lambda w + xh(x, w)$ WITH A CONSTANT DIAGONAL MATRIX $\Lambda$ OF SIGNATURE (1, 1)

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

MASAHIRO IWANO

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**1. Introduction.** 1°. *Assumptions.* Let there be given a system of $n$ nonlinear ordinary differential equations of the form

$$x^{\sigma+1}dw/dx = F(x, w)\,, \qquad F(0, 0) = 0\,, \tag{E}$$

where $\sigma$ is a positive integer; $x$ is a complex variable; $w$ is an $n$-vector; $F(x, w)$ is an $n$-vector function whose components are holomorphic functions of $(x, w)$ near the origin of the $(x, w)$-space. For many years of study, the author has been interested in the problem of constructing analytic expressions of bounded solutions for (E). A solution which is defined in a domain $D$ with the origin as an interior point or a boundary point will be said to be bounded in $D$ if, for an arbitrary point $x_0$ of $D$, there exists a smooth curve $\Gamma$ in $D$ starting from $x_0$ and extending to the origin such that this solution converges to zero as $x$ tends to the origin along the curve $\Gamma$.

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the eigenvalues of the Jacobian matrix $F_w(0, 0)$. We draw a straight line $L$ passing through the origin of the complex $\lambda$-plane. Denote by $S_1 = \{\lambda_1, \lambda_2, \cdots, \lambda_{n'}\}$ a set of the eigenvalues which are located on one side of $L$ and by $S_2 = \{\lambda_{n'+1}, \lambda_{n'+2}, \cdots, \lambda_{n''}\}$ those which are on the other side. Then each $\lambda_j$ of a set $S_3 = \{\lambda_{n''+1}, \lambda_{n''+2}, \cdots, \lambda_n\}$ is on $L$. As is well known by experts of this field, Malmquist [10, 11, 12] constructed analytic expressions for two kinds of particular bounded solutions which correspond to the sets $S_1$ and $S_2$, respectively. These analytic expressions are in terms of uniformly convergent power series of certain functions of $x$ with coefficients admitting asymptotic expansions in powers of $x$ as $x \to 0$ through some sectors $D_1$ and $D_2$, respectively (see, also, Hukuhara [1], Iwano [3, 4]). It is noteworthy that the sectors $D_1$ and $D_2$ have no common part. So, it seems to be hard to construct an analytic expression for a general solution unless certain special conditions are satisfied. This is certainly the case when we can find a straight line $L$ passing through the origin such that all the eigenvalues of the matrix $F_w(0, 0)$ are located on one side of $L$. When $\sigma = 0$, this condition has to be read as follows: all the eigenvalues and the unity are on one side of $L$. When some of the eigenvalues are zero, Iwano [4, 5, 6, 7, 8] constructed, under certain additional assumptions, analytic expressions for general solutions. But, as far as we know, in the case when there exists at least one pair of eigenvalues whose arguments differ from $\pi \pmod{2\pi}$, the problem of constructing an analytic expression for a general solution has not been studied yet. In this paper we will give an example of such equations which enables us to construct an analytic expression for a general solution.

We want to study a nonlinear 2-system of the form, in vector form,

$$x^2 dw/dx = \Lambda w + xh(x, w) \tag{1.1}$$

or, in scalar form,

$$\begin{aligned} x^2 dy/dx &= (\mu + \alpha x)y + xf(x, y, z) , \\ x^2 dz/dx &= (-\nu + \beta x)z + xg(x, y, z) . \end{aligned} \tag{1.2}$$

So, $w$, $\Lambda$ and $h(x, w)$ are given by

$$w = \begin{bmatrix} y \\ z \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \mu & 0 \\ 0 & -\nu \end{bmatrix}, \qquad h(x, w) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} w + \begin{bmatrix} f(x, y, z) \\ g(x, y, z) \end{bmatrix} .$$

We assume that:

(a) *$x$, $y$ and $z$ are complex variables;*

(b) *$\mu$ and $\nu$ are positive numbers, and $\alpha$ and $\beta$ are nonnegative numbers satisfying one of the following three conditions:*

$$\text{(i)}\ \alpha = 0, \beta > 0 ; \quad \text{(ii)}\ \alpha > 0, \beta > 0 ; \quad \text{(iii)}\ \alpha > 0, \beta = 0 . \tag{1.3}$$

(c) *$f(x, y, z)$ and $g(x, y, z)$ are holomorphic functions of $(x, y, z)$ for*

$$|x| \leqq a , \quad |y| \leqq b , \quad |z| \leqq b , \tag{1.4}$$

*and are both equal to zero identically at $y = z = 0$, $a$ and $b$ being positive constants. Moreover, all their first order partial derivatives with respect to $y$ and $z$ identically vanish at $(y, z) = (0, 0)$, namely*

$$f_y(x, 0, 0) = f_z(x, 0, 0) = g_y(x, 0, 0) = g_z(x, 0, 0) \equiv 0 . \tag{1.5}$$

2°. *Main result.* Our main result can be stated as follows.

MAIN THEOREM. *Given $\varepsilon > 0$, the differential equations* (1.2) *possess a general solution of the form*

$$y = \Phi(x, U(x), V(x)) , \qquad z = \Psi(x, U(x), V(x)) \tag{1.6}$$

*with the properties that:*

(i) *The pair* $(U(x), V(x)) = (C_1 x^\alpha \exp(-\mu/x), C_2 x^\beta \exp(\nu/x))$ *is the general solution of the differential equations*

$$x^2 du/dx = (\mu + \alpha x)u\,, \qquad x^2 dv/dx = (-\nu + \beta x)v\,, \tag{1.7}$$

$C_1$ *and* $C_2$ *being integration constants.*

(ii) $\Phi(x, u, v)$ *and* $\Psi(x, u, v)$ *are holomorphic functions of* $(x, u, v)$ *in a domain of the form*

$$0 < |x| < a''\,, \quad |\arg x - \pi/2| < \pi - \varepsilon\,, \quad |u| < b''\,, \quad |v| < b'' \tag{1.8}$$

*or*

$$0 < |x| < a''\,, \quad |\arg x + \pi/2| < \pi - \varepsilon\,, \quad |u| < b''\,, \quad |v| < b''\,, \tag{1.9}$$

$a''$ *and* $b''$ *being positive constants. Moreover, these functions have uniformly convergent expansions in powers of* $u$ *and* $v$

$$\begin{aligned} \Phi(x, u, v) &= u + \sum_{j,k} p_{jk}(x) u^j v^k \qquad (j + k \geqq 2)\,, \\ \Psi(x, u, v) &= v + \sum_{j,k} q_{jk}(x) u^j v^k \qquad (j + k \geqq 2)\,. \end{aligned} \tag{1.10}$$

*Here the coefficients* $p_{jk}(x)$ *and* $q_{jk}(x)$ *are holomorphic functions of* $x$ *for a sector of the form*

$$|\arg x - \pi/2| < \pi - \varepsilon\,, \qquad 0 < |x| < a'' \tag{1.11}$$

*or*

$$|\arg x + \pi/2| < \pi - \varepsilon\,, \qquad 0 < |x| < a'' \tag{1.12}$$

*and admit asymptotic expansions in powers of* $x$ *as* $x$ *tends to zero through the domain* (1.11) *or* (1.12).

To simplify the description, instead of (1.8) or (1.9) we write

$$0 < |x| < a''\,, \quad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \quad |u| < b''\,, \quad |v| < b''\,.$$

Analogously, a domain of the form (1.11) or (1.12) will be written as

$$|\arg x \mp \pi/2| < \pi - \varepsilon\,, \qquad 0 < |x| < a''\,. \tag{1.13}$$

3°. *Contents.* To prove Main Theorem, assuming that there exists a formal transformation of the form

$$y = u + \sum_{j,k} p_{jk}(x) u^j v^k\,, \quad z = v + \sum_{j,k} q_{jk}(x) u^j v^k \qquad (j + k \geqq 2)\,, \tag{1.10 bis}$$

which formally transforms the equations (1.2) into the differential equations (1.7), we want to determine the coefficients $p_{jk}(x)$ and $q_{jk}(x)$ as solutions of certain differential equations. To this end, by inserting (1.10 bis) into the equations (1.2) and replacing $x^2du/dx$, $x^2dv/dx$ by (1.7), we shall obtain differential equations which determine the functions $p_{jk}(x)$, $q_{jk}(x)$. For each pair $(j, k)$, a unique formal power series solution $\{p_{jk}(x), q_{jk}(x)\}$ will be obtained. To give analytic meaning to this formal solution, we put

$$p_{jk} = p_{jk}^0 + xP_{jk}\ , \quad q_{jk} = q_{jk}^0 + xQ_{jk}\ ,$$

where $p_{jk}^0 = p_{jk}(0)$, $q_{jk}^0 = q_{jk}(0)$. Then we have the differential equations satisfied by $\{P_{jk}, Q_{jk}\}$ and we can prove that there exists a unique solution $\{P_{jk}(x), Q_{jk}(x)\}$ which admits the formal power series solution as an asymptotic expansion for a domain of the form (1.13). Thus the coefficients $p_{jk}(x)$ and $q_{jk}(x)$ will be determined as analytic functions by the formulas

$$p_{jk}(x) = p_{jk}^0 + xP_{jk}(x)\ , \qquad q_{jk}(x) = q_{jk}^0(x) + xQ_{jk}(x)\ . \tag{1.14}$$

It will be proved that if the ratio $\mu/\nu$ is not equal to a rational number, we have for any pair $(j, k)$

$$p_{jk}^0 = 0\ , \qquad q_{jk}^0 = 0\ , \tag{1.15}$$

while if the ratio $\mu/\nu$ is equal to a rational number, say $m/n$, with relatively prime positive integers $m$ and $n$, we have

$$\begin{aligned} p_{jk}^0 &= 0 \quad \text{if} \quad (j, k) \neq (1 + ln, lm) \quad \text{for any} \quad l\ , \\ q_{jk}^0 &= 0 \quad \text{if} \quad (j, k) \neq (ln, 1 + lm) \quad \text{for any} \quad l\ . \end{aligned} \tag{1.16}$$

Thus the double power series $u + \sum p_{jk}^0 u^j v^k$, $v + \sum q_{jk}^0 u^j v^k$ have the form of single power series

$$\phi_0(u, v) = u(1 + \sum_l (u^n v^m)^l a_l)\ , \qquad \psi_0(u, v) = v(1 + \sum_l (u^n v^m)^l b_l)\ . \tag{1.17}$$

We will prove that the power series $\sum_l a_l w^l$, $\sum_l b_l w^l$ converge so that the sums $\phi_0(u, v)$ and $\psi_0(u, v)$ of the power series (1.17) define holomorphic functions of $(u, v)$ at $(0, 0)$.

If we replace $(u, v)$ by the general solution $(U(x), V(x))$ of the equations (1.7), the formal transformation (1.10 bis) with the relations (1.14) will produce a formal general solution (for the equations (1.2)) of the form

$$\begin{aligned} y &= \phi_0(U(x),\ V(x)) + \sum_{j,k} xP_{jk}(x)U(x)^j V(x)^k\ , \\ z &= \psi_0(U(x),\ V(x)) + \sum_{j,k} xQ_{jk}(x)U(x)^j V(x)^k\ . \end{aligned} \tag{1.18}$$

It will be proved that, for every $k$, the power series $\sum_j P_{jk}(x)u^j$ and $\sum_j Q_{jk}(x)u^j$ are convergent uniformly for $x$ in the domain (1.13) and, for every $j$, the power series $\sum_k P_{jk}(x)v^k$ and $\sum_k Q_{jk}(x)v^k$ are convergent uniformly for $x$ in the domain (1.13).

Let $N$ be an arbitrary but fixed positive integer. We make a transformation of the form

$$(1.19)\qquad \begin{aligned} y &= \phi_0(U(x),\ V(x)) + \textstyle\sum_{(N)} xP_{jk}(x)U(x)^j V(x)^k + Y\,, \\ z &= \psi_0(U(x),\ V(x)) + \textstyle\sum_{(N)} xQ_{jk}(x)U(x)^j V(x)^k + Z\,, \end{aligned}$$

where $\sum_{(N)}$ denotes the summation of all the arrangements $(j, k)$ such that $\min\{(j-1)/\nu, (k-1)/\mu\} < N$. Then it will be expected that the equations satisfied by $\{Y, Z\}$ admit a solution satisfying the order condition $Y = O(U(x)^{\nu N+1}V(x)^{\mu N+1})$, $Z = O(U(x)^{\nu N+1}V(x)^{\mu N+1})$. To prove this assertion, we put

$$(1.20)\qquad Y = U(x)\mathfrak{Y}\,, \qquad Z = V(x)\mathfrak{Z}\,.$$

We will prove that, for a given sufficiently small $\varepsilon > 0$, there exists a unique solution for the equations satisfied by $\{\mathfrak{Y}, \mathfrak{Z}\}$ such that

$$(1.21)\qquad \mathfrak{Y} = O(U(x)^{\nu N}V(x)^{\mu N+1})\,, \qquad \mathfrak{Z} = O(U(x)^{\nu N+1}V(x)^{\mu N})$$

whenever the values of $x$, $U(x)$, $V(x)$ belong to a domain in the $(x, u, v)$-space of the form

$$(1.22)\qquad 0 < |x| < a_N\,, \quad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \quad |u| < b_N\,, \quad |v| < b_N$$

with suitably chosen positive constants $a_N$ and $b_N$. In the course of the proof, it is convenient to replace this domain by a slightly modified domain of the form

$$(1.23)\qquad \begin{aligned} &0 < |x| < a'\omega(\arg x)\,, \qquad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \\ &|u| < b'\chi_\alpha(\arg x)\,, \qquad |v| < b'\chi_\beta(\arg x)\,, \end{aligned}$$

where $a'$ and $b'$ are positive constants. Here, the function $\omega(\tau)$ and $\chi_\delta(\tau)$ $(\delta = \alpha, \beta)$ are to be given by the formulas

$$\omega(\tau) = \begin{cases} (\sin\varepsilon)^{-1} & \text{if} \qquad |\tau \mp \pi/2| \leqq \pi/2\,, \\ |\cos\tau|(\sin\varepsilon)^{-1} & \text{if} \quad \pi/2 < |\tau \mp \pi/2| < \pi - \varepsilon \end{cases}$$

and

$$\chi_\delta(\tau) = \begin{cases} 1 & \text{if} \qquad |\tau \mp \pi/2| \leqq \pi/2\,, \\ |\cos\tau|^\delta & \text{if} \quad \pi/2 < |\tau \mp \pi/2| < \pi - \varepsilon\,. \end{cases}$$

A domain of the form (1.23) will be called *a stable domain for the equations* (1.7). Here is the reason: *Let* $(x_0, u^0, v^0)$ *be an arbitrary point of the domain* (1.23) *and determine the values of integration constants* $C_1$ *and* $C_2$ *so that we have* $U(x_0) = u^0$ *and* $V(x_0) = v^0$. *Then we can find a curve* $\Gamma_{x_0}$, *joining the point* $x_0$ *and the origin of the complex* $x$*-plane such that, when* $x$ *travels on this curve, the triple* $(x, U(x), V(x))$, *con-*

*sidered as a point of the $(x, u, v)$-space, always stays in the domain* (1.23). This property is a sort of stability of the general solution of the equations (1.7). A stability property like this always plays an important role in studying the analytic meaning of the formal solution. By using a fixed point theorem which was devised by Professor M. Hukuhara, we can prove the existence of a unique solution for equations satisfied by $\{\mathfrak{Y}, \mathfrak{Z}\}$ with the order condition (1.21) whenever the values of $x$, $U(x)$, $V(x)$ remain in the domain (1.23). By virtue of the uniqueness of such solutions, our standard analysis concludes the uniform convergence of the formal solution (1.18) and, consequently, the power series (1.10).

## Chapter I. Formal Transformation.

**2. Formal transformation.** Since $f(x, y, z)$ and $g(x, y, z)$ admit Taylor expansions in powers of $y$ and $z$, the equations (1.2) can be written as

$$\begin{aligned} x^2 dy/dx &= (\mu + \alpha x)y + \sum_{j,k}'' x a_{jk}(x) y^j z^k , \\ x^2 dz/dx &= (-\nu + \beta x)z + \sum_{j,k}'' x b_{jk}(x) y^j z^k , \end{aligned} \tag{2.1}$$

where the $a_{jk}(x)$'s and the $b_{jk}(x)$'s are holomorphic functions of $x$ for $|x| \leqq a$. $\sum''$ denotes the sum of all the arrangements $(j, k)$ of nonnegative integers $j$ and $k$ such that $j + k \geqq 2$.

When the ratio $\mu/\nu$ is equal to a rational number, we denote it by $m/n$ with relatively prime integers $m$ and $n$. We want to prove the following theorem.

THEOREM 1. *Let $\varepsilon$ be a sufficiently small positive number. Then there exists a formal transformation of the form*

$$y = u + \sum_{j,k}'' p_{jk}(x) u^j v^k , \qquad z = v + \sum_{j,k}'' q_{jk}(x) u^j v^k \tag{2.2}$$

*which formally changes the equations* (2.1) *into the equations*

$$x^2 du/dx = (\mu + \alpha x)u , \qquad x^2 dv/dx = (-\nu + \beta x)v . \tag{2.3}$$

*Here the $p_{jk}(x)$'s and the $q_{jk}(x)$'s have the form*

$$p_{jk}(x) = p_{jk}^0 + x P_{jk}(x) , \qquad q_{jk}(x) = q_{jk}^0 + x Q_{jk}(x) , \tag{2.4}$$

*where the $p_{jk}^0$'s and the $q_{jk}^0$'s are constants such that*

$$p_{jk}^0 \neq 0 \quad \textit{implies} \quad (j, k) = (1 + ln, lm) \quad \textit{for some integer} \quad l > 0 , \tag{2.5}$$

$$q_{jk}^0 \neq 0 \quad \textit{implies} \quad (j, k) = (ln, 1 + lm) \quad \textit{for some integer} \quad l > 0 , \tag{2.6}$$

*and the $P_{jk}(x)$'s and the $Q_{jk}(x)$'s are holomorphic functions of $x$ for a domain of the form*

$$0 < |x| < a' , \qquad |\arg x \mp \pi/2| < \pi - \varepsilon \tag{2.7}$$

*and, moreover, admit asymptotic expansions in powers of $x$ as $x$ tends to 0 through the domain* (2.7).

PROOF. Let $(u, v)$ be a solution of the equations (2.3). Put the power series (2.2) into both sides of (2.1) and formally rearrange them in the form of double power series in $u$ and $v$. Then we have the relations

$$x^2dy/dx = (\mu + \alpha x)u + \sum_{j,k}'' [x^2dp_{jk}/dx + (j\mu - k\nu + (j\alpha + k\beta)x)p_{jk}]u^jv^k ,$$

$$x^2dz/dx = (-\nu + \beta x)v + \sum_{j,k}'' [x^2dq_{jk}/dx + (j\mu - k\nu + (j\alpha + k\beta)x)q_{jk}]u^jv^k ,$$

$$\begin{aligned}(\mu + \alpha x)y + xf(x, y, z) = {} & (\mu + \alpha x)\Big(u + \sum_{j,k}'' p_{jk}u^jv^k\Big) \\ & + \sum_{j,k}'' x[a_{jk}(x) + A_{jk}(a_{st}(x), p_{st}(x), q_{st}(x))]u^jv^k ,\end{aligned}$$

$$\begin{aligned}(-\nu + \beta x)z + xg(x, y, z) = {} & (-\nu + \beta x)\Big(v + \sum_{j,k}'' q_{jk}u^jv^k\Big) \\ & + \sum_{j,k}'' x[b_{jk}(x) + B_{jk}(b_{st}(x), p_{st}(x), q_{st}(x))]u^jv^k .\end{aligned}$$

Here the $A_{jk}(a_{st}, p_{st}, q_{st})$'s (or the $B_{jk}(b_{st}, p_{st}, q_{st})$'s) are linear forms in the $a_{st}$'s (or the $b_{st}$'s) for all the arrangements $(s, t)$ such that $s + t < j + k$ with polynomial coefficients in the $P_{st}$'s and the $Q_{st}$'s for $s + t < j + k$. If we equate the coefficients in like terms, we have the linear differential equations

$$x^2dp_{jk}/dx = ((1-j)\mu + k\nu - ((j-1)\alpha + k\beta)x)p_{jk} + x(a_{jk}(x) + \mathfrak{A}_{jk}(x)) , \tag{2.8}$$

$$x^2dq_{jk}/dx = (-j\mu + (k-1)\nu - (j\alpha + (k-1)\beta)x)q_{jk} + x(b_{jk}(x) + \mathfrak{B}_{jk}(x)) , \tag{2.9}$$

which determine respectively the functions $p_{jk}(x)$ and $q_{jk}(x)$. Here, to simplify the description, we used the symbols

$$\mathfrak{A}_{jk}(x) = A_{jk}(a_{st}(x), p_{st}(x), q_{st}(x)) , \quad \mathfrak{B}_{jk}(x) = B_{jk}(b_{st}(x), p_{st}(x), q_{st}(x)) . \tag{2.10}$$

We shall define a transitive relation $<$ for the set of all arrangements $(j, k)$ of nonnegative integers $j$ and $k$ in such a way that we have $(j, k) < (j', k')$ if and only if either $j + k < j' + k'$ or $j + k = j' + k'$ and $j < j'$ holds. Suppose that the functions $p_{st}(x)$ and $q_{st}(x)$ for all arrangements $(s, t) < (j, k)$ have been determined as solutions of the equations (2.8) and (2.9) with $j = s$, $k = t$, respectively, in such a way that they are holomorphic in $x$ and admit asymptotic expansions in powers of $x$ as $x$ tends to 0 through the domain (2.7). Then, the functions $\mathfrak{A}_{jk}(x)$ and $\mathfrak{B}_{jk}(x)$ are thought of as known holomorphic functions of $x$ which admit asymptotic expansions in powers of $x$ as $x$ tends to 0 through the domain (2.7). There are two possibilities. If $(1-j)\mu + k\nu \neq 0$, then the equation (2.8) possesses a formal solution which is expressed by a power series in $x$ without a constant term. If $(1-j)\mu + k\nu = 0$, the quantity $(j-1)\alpha + k\beta$ is equal to neither zero nor a negative integer. Indeed, assume this is not the case. Since we must have $(j-1)\alpha + k\beta = (j-1)(\alpha + \beta\mu/\nu)$, the value of $j$ will necessarily be equal to 1 or 0. Thus the relation $(1-j)\mu + k\nu = 0$ will be reduced to the relation $k\nu = 0$ or $\mu + k\nu = 0$, which is a contradiction. Since the equation (2.8) is reduced to the equation

$$(2.11) \qquad x dp_{jk}/dx = -((j-1)\alpha + k\beta)p_{jk} + a_{jk}(x) + \mathfrak{A}_{jk}(x) ,$$

there is a formal solution which is expressed by a power series in $x$ with a constant term

$$(2.12) \qquad p_{jk}^0 = (a_{jk}(0) + \mathfrak{A}_{jk}(0))((j-1)\alpha + k\beta)^{-1} ,$$

where $a_{jk}(0) + \mathfrak{A}_{jk}(0) = \lim_{x\to 0}(a_{jk}(x) + \mathfrak{A}_{jk}(x))$. In any case, the differential equation (2.8) has a formal power series in $x$ as a formal solution. Similarly, this is also the case for the differential equation (2.9). Hence, it turns out that there exists a unique actual solution $p_{jk}(x)$ (holomorphic in $x$ in the domain (2.7)) which admits an asymptotic expansion of the formal solution as $x$ tends to 0 through the domain (2.7). Thus the function $p_{jk}(x)$ has been uniquely determined. Similarly, we can determine $q_{jk}(x)$ uniquely as a solution of the equation (2.9).

In order to get a solution of the form (2.4), we put, for example,

$$(2.13) \qquad p_{jk} = p_{jk}^0 + xP_{jk} .$$

If $(j, k) \neq (1 + ln, lm)$, we have $(1-j)\mu + k\nu \neq 0$ and $p_{jk}^0 = 0$. The equation (2.8) gives

$$(2.14) \quad x^2 dP_{jk}/dx = ((1-j)\mu + k\nu - ((j-1)\alpha + k\beta + 1)x)P_{jk} + a_{jk}(x) + \mathfrak{A}_{jk}(x) .$$

If $(j, k) = (1 + ln, lm)$, we have $(1-j)\mu + k\nu = 0$. The equation (2.11) implies

$$(2.15) \qquad \begin{aligned} x dP_{jk}/dx = & -((j-1)\alpha + k\beta + 1)P_{jk} \\ & + x^{-1}(a_{jk}(x) + \mathfrak{A}_{jk}(x) - ((j-1)\alpha + k\beta)p_{jk}^0) . \end{aligned}$$

One notes that, in view of the relation (2.12), the nonhomogeneous term of (2.15) is a bounded holomorphic function which admits an asymptotic expansion in powers of $x$ as $x$ tends to 0 through (2.7). In any case, by solving the equation (2.14) or (2.15) we can obtain a unique holomorphic solution $P_{jk}(x)$ admitting an asymptotic expansion in powers of $x$ for (2.7).

**3. Determination of holomorphic functions $\phi_0(u, v)$ and $\psi_0(u, v)$.** Let $(U(x), V(x))$ be a general solution of (2.3), namely

$$U(x) = C_1 \exp(-\mu/x) \cdot x^{\alpha}, \qquad V(x) = C_2 \exp(\nu/x) \cdot x^{\beta},$$

where $C_1$ and $C_2$ are integration constants. If we replace $(u, v)$ by $(U(x), V(x))$, the power series (2.2) represents a formal general solution of the equation (1.2). When the ratio $\mu/\nu$ is equal to a rational number, as was already proved, the constant terms $p_{jk}(0) = p^0_{jk}$ do not always vanish if we have $(j, k) = (1 + ln, lm)$ for some integer $l \geqq 1$ and the constant terms $q_{jk}(0) = q^0_{jk}$ are not always equal to zero if $(j, k) = (ln, 1 + lm)$ for some integers $l \geqq 1$, where $m/n$ is the relatively prime expression for $\mu/\nu$. This means that the asymptotic expansions of the coefficient functions $p_{jk}(x)$ and $q_{jk}(x)$ may begin with nonzero constant terms for some $(j, k)$. This situation will cause a certain trouble when we study an analytic meaning of the formal solution. So, we must consider a formal solution of a slightly different form. But, fortunately, we can prove the following theorem.

THEOREM 2. *The power series of $u$ and $v$ which consist of the terms independent of $x$ on the right hand sides of* (2.2), *namely*

$$\phi_0(u, v) = u + \sum_{j,k} p^0_{jk} u^j v^k, \qquad \psi_0(u, v) = v + \sum_{j,k} q^0_{jk} u^j v^k$$

*are convergent. If $\mu/\nu$ is equal to a rational number, $m/n$, the power series are reduced to*

$$\begin{aligned} \phi_0(u, v) &= u + u \sum_{l=1}^{\infty} (u^n v^m)^l p^0_{1+ln,lm}, \\ \psi_0(u, v) &= v + v \sum_{l=1}^{\infty} (u^n v^m)^l q^0_{ln,1+lm}. \end{aligned} \tag{3.1}$$

*If $\mu/\nu$ is not equal to a rational number, we have*

$$\phi_0(u, v) = u, \qquad \psi_0(u, v) = v. \tag{3.2}$$

*In any case, the sums $\phi_0(u, v)$ and $\psi_0(u, v)$ define holomorphic functions of $(u, v)$ at $(0, 0)$.*

PROOF. We assume that the ratio $\mu/\nu$ is a rational number. We formally rearrange the power series (2.2) in the form of single power series in $x$ as

(3.3) $\qquad y = \phi_0(u, v) + x\phi_1(u, v) + \cdots , \qquad z = \psi_0(u, v) + x\psi_1(u, v) + \cdots .$

In particular, the coefficients $\phi_0(u, v)$, $\psi_0(u, v)$, $\phi_1(u, v)$ and $\psi_1(u, v)$ are given by the power series

$$\phi_0(u, v) = u + \sum_{j,k} p^0_{jk} u^j v^k , \qquad \psi_0(u, v) = v + \sum_{j,k} q^0_{jk} u^j v^k ,$$

$$\phi_1(u, v) = \sum_{j,k} P_{jk}(0) u^j v^k , \qquad \psi_1(u, v) = \sum_{j,k} Q_{jk}(0) u^j v^k .$$

The power series $\phi_0(u, v)$ and $\psi_0(u, v)$ have the form

$$\phi_0(u, v) = u\Big(1 + \sum_{l=1}^{\infty} (u^n v^m)^l a_l\Big) , \qquad \psi_0(u, v) = v\Big(1 + \sum_{l=1}^{\infty} (u^n v^m)^l b_l\Big) .$$

In order to prove the convergence, we have to look for the equations which determine the functions $\phi_0(u, v)$ and $\psi_0(u, v)$. Insert (3.3) into the equations (1.2). Then a simple consideration gives $x^2 dy/dx = u(\mu + \alpha x)\partial\phi_0/\partial u + v(-\nu + \beta x)\partial\phi_0/\partial v + x\{u(\mu + \alpha x)\partial\phi_1/\partial u + v(-\nu + \beta x)\partial\phi_1/\partial v + x\phi_1\} + O(x^2)$, $(\mu + \alpha x)y + xf(x, y, z) = (\mu + \alpha x)(\phi_0 + x\phi_1) + xf(0, \phi_0, \psi_0) + O(x^2)$. Thus, equating the coefficients in like terms with respect to the powers of $x$, we obtain the equations

(3.4) $$\mu u\partial\phi_0/\partial u - \nu v\partial\phi_0/\partial v - \mu\phi_0 = 0 ,$$

(3.5) $$\mu u\partial\phi_1/\partial u - \nu v\partial\phi_1/\partial v - \mu\phi_1 = \alpha\phi_0 - \alpha u\partial\phi_0/\partial u - \beta v\partial\phi_0/\partial v + f(0, \phi_0, \psi_0) .$$

In quite a similar way, we can derive the equations

(3.6) $$\mu u\partial\psi_0/\partial u - \nu v\partial\psi_0/\partial v + \nu\psi_0 = 0 ,$$

(3.7) $$\mu u\partial\psi_1/\partial u - \nu v\partial\psi_1/\partial v + \nu\psi_1 = \beta\psi_0 - \alpha u\partial\psi_0/\partial u - \beta v\partial\psi_0/\partial v + g(0, \phi_0, \psi_0) .$$

The partial differential equations (3.4) and (3.6) have respectively formal general solutions of the form

(3.8) $$\phi_0 = u + u\sum_{l=1}^{\infty} (u^n v^m)^l a_l , \qquad \psi_0 = v + v\sum_{l=1}^{\infty} (u^n v^m)^l b_l$$

with undetermined coefficients $a_l$ and $b_l$.

Indeed, if one inserts formal power series $\phi_0 = u + \sum_{j,k} a_{jk} u^j v^k$ into (3.4), one finds the relation $\sum_{j,k} (\mu(j - 1) - \nu k) a_{jk} u^j v^k = 0$. Hence, we must have the relations $a_{jk} = 0$ if $(j, k) \neq (1 + ln, lm)$ for any $l \geqq 0$, and $a_{jk} =$ arbitrary if $(j, k) = (1 + ln, lm)$ for some $l \geqq 1$.

As was already shown, there exist for the linear equations (3.5) and (3.7), respectively, formal solutions which are expressed in terms of power series in $u$ and $v$. Therefore, when we expand the right hand side of (3.5) in powers of $u$ and $v$, no terms of the form $u(u^n v^m)^l$ must appear. This requirement is the condition for the undetermined $a_l$'s to satisfy. Similarly, no terms of the form $v(u^n v^m)^l$ must appear in the formal power series expansion of the right hand side of the equation (3.7). In this way, the power series (3.8) are uniquely determined. To prove the convergence of the power series (3.8), we have to look for the equations which define $\phi_0$ and $\psi_0$.

Let

$$f(0, \phi_0, \psi_0) = \sum_{j,k} f_{jk}\phi_0^j\psi_0^k\ , \quad g(0, \phi_0, \psi_0) = \sum_{j,k} g_{jk}\phi_0^j\psi_0^k \qquad (j + k \geqq 2) \tag{3.9}$$

be Taylor expansions. Put

$$\Phi_0 = u(1 + \phi_0)\ , \qquad \Psi_0 = v(1 + \psi_0)\ , \tag{3.10}$$

$$u^n v^m = w\ . \tag{3.11}$$

Corresponding to the power series (3.8), the equations in $\Phi_0$ and $\Psi_0$ admit formal solutions which are expressed by power series of $w$

$$\Phi_0 = \sum_l w^l a_l\ , \qquad \Psi_0 = \sum_l w^l b_l\ .$$

Hence, it is easy to see that the terms of the form $uw^k$ $(k = 1, 2, \cdots)$ come from only the partial sum (of (3.9)) of the form $\sum_l f_{1+ln,lm}\phi_0^{1+ln}\psi_0^{lm} = \sum_{l=1}^{\infty} f_{1+ln,lm}uw^l(1 + \Phi_0)^{1+ln}(1 + \Psi_0)^{lm}$. Thus the equation satisfied by $\Phi_0$ is given by a partial differential equation of the form

$$-\alpha u\partial\Phi_0/\partial u - \beta v\partial\Phi_0/\partial v + \sum_{l=1}^{\infty} f_{1+ln,lm}w^l(1 + \Phi_0)^{1+ln}(1 + \Psi_0)^{lm} = 0\ . \tag{3.12}$$

Analogously, we can derive, by using (3.7),

$$-\alpha u\partial\Psi_0/\partial u - \beta v\partial\Psi_0/\partial v + \sum_{l=1}^{\infty} g_{ln,1+lm}w^l(1 + \Phi_0)^{ln}(1 + \Psi_0)^{1+lm} = 0\ . \tag{3.13}$$

Since $u$ and $v$ are considered as the independent variables, we can regard them as general solutions of the differential equations

$$xdu/dx = \alpha u\ , \qquad xdv/dx = \beta v\ . \tag{3.14}$$

Then, $w = u^n v^m$ is a solution of the equation

$$xdw/dx = (\alpha n + \beta m)w\ .$$

If we consider $w$ as the independent variable, $x$, $u$ and $v$ are functions of $w$, so that they satisfy the differential equations

$$wdx/dw = (\alpha n + \beta m)^{-1}x\ , \tag{3.15}$$

$$wdu/dw = (\alpha n + \beta m)^{-1}\alpha u\ , \qquad wdv/dw = (\alpha n + \beta m)^{-1}\beta v\ . \tag{3.16}$$

Hence, if $\Phi_0$ is regarded as a function of $w$, we have $(\alpha n + \beta m)wd\Phi_0/dw =$

$(\partial\Phi_0/\partial u)(\alpha n+\beta m)wdu/dw+(\partial\Phi_0/\partial v)(\alpha n+\beta m)wdv/dw = \alpha u\partial\Phi_0/\partial u+\beta v\partial\Phi_0/\partial v$. Thus, it turns out that the 2-system of partial equations (3.12) and (3.13) is reduced to the 2-system of ordinary differential equations

$$(3.17)\quad \begin{aligned} (\alpha n+\beta m)wd\Phi_0/dw &= \sum_{l=1}^{\infty} f_{1+ln,lm}w^l(1+\Phi_0)^{1+ln}(1+\Psi_0)^{lm}, \\ (\alpha n+\beta m)wd\Psi_0/dw &= \sum_{l=1}^{\infty} g_{ln,1+lm}w^l(1+\Phi_0)^{ln}(1+\Psi_0)^{1+lm}, \end{aligned}$$

or, removing the common factor $w$,

$$(3.18)\quad \begin{aligned} d\Phi_0/dw &= \sum_{l=1}^{\infty}(\alpha n+\beta m)^{-1}f_{1+ln,lm}w^{l-1}(1+\Phi_0)^{1+ln}(1+\Psi_0)^{lm}, \\ d\Psi_0/dw &= \sum_{l=1}^{\infty}(\alpha n+\beta m)^{-1}g_{ln,1+lm}w^{l-1}(1+\Phi_0)^{ln}(1+\Psi_0)^{1+lm}. \end{aligned}$$

It is the 2-system of equations (3.18) that determines the functions $\Phi_0(w)$ and $\Psi_0(w)$. This system has no singularity at $w=0$. Hence there exists a unique holomorphic solution $\Phi_0(w)$, $\Psi_0(w)$ satisfying the initial condition $(\Phi_0, \Psi_0)=(0, 0)$ at $w=0$. Thus, by virtue of the uniqueness of formal solutions, the functions defined by

$$\phi_0(u, v) = u(1+\Phi_0(u^n v^m)), \qquad \psi_0(u, v) = v(1+\Psi_0(u^n v^m))$$

admit the convergent expansions (3.1).

It is immediately seen that there is no integer $l$ such that $(j, k) = (1+ln, lm)$ when the ratio $\mu/\nu$ is not equal to a rational number.

## Chapter II. Proof of Main Theorem
(Convergence of the formal solution).

**4. Formal solution.** In view of Theorems 1 and 2, we obtain the following:

THEOREM 3. *The equations* (1.2) *possess a formal solution of the form*

$$(4.1)\quad \begin{aligned} y &= \phi_0(U(x), V(x)) + \sum_{j,k}{}'' xP_{jk}(x)U(x)^j V(x)^k, \\ z &= \psi_0(U(x), V(x)) + \sum_{j,k}{}'' xQ_{jk}(x)U(x)^j V(x)^k \end{aligned}$$

*with*

$$U(x) = C_1 x^\alpha \exp(-\mu/x), \qquad V(x) = C_2 x^\beta \exp(\nu/x).$$

*Here,*

(i) $\phi_0(u, v)$ *and* $\psi_0(u, v)$ *have the form*

$$(4.2)\quad \phi_0(u, v) = u(1+\Phi_0(u^n v^m)), \qquad \psi_0(u, v) = v(1+\Psi_0(u^n v^m)),$$

*where* $\Phi_0(w)$ *and* $\Psi_0(w)$ *are holomorphic functions of* $w$ *in a neighborhood* *of* $w = 0$ *and are identically zero when the ratio* $\mu/\nu$ *is equal to an irrational number.*

ii) *The* $P_{jk}(x)$*'s and the* $Q_{jk}(x)$*'s are functions admitting asymptotic expansions in powers of* $x$ *in a domain of the form* (2.7).

When we study an analytic meaning of the formal solution (4.1), the following theorem will be very important.

THEOREM 4. (i) *For each* $k$, *the power series* $\sum_j P_{jk}(x)u^j$ *and* $\sum_j Q_{jk}(x)u^j$ *are uniformly convergent for a domain of the form*

$$(4.3)\qquad 0 < |x| < a_1\,, \quad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \quad |u| < b_1\,,$$

*corresponding to the domain of validity of the asymptotic expansions for the coefficients* $P_{jk}(x)$ *and* $Q_{jk}(x)$, *where* $a_1$ *and* $b_1$ *are positive constants.*

(ii) *Analogously, for each* $j$, *the power series* $\sum_k P_{jk}(x)v^k$ *and* $\sum_k Q_{jk}(x)v^k$ *uniformly converge for a domain of the form*

$$(4.4)\qquad 0 < |x| < a_1\,, \quad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \quad |v| < b_1\,.$$

PROOF. We give the proof of Assertion (i) only, because that of Assertion (ii) can be carried out in quite a similar way.

To prove this theorem, let us introduce the new variables $\{\eta, \zeta\}$ by

$$(4.5)\qquad y = \phi_0(U(x),\ V(x)) + \eta\,, \qquad z = \psi_0(U(x),\ V(x)) + \zeta\,.$$

Then the equations satisfied by $(\eta, \zeta)$ can be written in the form

$$(4.6)\qquad \begin{aligned} x^2 d\eta/dx &= (\mu + \alpha x)\eta + xF(x,\ U(x),\ V(x),\ \eta,\ \zeta)\,,\\ x^2 d\zeta/dx &= (-\nu + \beta x)\zeta + xG(x,\ U(x),\ V(x),\ \eta,\ \zeta)\,, \end{aligned}$$

where the $F(x, u, v, \eta, \zeta)$ and the $G(x, u, v, \eta, \zeta)$ are holomorphic functions of $(x, u, v, \eta, \zeta)$ for a domain of the form

$$(4.7)\qquad |x| < a_2\,, \quad |u| < b_2\,, \quad |v| < b_2\,, \quad |\eta| < c_2\,, \quad |\zeta| < c_2$$

with suitably chosen positive constants $a_2$, $b_2$ and $c_2$.

Indeed, a simple consideration gives $x^2 d\eta/dx = x^2(dy/dx) - x^2(d/dx)\phi_0(U(x), V(x)) = (\mu+\alpha x)(\phi_0(U(x), V(x)) + \eta) + xf(x, \phi_0+\eta, \psi_0+\zeta) - (\mu+\alpha x)U(x)\partial\phi_0/\partial U - (-\nu + \beta x)V(x)\partial\phi_0/\partial V$. Here $\partial\phi_0/\partial U$ means $(\partial\phi_0/\partial u)(U(x), V(x))$. In view of the equations (3.4), we have $F(x, u, v, \eta, \zeta) = \alpha\phi_0(u, v) - \alpha u(\partial/\partial u)\phi_0(u, v) - \beta v(\partial/\partial v)\phi_0(u, v) + f(x, \phi_0(u, v) + \eta, \psi_0(u, v) + \zeta)$. This shows that the $F$ satisfies our requirement.

Obviously, the power series

$$(4.8)\qquad \eta = x \sum_{j\,k}{}'' P_{jk}(x) U(x)^j V(x)^k\,, \qquad \zeta = x \sum_{j,k}{}'' Q_{jk}(x) U(x)^j V(x)^k$$

is a formal solution of the equations (4.6). We formally rewrite (4.8) in the form of single power series in $V$ in such a way that

(4.9) $$\eta = x\sum_{k=0}^{\infty} A_k(x,\ U(x))V(x)^k\ , \qquad \zeta = x\sum_{k=0}^{\infty} B_k(x,\ U(x))V(x)^k\ ,$$

where the $A_k(x, u)$'s and the $B_k(x, u)$'s are power series in $u$. Let us look for the differential equations which determine the coefficients $A_k(x, u)$ and $B_k(x, u)$. A direct calculation yields $x^2d\eta/dx = x\sum_{k=0}^{\infty}\{x^2dA_k/dx + (-k\nu + (k\beta + 1)x)A_k\}V^k$, $(\mu + \alpha x)\eta + xF(x, U, V, \eta, \zeta) = x(\mu + \alpha x)A_0 + xF(x, U, 0, xA_0, xB_0) + x\sum_{k=1}^{\infty}\{(\mu + \alpha x + xF_\eta(x, U, 0, xA_0, xB_0))A_k + xF_\zeta(x, U, 0, xA_0, xB_0)B_k + x\mathfrak{A}_k(x, U; A_1, \cdots, A_{k-1}, B_1, \cdots, B_{k-1})\}V^k$. Here the $\mathfrak{A}_k$'s are linear forms in the expressions $(\partial^{r+s+t}/\partial v^r\partial\eta^s\partial\zeta^t)F(x, U(x), 0, xA_0(x, U(x)), xB_0(x, U(x)))$, $2 \leqq r + s + t \leqq k$ with polynomial coefficients in $(x, A_1(x, U), \cdots, A_{k-1}(x, U), B_1(x, U), \cdots, B_{k-1}(x, U))$. In particular, if $k = 1$, we have $\mathfrak{A}_1 = (\partial F/\partial v)(x, U(x), 0, xA_0(x, U(x)), xB_0(x, U(x)))$. Thus we are lead to the relations

(4.10) $$x^2dA_0/dx = (\mu + (\alpha - 1)x)A_0 + F(x,\ U(x),\ 0,\ xA_0,\ xB_0)$$

and

(4.11.$k$) $$\begin{aligned} x^2dA_k/dx = {} & (k\nu + \mu + (\alpha - k\beta - 1)x)A_k \\ & + x[F_\eta]A_k + x[F_\zeta]B_k + x[\mathfrak{A}_k]\ , \end{aligned}$$

where, to simplify the description, we adopted the symbol [ ], for example, to mean

$$[F_\eta] = F_\eta(x,\ U(x),\ 0,\ xA_0(x,\ U(x)),\ xB_0(x,\ U(x)))\ ,$$

$$[\mathfrak{A}_k] = \mathfrak{A}_k(x,\ U(x),\ A_1(x,\ U(x)),\ \cdots,\ B_{k-1}(x,\ U(x)))\ .$$

Similarly, by using the power series expression for $\zeta$, we can derive equations of the form

(4.12) $$x^2dB_0/dx = (-\nu + (\beta - 1)x)B_0 + G(x,\ U(x),\ 0,\ xA_0,\ xB_0)\ ,$$

(4.13.$k$) $$\begin{aligned} x^2dB_k/dx = {} & ((k-1)\nu + (\beta - k\beta - 1)x)B_k \\ & + x[G_\eta]A_k + x[G_\zeta]B_k + [\mathfrak{B}_k]\ . \end{aligned}$$

The meaning of the functions $[G_\eta]$, $[G_\zeta]$, $[\mathfrak{B}_k]$ will be almost clear.

The following are noteworthy.

( i ) *The system of equations* (4.10) *and* (4.12) *determines the functions* $A_0(x, u)$ *and* $B_0(x, u)$ *and it is a nonlinear* 2-*system with an irregular type singularity at* $x = 0$. *Moreover, there exists a formal solution of the form*

(4.14) $$A_0 = \sum_j P_{j0}(x)U(x)^j\ , \qquad B_0 = \sum_j Q_{j0}(x)U(x)^j\ .$$

( ii ) *For every* $k \geqq 1$, *the* 2-*system of linear differential equations* (4.11.$k$) *and* (4.13.$k$) *determines the functions* $A_k(x, u)$ *and* $B_k(x, u)$. $x = 0$

*is an irregular singularity of this linear system. There is a formal solution of the form*

$$(4.15.k) \qquad A_k = \sum_j P_{jk}(x)U(x)^j\,, \qquad B_k = \sum_j Q_{jk}(x)U(x)^j\,.$$

(iii) *These differential equations involve an arbitrary function $U(x)$ in the sense that $U(x)$ contains an arbitrary constant.*

Differential equations of this type were already studied by the author. By applying a theorem due to Iwano [3] (in which a nonlinear $n$-system of much more general type was studied), we see that the formal solution (4.14) is uniformly convergent when the values of $(x, U(x))$ belong to a domain of the form (4.3) and its sum $\{A_0(x, U(x)), B_0(x, U(x))\}$ represents a solution of the equations (4.10) and (4.12) whenever the values of $x$ and $U(x)$ stay in the domain (4.3). Thus, $A_0(x, u)$ and $B_0(x, u)$, considered as functions of $(x, u)$, are holomorphic in $(x, u)$ for (4.3). In order to apply an induction technique, suppose that the functions $A_l(x, u)$ and $B_l(x, u)$ have been already determined for $l < k$ in such a way that, for every $l$, they are holomorphic functions of $(x, u)$ for (4.3) and the pair $\{A_l(x, U(x)), B_l(x, U(x))\}$ is a solution of the 2-system of equations (4.11.$l$) and (4.13.$l$) whenever the values of $x$ and $U(x)$ remain in the domain (4.3). Then the coefficients of the linear differential equations (4.11.$k$) and (4.13.$k$) are considered as holomorphic functions of $(x, U(x))$ provided the values of $x$ and $U(x)$ remain in the domain (4.3). Moreover, there exists a formal solution given by the power series (4.15.$k$). So, if we again apply the same theorem to our linear system, it can be proved that this formal solution is uniformly convergent so that the sum $\{A_k(x, U(x)), B_k(x, U(x))\}$ represents a solution of the linear system (4.11.$k$)-(4.13.$k$) provided the values of $x$ and $U(x)$ stay in the domain (4.3). Hence, $A_k(x, u)$ and $B_k(x, u)$ have been determined as holomorphic functions of $(x, u)$ for the domain (4.3).

Thus, proceeding inductively, we see that, for each $k$, the coefficients $A_k(x, u)$ and $B_k(x, u)$ are uniquely determined as holomorphic functions of $(x, u)$ in a domain of the form (4.3) in such a way that the pair $\{A_k(x, U(x)), B_k(x, U(x))\}$ is a solution of the linear system (4.11.$k$)-(4.13.$k$). This proves Assertion (i) of Theorem 4.

**5. Truncated differential equations.** In order to prove the convergence of the power series (2.2), it suffices that the formal solution (4.1) is convergent, or what is the same thing, that the formal solution (4.8) of the equations (4.6) is convergent whenever the values of $x$, $U(x)$, $V(x)$ remain in a domain of the form (1.8) (or (1.9)). Observe that the differential equations and their formal solutions under consideration are given by

$$(5.1)\qquad \begin{aligned} x^2 d\eta/dx &= (\mu + \alpha x)\eta + xF(x,\ U(x),\ V(x),\ \eta,\ \zeta)\ , \\ x^2 d\zeta/dx &= (-\nu + \beta x)\zeta + xG(x,\ U(x),\ V(x),\ \eta,\ \zeta) \end{aligned}$$

and

$$(5.2)\qquad \eta = x\sum_{j,k}'' P_{jk}(x)U(x)^j V(x)^k\ , \qquad \zeta = x\sum_{j,k}'' Q_{jk}(x)U(x)^j V(x)^k\ ,$$

where

$$U(x) = C_1 x^{\alpha}\exp(-\mu/x)\ , \quad V(x) = C_2 x^{\beta}\exp(\nu/x)\ .$$

As was shown in Section 4, the function $F(x, u, v, \eta, \zeta)$ takes the form $F(x, u, v, \eta, \zeta) = \alpha\phi_0(u, v) - \alpha u(\partial/\partial u)\phi_0(u, v) - \beta v(\partial/\partial v)\phi_0(u, v) + f(x, \phi_0(u, v) + \eta, \psi_0(u, v) + \zeta)$. By definition, we have the order relations

$$\begin{aligned} f(x,\ y,\ z) &= O(y^2) + O(yz) + O(z^2)\ , \\ \phi_0(u,\ v) &= u(1 + O(u^n v^m))\ , \\ \psi_0(u,\ v) &= v(1 + O(u^n v^m))\ . \end{aligned}$$

Hence, it is concluded that the functions $F(x, u, v, \eta, \zeta)$ and $G(x, u, v, \eta, \zeta)$ satisfy an inequality of the form

$$(5.3)\qquad \begin{aligned} &\max\{|F(x,u,v,\eta,\zeta) - F(x,u,v,0,0)|, |G(x,u,v,\eta,\zeta) - G(x,u,v,0,0)|\} \\ &\qquad \leqq M_0((|u| + |v|)(|\eta| + |\zeta|) + |\eta|^2 + |\zeta|^2) \end{aligned}$$

for $(x, u, v, \eta, \zeta)$ in the domain (4.7), where $M_0$ is a suitably chosen positive constant.

Let $N$ be an arbitrary but fixed positive integer, and put

$$(5.4)\qquad P_N(x,\ u,\ v) = x\sum\nolimits_{(N)} P_{jk}(x)u^j v^k\ , \quad Q_N(x,\ u,\ v) = x\sum\nolimits_{(N)} Q_{jk}(x)u^j v^k\ ,$$

where $\sum_{(N)}$ denotes the summation of all the arrangements $(j, k)$ of non-negative integers $j$ and $k$ such that $\min\{(j-1)/\nu, (k-1)/\mu\} < N$. Now we apply the transformation

$$(5.5)\qquad \eta = P_N(x,\ U(x),\ V(x)) + Y\ , \qquad \zeta = Q_N(x,\ U(x),\ V(x)) + Z\ .$$

Since

$$x^2 dY/dx = x^2 d\eta/dx - x^2\partial P_N/\partial x - U(x)(\mu + \alpha x)\partial P_N/\partial U - V(x)(-\nu + \beta x)\partial P_N/\partial V\ ,$$

we see that the transformed equations take the form

$$(5.6)\qquad \begin{aligned} x^2 dY/dx &= (\mu + \alpha x)Y + xA(x,\ U(x),\ V(x),\ Y,\ Z)\ , \\ x^2 dZ/dx &= (-\nu + \beta x)Z + xB(x,\ U(x),\ V(x),\ Y,\ Z)\ , \end{aligned}$$

where, $A(x, u, v, Y, Z)$ and $B(x, u, v, Y, Z)$ are holomorphic and bounded functions of $(x, u, v, Y, Z)$ in a domain in the $(x, u, v, Y, Z)$-space of the form

$$(5.7)\qquad \begin{aligned} &0 < |x| < a_N\,, \quad |\arg x \mp \pi/2| < \pi - \varepsilon\,, \\ &|u| < b_N\,, \quad |v| < b_N\,, \quad |Y| < c_N\,, \quad |Z| < c_N\,, \end{aligned}$$

where $a_N$, $b_N$ and $c_N$ are suitably chosen positive constants. It is obvious that the equations (5.6) have a formal solution of the form

$$(5.8)\qquad Y = x \sum_{\substack{j \geqq \nu N+1 \\ k \geqq \mu N+1}}^{(N)} R_{jk}(x) U(x)^j V(x)^k\,, \quad Z = x \sum_{\substack{j \geqq \nu N+1 \\ k \geqq \mu N+1}}^{(N)} S_{jk}(x) U(x)^j V(x)^k\,,$$

where the coefficients $R_{jk}(x)$ and $S_{jk}(x)$ have analytic meanings analogous to those of the $P_{jk}(x)$'s and the $Q_{jk}(x)$'s.

Note that the functions $P_N(x, u, v)$ and $Q_N(x, u, v)$ satisfy the order relations $P_N(x, u, v) = O(u^2) + O(uv) + O(v^2)$, $Q_N(x, u, v) = O(u^2) + O(uv) + O(v^2)$. By using the inequality (5.3) and the formal solution (5.8), we can easily verify that the functions $A(x, u, v, Y, Z)$ and $B(x, u, v, Y, Z)$ satisfy an inequality of the form

$$(5.9)\qquad \begin{aligned} &\max\{|A(x, u, v, Y, Z)|, |B(x, u, v, Y, Z)|\} \\ &\qquad \leqq M((|u| + |v|)(|Y| + |Z|) + |Y|^2 + |Z|^2) + L_N |u|^{\nu N+1} |v|^{\mu N+1} \end{aligned}$$

for $(x, u, v, Y, Z)$ in the domain (5.7) and Lipschitz' inequality of the form

$$(5.10)\qquad \begin{aligned} &\max\{|A(x, u, v, Y_1, Z_1) - A(x, u, v, Y_2, Z_2)|, \\ &\qquad |B(x, u, v, Y_1, Z_1) - B(x, u, v, Y_2, Z_2)|\} \\ &\qquad \leqq M(|u| + |v| + |Y_1| + |Y_2| + |Z_1| + |Z_2|)(|Y_1 - Y_2| + |Z_1 - Z_2|)\,, \end{aligned}$$

where $L_N$ is a positive constant and $M \geqq M_0$ is a positive constant independent of $N$.

We expect that the equations (5.6) possess an actual solution satisfying the order condition

$$(5.11)\qquad Y = O(xU(x)^{\nu N+1} V(x)^{\mu N+1})\,, \quad Z = O(xU(x)^{\nu N+1} V(x)^{\mu N+1})\,.$$

We make a further transformation of the form

$$(5.12)\qquad Y = U(x)\mathfrak{Y}\,, \quad Z = V(x)\mathfrak{Z}\,,$$

so that we have the equations

$$(5.13)\qquad \begin{aligned} d\mathfrak{Y}/dx &= x^{-1} U(x)^{-1} A(x, U(x), V(x), U(x)\mathfrak{Y}, V(x)\mathfrak{Z})\,, \\ d\mathfrak{Z}/dx &= x^{-1} V(x)^{-1} B(x, U(x), V(x), U(x)\mathfrak{Y}, V(x)\mathfrak{Z})\,. \end{aligned}$$

**6. Fundamental lemma and the proof of the convergence.** To prove the existence of a solution for the equations (5.6) satisfying the order condition (5.11), we have to borrow a lemma which is usually called a fundamental lemma.

FUNDAMENTAL LEMMA. *For a preassigned sufficiently small positive* $\varepsilon$, *the differential equations* (5.13) *possess a solution of the form*

$$\mathfrak{Y} = \mathfrak{Y}_N(x, U(x), V(x)), \quad \mathfrak{Z} = \mathfrak{Z}_N(x, U(x), V(x)) \tag{6.1}$$

*such that the* $\mathfrak{Y}_N(x, u, v)$ *and the* $\mathfrak{Z}_N(x, u, v)$ *are holomorphic functions of* $(x, u, v)$ *in a domain of the form*

$$(6.2)_N \qquad 0 < |x| < a_N^0, \quad |\arg x \mp \pi/2| < \pi - \varepsilon, \quad |u| < b_N^0, \quad |v| < b_N^0$$

*with some positive constants* $a_N^0(\leqq a_N)$ *and* $b_N^0(\leqq b_N)$. *Moreover, a solution of the equations* (5.13) *such that*

$$\mathfrak{Y} = O(U(x)^{\nu N} V(x)^{\mu N+1}), \quad \mathfrak{Z} = O(U(x)^{\nu N+1} V(x)^{\mu N}) \tag{6.3}$$

*is unique.*

Conditional on this fundamental lemma, we shall prove the uniform convergence of the formal solution (5.2), which produces the proof of our main theorem. Obviously the expressions

$$\begin{aligned} \eta_N(x, U(x), V(x)) &= P_N(x, U(x), V(x)) + U(x)\mathfrak{Y}_N(x, U(x), V(x)) \\ \zeta_N(x, U(x), V(x)) &= Q_N(x, U(x), V(x)) + V(x)\mathfrak{Z}_N(x, U(x), V(x)) \end{aligned} \tag{6.4}$$

are solutions of the equations (5.1). If we can prove that these sums are independent of $N$, the convergence of our formal solution will be established.

Indeed, the functions $P_N(x, u, v)$ and $\mathfrak{Y}_N(x, u, v)$ are both holomorphic with respect to $(u, v)$ at $(0, 0)$. Hence, by virtue of the Cauchy Theorem, the sums $\eta_N(x, U(x), V(x))$ and $\zeta_N(x, U(x), V(x))$, which are independent of $N$, have uniformly convergent expansions in double power series of $U(x)$ and $V(x)$. By the uniqueness of the Hartogs series expansions, such double power series expansions must coincide with the power series (5.2).

Thus the solution $\{\Phi(x, U(x), V(x)), \Psi(x, U(x), V(x))\}$ mentioned in the introduction is given by the formulas

$$\begin{aligned} \Phi(x, U(x), V(x)) &= \phi_0(U(x), V(x)) + \eta(x, U(x), V(x)), \\ \Psi(x, U(x), V(x)) &= \psi_0(U(x), V(x)) + \zeta(x, U(x), V(x)), \end{aligned}$$

where $\eta(x, u, v) = \eta_N(x, u, v)$ and $\zeta(x, u, v) = \zeta_N(x, u, v)$ are supposed to be independent of $N$.

To prove that the solution $\{\eta_N(x, U(x), V(x)), \zeta_N(x, U(x), V(x))\}$ is independent of $N$, take $N' > N$. Then, the pair

$$\eta = \eta_{N'}(x, U(x), V(x)) - P_N(x, U(x), V(x)) ,$$
$$\zeta = \zeta_{N'}(x, U(x), V(x)) - Q_N(x, U(x), V(x))$$

is a solution of the equations (5.13) because $\{\eta_{N'}(x, U(x), V(x)), \zeta_{N'}(x, U(x), V(x))\}$ is a solution of (5.1). However, this solution satisfies the order condition $U(x)^{-1}(\eta_{N'}(x, U(x), V(x)) - P_N(x, U(x), V(x))) = U(x)^{-1}(P_{N'}(x, U(x), V(x)) - P_N(x, U(x), V(x)) + U(x)\mathfrak{Y}_{N'}(x, U(x), V(x))) = O(xU(x)^{\nu N}V(x)^{\mu N+1}) + O(U(x)^{\nu N}V(x)^{\mu N+1}) = O(U(x)^{\nu N}V(x)^{\mu N+1})$ in the common part of the domains $(6.2)_N$ and $(6.2)_{N'}$. Similarly, we have the order condition $V(x)^{-1}(\zeta_{N'}(x, U(x), V(x)) - Q_N(x, U(x), V(x))) = O(U(x)^{\nu N+1}V(x)^{\mu N})$ in the same common part. Owing to the fundamental lemma such a solution must coincide with our solution $\{\mathfrak{Y}_N(x, U(x), V(x)), \mathfrak{Z}_N(x, U(x), V(x))\}$. Hence we have the identities

$$\eta_{N'}(x, U(x), V(x)) = P_N(x, U(x), V(x)) + U(x)\mathfrak{Y}_N(x, U(x), V(x)) ,$$
$$\zeta_{N'}(x, U(x), V(x)) = Q_N(x, U(x), V(x)) + V(x)\mathfrak{Z}_N(x, U(x), V(x))$$

in the common part of the domains $(6.2)_N$ and $(6.2)_{N'}$, which immediately implies that we have the identity relations

$$\eta_{N'}(x, u, v) \equiv \eta_N(x, u, v) , \quad \zeta_{N'}(x, u, v) \equiv \zeta_N(x, u, v) ,$$

in the common part of the domains $(6.2)_N$ and $(6.2)_{N'}$. Thus the functions $\eta(x, u, v)$ and $\zeta(x, u, v)$ are defined as holomorphic functions of $(x, u, v)$ in a domain of the form

$$0 < |x| < a'' , \quad |\arg x \mp \pi/2| < \pi - \varepsilon , \quad |u| < b'' , \quad |v| < b''$$

with $a'' = \sup a_N$, $b'' = \sup b_N$.

## Chapter III. Proof of the Fundamental Lemma.

**7. Family $\mathfrak{F}$ and mapping $\mathfrak{T}$.** We need a lengthy reasoning to prove the fundamental lemma. It is very convenient to consider, instead of the domain $(6.2)_N$, a domain of the form

$$\begin{aligned} 0 < |x| < a^0\omega(\arg x) , \quad |\arg x \mp \pi/2| < \pi - \varepsilon , \\ |u| < b^0\chi_\alpha(\arg x) , \quad |v| < b^0\chi_\beta(\arg x) , \end{aligned} \tag{7.1}$$

where the $\omega(\tau)$ and the $\chi_\delta(\tau)$'s $(\delta = \alpha, \beta)$ are continuous functions defined by

$$\omega(\tau) = \begin{cases} (\sin\varepsilon)^{-1} , & |\tau \mp \pi/2| \leqq \pi/2 , \\ |\cos\tau|(\sin\varepsilon)^{-1} , & \pi/2 < |\tau \mp \pi/2| < \pi - \varepsilon \end{cases} \tag{7.2}$$

$$(7.3)\qquad \chi_\delta(\tau) = \begin{cases} 1\,, & |\tau \mp \pi/2| \leqq \pi/2\,, \\ |\cos\tau|^\delta\,, & \pi/2 < |\tau \mp \pi/2| < \pi - \varepsilon\,. \end{cases}$$

The functions are bounded from below by a positive constant. It is to be noted that, for a preassigned positive numbers $a^0$ and $b^0$, if we choose the values of $a_N^0$ and $b_N^0$ in a suitable way, every point $(x, u, v)$ of the domain $(6.2)_N$ belongs to the domain (7.1).

We will prove that the fundamental lemma holds in a domain of the form (7.1). Let $\mathfrak{F}$ be the set of pairs $\{\phi, \psi\}$ of functions $\phi(x, u, v)$ and $\psi(x, u, v)$ which are holomorphic in $(x, u, v)$ for the domain (7.1) and satisfy there an inequality of the form

$$(7.4)\qquad \max\{|v|^{-1}|\phi(x, u, v)|, |u|^{-1}|\psi(x, u, v)|\} \leqq K_N|u^\nu v^\mu|^N\,,$$

$K_N$ being a certain positive constant. Since $\{0, 0\} \in \mathfrak{F}$, $\mathfrak{F}$ is not empty. It is almost evident that this family is closed and normal with respect to the topology of uniform convergence on every compact subsets. Moreover, $\mathfrak{F}$ is convex.

In order to define a mapping $\mathfrak{T}$, let $(x_0, u^0, v^0)$ be an arbitrary point of the domain (7.1). Choose the values of integration constants $C_1$ and $C_2$ satisfying the conditions $U(x_0) = u^0$, $V(x_0) = v^0$, so that we have

$$(7.5)\qquad C_1 u^0(x_0)^{-\alpha}\exp(\mu/x_0)\,,\quad C_2 = v^0(x_0)^{-\beta}\exp(-\nu/x_0)\,.$$

We define the functions $\Phi(x, u, v)$ and $\Psi(x, u, v)$ by the integrals

$$(7.6)\qquad \Phi(x_0, u^0, v^0) = \int_0^{x_0} \Xi(x)dx\,,\quad \Psi(x_0, u^0, v^0) = \int_0^{x_0} \Pi(x)dx\,,$$

where the integration is to be carried out along a curve $\Gamma_{x_0}$ which will be defined later. Here, to simplify the description, we introduced the functions $\Xi(x)$ and $\Pi(x)$ by

$$\Xi(x) = x^{-1}U(x)^{-1}A(x, U(x), V(x), U(x)\phi(x, U(x), V(x)), V(x)\psi(x, U(x), V(x)))\,,$$
$$\Pi(x) = x^{-1}V(x)^{-1}B(x, U(x), V(x), U(x)\phi(x, U(x), V(x)), V(x)\psi(x, U(x), V(x)))\,.$$

The mapping $\mathfrak{T}$ is to be defined by

$$\{\phi(x, u, v), \psi(x, u, v)\} \to \{\Phi(x, u, v), \Psi(x, u, v)\}\,.$$

We want to prove that, if the curve $\Gamma_{x_0}$ is chosen in a suitable way, there exists a fixed point of the mapping $\mathfrak{T}$. But, since the definition of the curve $\Gamma_{x_0}$ is very complicated, we will first list up, without proofs, some important properties concerning this curve. Those properties will be verified in the last chapter.

**8. Curve $\Gamma_{x_0}$ and its properties.** The domain of the independent variable $x$ is given either by

(8.1) $$D:\quad 0<|x|<a^0\omega(\arg x)\ ,\quad |\arg x-\pi/2|<\pi-\varepsilon$$

or by

(8.1 bis) $$0<|x|<a^0\omega(\arg x)\ ,\quad |\arg x+\pi/2|<\pi-\varepsilon\ .$$

As is illustrated in the figure, we divide the domain into three subsets either of the form

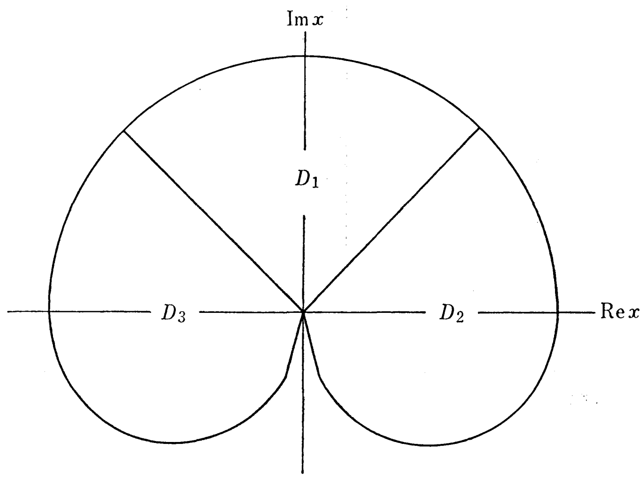


FIGURE 1

(8.2) $$\begin{aligned} D_1&:\ |\mathrm{Re}\,x|\leqq \mathrm{Im}\,x\ ,\quad \mathrm{Im}\,x>0\ ,\\ D_2&:\ \mathrm{Re}\,x>\mathrm{Im}\,x\ ,\quad \mathrm{Re}\,x>0\ ,\\ D_3&:\ -\mathrm{Re}\,x>\mathrm{Im}\,x\ ,\quad \mathrm{Re}\,x<0\end{aligned}$$

or of the form

(8.2 bis) $$\begin{aligned} D_1&:\ |\mathrm{Re}\,x|\leqq -\mathrm{Im}\,x\ ,\quad \mathrm{Im}\,x<0\ ,\\ D_2&:\ \mathrm{Re}\,x>-\mathrm{Im}\,x\ ,\quad \mathrm{Re}\,x>0\ ,\\ D_3&:\ -\mathrm{Re}\,x>-\mathrm{Im}\,x\ ,\quad \mathrm{Re}\,x<0\ .\end{aligned}$$

We discuss only the former case. Because, if we notice that the domain (8.1 bis) is the symmetric image of the domain (8.1) with respect to the real axis, the latter will be treated in quite a similar way.

1°. We consider the case when $\alpha\geqq 0$ and $\beta>0$. Then we define the positive number $\kappa$ by

(8.3) $$\kappa=4\nu\beta^{-1}+\sqrt{2}\,a_0(\sin\varepsilon)^{-1}\ .$$

Our curve $\Gamma_{x_0}$ consists generally of two parts $\Gamma'$ and $\Gamma''$ (See Figure 2).

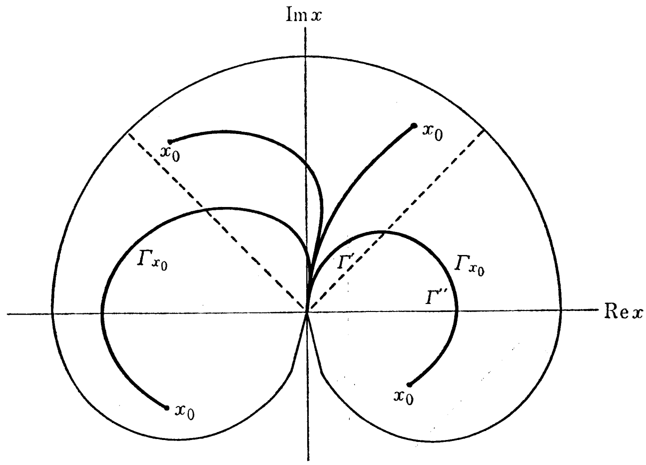


FIGURE 2

(I) The case $x_0 \in D_1$. Let $x_0 = A_0 + iB_0$ ($i = \sqrt{-1}$). Then, $|A_0| \leqq B_0$, $B_0 > 0$. The curve $\Gamma_{x_0}$ consists of a part $\Gamma'$ only. The variable point $x = \gamma(\sigma, x_0)$ on $\Gamma'$ is expressed by

$$\gamma(\sigma, x_0)^{-1} = \sigma + A - iBe^{i\sigma}\,, \qquad 0 \leqq \sigma < \infty\,, \tag{8.4}$$

if we put

$$A = A_0/(A_0^2 + B_0^2)\,, \quad B = B_0/(A_0^2 + B_0^2)\,. \tag{8.5}$$

(II) The case $x_0 \in D_2$. Let $x_0 = A_0 + iB_0$. Then, $A_0 > B_0$ and $A_0 > 0$. The curve $\Gamma_{x_0}$ consists of two parts $\Gamma'$ and $\Gamma''$. The variable point $x$ on $\Gamma''$ is expressed in the polar coordinates by

$$x = (|x_0| \cos \tau/\cos \theta)e^{i\tau}\,, \qquad \theta \leqq \tau \leqq \pi/4\,, \tag{8.6}$$

where $\tau = \arg x$, $\theta = \arg x_0$. The end point of the curve $\Gamma'$ is $|x_0|(2\cos\theta)^{-1}(1+i)$ ($i = \sqrt{-1}$). Then we switch to a curve $\Gamma'$ of the form (8.4) with $A = B = |x_0|^{-1}\cos\theta > 0$ since $A_0 = B_0 = |x_0|(2\cos\theta)^{-1}$.

(III) The case $x_0 \in D_3$. Let $x_0 = A_0 + iB_0$. Then, $-A_0 > B_0$ and $A_0 < 0$. The curve $\Gamma_{x_0}$ is made of two parts $\Gamma'$ and $\Gamma''$. The variable point $x$ on $\Gamma''$ is expressible as

$$x = (|x_0| \cos \tau/\cos \theta)e^{i\tau} \qquad 3\pi/4 \leqq \tau \leqq \theta\,. \tag{8.7}$$

At the end point of the curve $\Gamma''$, namely $|x_0|(2\cos\theta)^{-1}(1-i)$ ($i = \sqrt{-1}$), we switch to a curve $\Gamma'$ of the form (8.4) with $A = -B = |x_0|^{-1}\cos\theta < 0$ since we have $A_0 = -B_0 = |x_0|(2\cos\theta)^{-1}$.

The curve $\Gamma_{x_0}$ furnishes the following properties stated in the proposition below.

PROPOSITION A. (i) *The curve $\Gamma_{x_0}$ is contained in the domain* (8.1) *as long as its start point $x_0$ is.*

(ii) *The part of this curve for $\sigma > \max\{0, -A\}$ is in the open first quadrant and is tangent to the imaginary axis at the origin.*

(iii) *This curve never intersects itself and, for any two points $x_0$ and $x_1$, the curves $\Gamma_{x_0}$ and $\Gamma_{x_1}$ cannot have common points unless one of them is a subset of the other as point sets.*

(iv) *When $x$ travels on the part $\Gamma'$, $|x|$ is a decreasing function in the parameter $\sigma$.*

Let $s$ be the arc length of the curve $\Gamma_{x_0}$ measured from the origin to the variable point $x$. When $x$ is on $\Gamma''$, $s$ is considered as a decreasing function or an increasing function in the argument $\tau = \arg x$ according as we have $x_0 \in D_2$ or $x_0 \in D_3$. Hence, $ds = \mp\{(d|x|/d\tau)^2 + |x|^2\}^{1/2}d\tau = \mp\{(|x_0|\sin\tau/\cos\theta)^2 + (|x_0|\cos\tau/\cos\theta)^2\}^{1/2}d\tau = \mp|x_0||\cos\theta|^{-1}d\tau = -|x_0|(\cos\theta)^{-1}d\tau$. Thus we have

$$ds = -|x_0|(\cos\theta)^{-1}d\tau \quad \text{on} \quad \Gamma'' . \tag{8.8}$$

By differentiating (8.4), we have

$$dx/d\sigma = -x^2(1 - i\kappa Be^{\kappa\sigma}) \quad \text{on} \quad \Gamma' . \tag{8.9}$$

Since $s$ is a decreasing function in the parameter $\sigma$, we have $ds = -|dx/d\sigma|d\sigma$ and hence

$$ds = -|x|^2\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{1/2}d\sigma \quad \text{on} \quad \Gamma' . \tag{8.10}$$

The growth order of the general solution of (2.3) along the curve $\Gamma_{x_0}$ will be clarified in the proposition below.

PROPOSITION B. (i) *The functions $U(x)$ and $V(x)$ with the initial condition* (7.5) *satisfy the inequalities*

$$|U(x)| < b^0\chi_\alpha(\arg x) , \quad |V(x)| < b^0\chi_\beta(\arg x) \quad \textit{on} \quad \Gamma_{x_0} . \tag{8.11}$$

(ii) *In particular, as $x$ is on the curve $\Gamma'$, we have*

$$|U(x)|^{-1}d|U(x)|/ds \geqq \mu|x|^{-2}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} , \tag{8.12}$$

$$|V(x)|^{-1}d|V(x)|/ds \geqq \nu|x|^{-2}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} . \tag{8.13}$$

(iii) *We have moreover inequalities of the form*

(8.14) $$|U(x)| \leqq |u^0|(\sin \varepsilon)^{-\alpha}, \quad |V(x)| \leqq |v^0|(\sin \varepsilon)^{-\beta} \quad on \quad \Gamma_{x_0}.$$

The inequalities (8.11) show that, as $x$ travels from the start point $x_0$ to the origin along the curve $\Gamma_{x_0}$, the values of $x$, $U(x)$ and $V(x)$ continue to remain in the domain (7.1) provided their initial values $x_0$, $u^0$ and $v^0$ belong to the same one. For this reason, a domain of the form (7.1) will be usually called a *stable domain for the reduced equations* (2.3). Such a stability condition will be necessary when we want to prove the existence of solutions by means of the fixed point technique. The inequalities (8.12) and (8.13) mean that, on the curve $\Gamma'$, the functions $|U(x)|$ and $|V(x)|$ are monotone decreasing functions in the parameter $\sigma$ associated with the curve $\Gamma'$.

2°. In the case when $\alpha > 0$ and $\beta = 0$, we define the positive number $\kappa$ by

(8.15) $$\kappa = 4\mu\alpha^{-1} + \sqrt{2}\, a^0(\sin \varepsilon)^{-1}.$$

A curve $\Gamma'$ is to be defined by

(8.16) $$\gamma(\sigma, x_0)^{-1} = -\sigma + A - iBe^{\kappa\sigma}, \qquad 0 \leqq \sigma < \infty.$$

Then if $\sigma > \max\{0, A\}$, the part of the curve $\Gamma'$ is in the open second quadrant and is tangent to the imaginary axis at the origin. A curve $\Gamma''$ will be defined in the same way as in 1°. Thus we have a similar curve $\Gamma_{x_0}$ as in 1°. So we do not go into any details in this case.

**9. Existence of a fixed point.** 1°. First of all we must prove the following assertion.

(a) *The mapping $\mathfrak{T}$ is well defined.*

To prove this, one notes that, if the positive constants $a^0$ and $b^0$ are chosen appropriately, the domain (7.1) becomes a subset of the domain in the $(x, u, v)$-space which is expressed by the $(x, u, v)$-component of the domain (5.7) in the $(x, u, v, Y, Z)$-space. Suppose that the constants $K_N$ and $b^0$ satisfy

(9.1) $$K_N(b^0)^{(\nu+\mu)N+2} < b_N, \quad K_N(b^0)^{(\nu+\mu)N+1} < 1.$$

By virtue of Proposition B, the values of $x$, $U(x)$, $V(x)$ belong to the domain (7.1) as $x$ moves on the curve $\Gamma_{x_0}$. Hence, the functions $\phi(x, U(x), V(x))$ and $\psi(x, U(x), V(x))$ become holomorphic functions of $x$ on the curve $\Gamma_{x_0}$. Therefore, owing to (9.1), the values of $x$, $U(x)$, $V(x)$, $U(x)\phi(x, U(x), V(x))$ and $V(x)\psi(x, U(x), V(x))$ remain in the domain (5.7) as $x$ travels on $\Gamma_{x_0}$. This fact shows that the integrands $\Xi(x)$ and $\Pi(x)$ are holomorphic functions of $x$ on the curve $\Gamma_{x_0}$.

We shall show that the integrals (7.6) are convergent. Note that, in view of (5.9), we have, for example,

$$
\begin{aligned}
(9.2)\quad |\Xi(x)| &\leqq |x|^{-1}|U|^{-1}\{M(|U|+|V|)(2K_N|U|^{\nu N+1}|V|^{\mu N+1}) \\
&\quad + M(2K_N^2)|U|^{2\nu N+2}|V|^{2\mu N+2} + L_N|U|^{\nu N+1}|V|^{\mu N+1}\} \\
&= \{2M(|U|+|V|+K_N|U|^{\nu N+1}|V|^{\mu N+1})K_N + L_N\}|x|^{-1}|U|^{\nu N}|V|^{\mu N+1} \\
&\leqq \{2M(2b^0 + K_N(b^0)^{(\nu+\mu)N+2})K_N + L_N\}|x|^{-1}|U|^{\nu N}|V|^{\mu N+1} \\
&\leqq ((6Mb^0)K_N + L_N)|x|^{-1}|U(x)|^{\nu N}|V(x)|^{\mu N+1} \qquad (\text{by } (9.1)) .
\end{aligned}
$$

In quite a similar way, we can derive an inequality of the form

$$|\Pi(x)| \leqq ((6Mb^0)K_N + L_N)|x|^{-1}|U(x)|^{\nu N+1}|V(x)|^{\mu N} .$$

However, thanks to (8.10), the inequalities (8.12) and (8.13) give $-|U|^{-1}d|U|/d\sigma \geqq \mu$, $-|V|^{-1}d|V|/d\sigma \geqq \nu$ on $\Gamma'$, which implies that $|U(x)| = O(e^{-\mu\sigma})$, $|V(x)| = O(e^{-\nu\sigma})$ on $\Gamma'$. And, (8.4) and (8.10) imply that we have $|x|^{-1}ds = O(1)d\sigma$ for large $\sigma$. Thus we have $|\Xi(x)|ds = O(\exp(-(\mu\nu N + \nu + \nu\mu N)\sigma))d\sigma$ for large $\sigma$. Hence the first integral of (7.6) is convergent.
Analogously we can prove the convergence of the second.

2°. We shall prove the following assertion.

(b) $\mathfrak{T}$ *maps* $\mathfrak{F}$ *into itself,*

or what is the same thing,

(b.1) *The functions* $\Phi(x, u, v)$ *and* $\Psi(x, u, v)$ *satisfy the inequality*

$$(9.3)\qquad \max\{|v|^{-1}|\Phi(x, u, v)|, |u|^{-1}|\Psi(x, u, v)|\} \leqq K_N|u^\nu v^\mu|^N .$$

(b.2) *These functions are holomorphic in* $(x, u, v)$ *in the domain* (7.1).

Proof of Assertion (b.1). It is sufficient to prove that

$$
(9.4)\qquad
\begin{aligned}
|\Phi(x_0, u^0, v^0)| &\leqq K_N(|u^0|^\nu|v^0|^\mu)^N|v^0| , \\
|\Psi(x_0, u^0, v^0)| &\leqq K_N(|u^0|^\nu|v^0|^\mu)^N|u^0| ,
\end{aligned}
$$

because the point $(x_0, u^0, v^0)$ was arbitrarily chosen from the domain (7.1).
In view of the inequality (9.2), it will be sufficient to prove that

$$(9.5)\qquad ((6Mb^0)K_N + L_N)\int_{\Gamma x_0}|x|^{-1}|U(x)|^{\nu N}|V(x)|^{\mu N+1}ds \leqq K_N|u^0|^{\nu N}|v^0|^{\mu N+1} .$$

To estimate this integral, we estimate two integrals

$$(9.6)\qquad \int_{\Gamma''}|x|^{-1}|U(x)|^{\nu N}|V(x)|^{\mu N+1}ds ,$$

$$(9.7)\qquad \int_{\Gamma'}|x|^{-1}|U(x)|^{\nu N}|V(x)|^{\mu N+1}ds .$$

By virtue of (8.14) in Proposition B, we get two types of estimation

$$(9.8)\qquad |U(x)|^{\nu N}|V(x)|^{\mu N+1} \leqq \begin{cases} |u^0|^{\nu N}|v^0|^{\mu N+1}(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\beta} & \text{on } \Gamma'' , \\ |v^0|(\sin\varepsilon)^{-\beta}|U(x)|^{\nu N}|V(x)|^{\mu N} & \text{on } \Gamma' . \end{cases}$$

Since

$$\begin{aligned} x^2(d/dx)(U(x)^\nu V(x)^\mu) &= (\nu(\mu+\alpha x)+\mu(-\nu+\beta x))U(x)^\nu V(x)^\mu \\ &= (\alpha\nu+\beta\mu)xU(x)^\nu V(x)^\mu , \end{aligned}$$

we have, by solving the linear equation,

$$(9.9)\qquad U(x)^\nu V(x)^\mu = (u^0)^\nu(v^0)^\mu(x/x_0)^{\alpha\nu+\beta\mu} .$$

On the curve $\Gamma''$, by virtue of (8.8), we have

$$|x|^{-1}ds = -(|x_0|^{-1}\cos\theta/\cos\tau)(|x_0|/\cos\theta)d\tau = -(\cos\tau)^{-1}d\tau .$$

When $x$ is on $\Gamma''$, $\tau=\arg x$ belongs to a $\tau$-interval $[\arg x_0, \pi/4]$ or $[3\pi/4, \arg x_0]$. Hence, we have

$$\int_{\Gamma''}|x|^{-1}ds = \int_{\pi/4}^{\theta} -\frac{d\tau}{\cos\tau} \quad\text{or}\quad = \int_{3\pi/4}^{\theta} -\frac{d\tau}{\cos\tau} \leqq \frac{3\pi}{4}\frac{1}{\sin\varepsilon} .$$

Thus the integral (9.6) is bounded by

$$(9.10)\qquad \int_{\Gamma''}|x|^{-1}|U|^{\nu N}|V|^{\mu N+1}ds \leqq (3\pi/4)(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\beta-1}|u^0|^{\nu N}|v^0|^{\mu N+1} .$$

On the curve $\Gamma'$, as is seen from Proposition B, the functions $|U(x)|$ and $|V(x)|$ are increasing functions in the arc length $s$. By (8.12) and (8.13), we see that the function $W(x)=(|U(x)|^\nu|V(x)|^\mu)^N$ satisfies the inequality $dW(x)/ds \geqq N(\nu\mu+\mu\nu)W(x)|x|^{-2}\{1+\kappa^2B^2e^{2\kappa\sigma}\}^{-1/2}$. We shall prove later that

$$(9.11)\qquad \kappa B \equiv \kappa B_0/(A_0^2+B_0^2) > 1 \quad\text{if}\quad |A_0| \leqq B_0 .$$

Hence we get $|x|^{-1}\{1+\kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} = \{(\sigma+A)^2+B^2e^{2\kappa\sigma}\}^{1/2}\{1+\kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} \geqq Be^{\kappa\sigma}\{2\kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} = (\sqrt{2}\,\kappa)^{-1}$. Thus we have $dW(x)/ds \geqq N(\nu\mu+\mu\nu)(\sqrt{2}\,\kappa)^{-1}|x|^{-1}W(x) \geqq (N\nu\mu/\kappa)|x|^{-1}W(x)$. Hence it turns out that the integral (9.7) is bounded by $\int_{\Gamma'} \leqq \int_{\Gamma'}|v^0|(\sin\varepsilon)^{-\beta}|x|^{-1}W(x)ds \leqq |v^0|(\sin\varepsilon)^{-\beta}\kappa(N\nu\mu)^{-1}W(x_1)$, where $x_1$ is the start point of the curve $\Gamma'$. In view of (9.9), we have $W(x_1)=|u^0|^{\nu N}|v^0|^{\mu N}|x_1/x_0|^{(\alpha\nu+\beta\mu)N}$. The point $x_1$ is given by

$$x_1 = \begin{cases} x_0 & \text{if}\quad x_0\in D_1 , \\ |x_0|(\sqrt{2}\cos\theta)^{-1}\exp(\pi i/4) & \text{if}\quad x_0\in D_2 , \\ |x_0|(\sqrt{2}\,|\cos\theta|)^{-1}\exp(3\pi i/4) & \text{if}\quad x_0\in D_3 . \end{cases}$$

Since $|\cos\theta| \geqq \sin\varepsilon$, it turns out that the value $W(x_1)$ is bounded by $W(x_1) \leqq |u^0|^{\nu N}|v^0|^{\mu N}(\sqrt{2}\,\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N} \leqq |u^0|^{\nu N}|v^0|^{\mu N}(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N}$. Thus we have the estimate

$$(9.12)\qquad \int_{\Gamma'} |x|^{-1}|U|^{\nu N}|V|^{\mu N+1}ds \leqq \kappa(N\nu\mu)^{-1}(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\beta}|u^0|^{\nu N}|v^0|^{\mu N+1}\,.$$

It follows from (9.10) and (9.12) that the inequality (9.5) certainly holds if we take $K_N$ in such a way, for example, that it satisfies the equation

$$(9.13)\qquad ((6Mb^0)K_N + L_N)(\kappa(N\nu\mu)^{-1} + (3\pi/4)(\sin\varepsilon)^{-1})(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\alpha-\beta} = K_N\,.$$

In quite a similar way, we can prove that the inequality

$$(9.14)\qquad ((6Mb^0)K_N + L_N)\int_{\Gamma x_0} |x|^{-1}|U(x)|^{\nu N+1}|V(x)|^{\mu N}ds \leqq K_N(|u^0|^{\nu}|v^0|^{\mu})^N|u^0|$$

is satisfied for the $K_N$ defined by the equation (9.13).

To complete the proof of Assertion (b.1), we shall prove the inequality (9.11). One observes that, since $A_0 + iB_0 \in D_1$, $A_0$ and $B_0$ satisfy the inequalities $|A_0| \leqq B_0 < \{(a^0/\sin\varepsilon)^2 - A_0^2\}^{1/2}$, $|A_0| < a^0(\sqrt{2}\,\sin\varepsilon)^{-1}$. However, since $B = B_0/(A_0^2 + B_0^2)$, we have $\partial B/\partial B_0 = (A_0^2 - B_0^2)/(A_0^2 + B_0^2)^2 \leqq 0$. Hence, $B > (a^0/\sin\varepsilon)^{-2}\{(a^0/\sin\varepsilon)^2 - A_0^2\}^{1/2} > (a^0/\sin\varepsilon)^{-2}\{2^{-1}(a^0/\sin\varepsilon)^2\}^{1/2} = (\sqrt{2}\,a^0)^{-1}\sin\varepsilon$. Thus the definition of $\kappa$ proves the inequality (9.11).

PROOF OF ASSERTION (b.2). The inequalities (9.4) show that, for each fixed $x_0$, the integrals (7.6) are uniformly convergent with respect to $(u^0, v^0)$. Hence the functions $\Phi(x_0, u, v)$ and $\Psi(x_0, u, v)$ are holomorphic with respect to $(u, v)$ at $(u^0, v^0)$. To prove that, for each fixed pair $(u^0, v^0)$, the functions $\Phi(x, u^0, v^0)$ and $\Psi(x, u^0, v^0)$ are holomorphic with respect to $x$ at $x = x_0$, it is sufficient to prove that, for $x_1$ sufficiently near $x_0$, we have, for example, the relation

$$(9.15)\qquad \Phi(x_0, u^0, v^0) = \int_{\Gamma x_1} \Xi(x)dx + \int_{x_1}^{x_0} \Xi(x)dx\,,$$

where the second integration must be carried out along the segment $\overline{x_1x_0}$. Let $t_0$ and $t_1$ be intersection points of the curves $\Gamma_{x_0}$ and $\Gamma_{x_1}$ with a circle $|x| = \rho$ of a sufficiently small radius $\rho$. Then the relation (9.15) will be established if we can prove that

$$(9.16)\qquad \left|\int_{t_0}^{t_1} \Xi(x)dx\right| \to 0 \quad \text{as} \quad \rho \to 0\,,$$

where the integration is to be carried out along an arc of the circle $|x| = \rho$. Since the arc length is less than $\rho\pi/4$, the integral appearing on the left hand side of (9.16) is bounded by $((6Mb^0)K_N +$

$L_N)\rho^{-1}|U(x)|^{\nu N}|V(x)|^{\mu N+1}(\pi\rho/4) = O(|U(x)|^{\nu N}|V(x)|^{\mu N+1}) = O(\exp(-(\mu\nu N+\nu+\nu\mu N)\sigma)) \to 0$ as $\rho \to 0$. Analogously, we can prove a relation of the form (9.15) for the function $\Psi(x_0, u^0, v^0)$. This proves the assertion (b.2).

3°. We have to prove the following assertion.

(c) *The mapping* $\mathfrak{T}$ *is continuous with respect to the topology of uniform convergence on compact subsets.*

To prove this assertion, let $\{\phi_l(x, u, v), \psi_l(x, u, v)\}$ be any sequence of the family $\mathfrak{F}$ which converges uniformly $\{\phi(x, u, v), \psi(x, u, v)\}$. Put

$$\mathfrak{T}\{\phi_l(x, u, v), \psi_l(x, u, v)\} = \{\Phi_l(x, u, v), \Psi_l(x, u, v)\},$$
$$\mathfrak{T}\{\phi(x, u, v), \psi(x, u, v)\} = \{\Phi(x, u, v), \Psi(x, u, v)\}.$$

Then, the continuity of $\mathfrak{T}$ follows from the assertion that we have uniformly $\lim_{l\to\infty}\{\Phi_l(x, u, v), \Psi_l(x, u, v)\} = \{\Phi(x, u, v), \Psi(x, u, v)\}$. If we use the Lipschitz inequality (5.10), this assertion can be easily verified. We would like to leave the proof to the reader.

Owing to a fixed point theorem in the functional space (see, for example, Hukuhara [2], pp. 14-15), there exist a pair of functions, say $\{\mathfrak{Y}_N(x, u, v), \mathfrak{Z}_N(x, u, v)\}$, which correspond to a fixed point of the mapping $\mathfrak{T}$. Then we can prove that

(d) *The pair* $\{\mathfrak{Y}_N(x, U(x), V(x)), \mathfrak{Z}_N(x, U(x), V(x))\}$ *is a solution of the equations* (5.13).

Since the proof of this assertion is easy, the author would like to leave the proof to the reader (See, Iwano [5]).

Thus we have obtained a proof of the fundamental lemma.

Finally we have to prove the uniqueness of our solutions.

(e) *A solution of the equations* (5.13) *such that*

$$\mathfrak{Y} = O(|U(x)|^{\nu N}|V(x)|^{\mu N+1}), \quad \mathfrak{Z} = O(|U(x)|^{\nu N+1}|V(x)|^{\mu N}) \tag{9.17}$$

*is unique.*

To prove this, we impose the condition

$$24Mb^0(\kappa(N\nu\mu)^{-1} + (3\pi/4)(\sin\varepsilon)^{-1})(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\alpha-\beta} < 1 \tag{9.18}$$

on the value of $b^0$, where $M$ is the constant appearing in (5.10). It is easy to see that this condition also enables us to solve the $K_N$ as a positive solution of the equation (9.13). Suppose that there exist two pairs of solutions for the equations (5.13) satisfying the order condition (9.17). Let $\{\mathfrak{Y}(x, U(x), V(x)), \mathfrak{Z}(x, U(x), V(x))\}$ be the difference between these two solutions. Then we have

(9.19) $\max\{|u||\mathfrak{Y}(x, u, v)|, |v||\mathfrak{Z}(x, u, v)|\} \leqq 2K_N|u|^{\nu N+1}|v|^{\mu N+1}$

for $(x, u, v)$ in the domain (7.1). On the other hand, in view of (5.10) with $Y_1 - Y_2 = U\mathfrak{Y}$ and $Z_1 - Z_2 = V\mathfrak{Z}$, we have the inequality, for example,

$$|\mathfrak{Y}(x_0, u^0, v^0)|$$
$$\leqq M\int_{\Gamma_{x_0}} (|U| + |V| + 4K_N|U|^{\nu N+1}|V|^{\mu N+1})(4K_N)|x|^{-1}|U|^{\nu N}|V|^{\mu N+1}ds\ ,$$

where the last integral is estimated as

$$\leqq M\int_{\Gamma_{x_0}} (2b^0 + 4K_N(b^0)^{(\nu+\mu)N+2})(4K_N)|x|^{-1}|U|^{\nu N}|V|^{\mu N+1}ds$$
$$\leqq (6Mb^0)(4K_N)\Big(\int_{\Gamma'} + \int_{\Gamma''}\Big)|x|^{-1}|U|^{\nu N}|V|^{\mu N+1}ds \quad \text{(by (9.1))}$$
$$\leqq (6Mb^0)(4K_N)(\kappa(N\nu\mu)^{-1} + (3\pi/4)(\sin\varepsilon)^{-1})(\sin\varepsilon)^{-(\alpha\nu+\beta\mu)N-\alpha-\beta}|u^0|^{\nu N}|v^0|^{\mu N+1} \quad \text{(by (9.10), (9.12))}$$
$$\leqq K_N|u^0|^{\nu N}|v^0|^{\mu N+1} \quad \text{(by (9.18))}\ .$$

Hence, the inequality (9.18) implies that $|\mathfrak{Y}(x_0, u^0, v^0)| < K_N|u^0|^{\nu N}|v^0|^{\mu N+1}$. Thus the constant $2K_N$ appearing in (9.19) has been halved. Applying this procedure repeatedly, we have an inequality of the form $|\mathfrak{Y}(x_0, u^0, v^0)| \leqq (K_N/2^l)|u^0|^{\nu N}|v^0|^{\mu N+1}$ for each integer $l$, which implies $\mathfrak{Y}(x, u, v) \equiv 0$, because $(x_0, u^0, v^0)$ is an arbitrary point of the domain (7.1). Similarly, we can prove that $\mathfrak{Z}(x, u, v) \equiv 0$. This completes the proof of the fundamental lemma conditional on Propositions A and B.

We want to show how to determine various constants in order to obtain a unique solution of the fundamental lemma for the domain (7.1). For any fixed positive integer $N$, there exist positive constants $(a_N, b_N, c_N)$ and $(M, L_N)$ which are associated with the domain (5.7) and the inequalities (5.9), (5.10). Take the value of $a^0$ small enough to have $a^0/\sin\varepsilon < a_N$. The positive constant $\kappa$ is to be defined by the formula (8.3). Choose the value of $b^0$ so that the inequalities $b^0 < b_N$ and (9.18) are both satisfied. Finally we determine the positive constant $K_N$ as a solution of the equation (9.13). Since $K_N$ is considered as an increasing function of $b^0$, if the value of $b^0$ is sufficiently small, the second inequality of (9.1) certainly holds together with $b^0 < b_N$. Then the first of (9.1) is automatically satisfied. Thus we have a unique solution of the fundamental lemma for the domain (7.1).

## Chapter IV. Verification of Propositions A and B.

**10. Verification of Proposition A.** Conditional on Propositions A and B which are the conditions imposed on our curve $\Gamma_{x_0}$, the fundamental lemma has been established. These two propositions will be proved in the rest of this paper (also, refer to Iwano [9]). But, since Proposition A is geometrically understandable except for Property (iv), we are satisfied with verifying that $|x|$ is a monotone decreasing function in the parameter $\sigma$.

Since the start point $A_0 + iB_0$ of the curve $\Gamma'$ always is in the set $D_1$, we have the inequalities

$$(10.1) \qquad |A_0| \leqq B_0 < \{(a^0/\sin\varepsilon)^2 - A_0^2\}^{1/2}\,, \quad |A_0| \leqq a^0/(\sqrt{2}\sin\varepsilon)\,.$$

On the curve $\Gamma'$, $|x|^{-2}$ is a function of $\sigma$, say $L(\sigma)$,

$$(10.2) \qquad L(\sigma) = (\sigma + A)^2 + B^2e^{2\kappa\sigma}\,.$$

Hence it is sufficient to prove that the derivative $L'(\sigma)$ is positive valued for $0 \leqq \sigma < \infty$.

Since $L'(\sigma) = 2(\sigma + A) + 2\kappa B^2e^{2\kappa\sigma}$, we have

$$\begin{aligned} L'(0) &= 2A + 2\kappa B^2 > 2A + 2B \quad &&\text{(by (9.11))} \\ &\geqq 0 &&\text{(by (10.1))}\,. \end{aligned}$$

Moreover, it is obvious that the second order derivative $L''(\sigma)$ satisfies

$$L''(\sigma) = 2 + 4\kappa^2B^2e^{2\kappa\sigma} > 0\,.$$

Hence, it follows that $L'(\sigma) > 0$, which is our required inequality.

**11. Verification of the inequalities** (8.12) **and** (8.13). In this section, we write $U$ and $V$ in place of $U(x)$ and $V(x)$. One notes that

$$\begin{aligned} (11.1) \qquad |U|^{-1}d|U|/ds &= (d/ds)\log|U| = (d/ds)\,\mathrm{Re}\,(\log U) \\ &= \mathrm{Re}\,\{(d/ds)\log U\} = \mathrm{Re}\,(U^{-1}dU/ds) \\ &= \mathrm{Re}\,(U^{-1}dU/ds\cdot dx/ds) = \mathrm{Re}\,\{(\mu + \alpha x)x^{-2}dx/ds\}\,. \end{aligned}$$

In view of (8.9) and (8.10), it is verified that

$$(11.2) \qquad x^{-2}dx/ds = x^{-2}dx/d\sigma\cdot d\sigma/ds = |x|^{-2}(1 - i\kappa Be^{\kappa\sigma})\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2}\,.$$

Hence, we have

$$\begin{aligned} (11.3) \qquad |U|^{-1}d|U|/ds &= |x|^{-2}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} \\ &\quad\times \mathrm{Re}\,[\{\mu + \alpha\{(\sigma + A)^2 + B^2e^{2\kappa\sigma}\}^{-1}(\sigma + A + iBe^{\kappa\sigma})\}(1 - i\kappa Be^{\kappa\sigma})] \\ &= \{(\kappa\alpha + \mu)B^2e^{2\kappa\sigma} + \mu(\sigma + A)^2 + \alpha(\sigma + A)\}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2}\,. \end{aligned}$$

If we can prove that the last expression is bounded from below by $\mu|x|^{-2}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2}$, namely the function $H(\sigma)$ defined by

$$(11.4) \qquad H(\sigma) = \kappa\alpha B^2e^{2\kappa\sigma} + \alpha(\sigma + A)$$

is positive valued for $0 \leqq \sigma < \infty$, the inequality (8.12) will be established. To prove this, we observe first that there holds a trivial inequality $H'(0) = 2\kappa^2\alpha B^2 e^{2\kappa\sigma} + \alpha \geqq 0$ for $0 \leqq \sigma < \infty$. On the other hand, we have

$$\begin{aligned} H(0) &= \kappa\alpha B^2 + \alpha A = \{\alpha/(A_0^2 + B_0^2)\}(\kappa B_0^2/(A_0^2 + B_0^2) + A_0) \\ &\geqq \{\alpha/(A_0^2 + B_0^2)\}\{\kappa/2 - a^0/(\sqrt{2}\sin\varepsilon)\} \quad \text{(by (10.1))} \\ &= \{\alpha/(A_0^2 + B_0^2)\}(2\nu/\beta) \quad \text{(by (8.3))} . \end{aligned}$$

This implies that we have $H(\sigma) \geqq 0$ for $0 \leqq \sigma < \infty$. Thus the inequality (8.12) holds.

By a similar consideration, we can derive the relation

$$\begin{aligned} \text{(11.5)} \qquad & |V|^{-1}d|V|/ds \\ &= \{(\kappa\beta - \nu)B^2e^{2\kappa\sigma} - \nu(\sigma + A)^2 + \beta(\sigma + A)\}\{1 + \kappa^2B^2e^{2\kappa\sigma}\}^{-1/2} . \end{aligned}$$

In order to have the inequality (8.13), it will be sufficient to prove that the function $K(\sigma)$ defined by

$$\text{(11.6)} \qquad K(\sigma) = (\kappa\beta - 2\nu)B^2e^{2\kappa\sigma} - 2\nu(\sigma + A)^2 + \beta(\sigma + A)$$

is positive valued for $0 \leqq \sigma < \infty$. We notice first that $K(0) \geqq 0$. Indeed,

$$\begin{aligned} K(0) &= (\kappa\beta - 2\nu)B^2 - 2\nu A^2 + \beta A \\ &= \kappa\beta B^2 - 2\nu/(A_0^2 + B_0^2) + \beta A \quad \text{(by (8.5))} \\ &= \{\beta/(A_0^2 + B_0^2)\}\{\kappa B_0^2/(A_0^2 + B_0^2) - 2\nu/\beta + A_0\} \\ &\geqq \{\beta/(A_0^2 + B_0^2)\}\{\kappa/2 - 2\nu/\beta - a^0/(\sqrt{2}\sin\varepsilon)\} \quad \text{(by (10.1))} \\ &= 0 \quad \text{(by (8.3))} . \end{aligned}$$

Since $K'(0) = 2\kappa(\kappa\beta - 2\nu)B^2e^{2\kappa\sigma} - 4\nu(\sigma + A) + \beta$, we have $K'(0) > 0$. Because,

$$\begin{aligned} K'(0) &= 2\kappa(\kappa\beta - 2\nu)B^2 - 4\nu A + \beta \\ &> 2(\kappa\beta - 2\nu)B - 4\nu A + \beta \quad \text{(by (9.11))} \\ &\geqq 2(\kappa\beta - 4\nu)B + \beta \quad \text{(by (10.1))} \\ &= 2\sqrt{2}\,\beta(a^0/\sin\varepsilon)B + \beta \quad \text{(by (8.3))} \\ &> 0 . \end{aligned}$$

Moreover, we have $K''(\sigma) = 4\kappa^2(\kappa\beta - 2\nu)B^2e^{2\kappa\sigma} - 4\nu > 0$. Indeed,

$$\begin{aligned} K''(\sigma) \geqq K''(0) &= 4\kappa^2(\kappa\beta - 2\nu)B^2 - 4\nu \\ &> 4(\kappa\beta - 2\nu) - 4\nu \quad \text{(by (9.11))} \\ &= 4(\kappa\beta - 3\nu) > 0 \quad \text{(by (8.3))} . \end{aligned}$$

Thus we have proved that $K'(\sigma) > 0$ holds for $0 \leqq \sigma < \infty$. Consequently, the inequality $K(\sigma) > 0$ is satisfied in the same $\sigma$-interval. This proves the inequality (8.13).

**12. Verification of the inequality** (8.11). The functions $U(x)$ and $V(x)$ are expressed by

$$(12.1)\qquad \begin{aligned} U(x) &= u^0(x/x_0)^\alpha \exp(\mu/x_0 - \mu/x)\ , \\ V(x) &= v^0(x/x_0)^\beta \exp(\nu/x - \nu/x_0)\ . \end{aligned}$$

Thanks to the inequalities (8.12) and (8.13), the functions $|U(x)|$ and $|V(x)|$ are monotone decreasing functions in the parameter $\sigma$. So, we have

$$(12.2)\qquad |U(x)| \leqq |U(x_1)|\ ,\quad |V(x)| \leqq |V(x_1)| \quad \text{on } \Gamma'\ ,$$

where $x_1$ denotes the start point of the curve $\Gamma'$.

First of all we will prove that

$$(12.3)\qquad \mathrm{Re}\,(1/x) = \mathrm{Re}\,(1/x_0) \quad \text{on} \quad \Gamma''\ .$$

Indeed, in view of (8.7) and (8.8), we have

$$\begin{aligned} x &= (|x_0| \cos\tau/\cos\theta)e^{i\tau} \quad \text{on} \quad \Gamma''\ , \\ ds &= -(|x_0|/\cos\theta)d\tau \quad \text{on} \quad \Gamma''\ . \end{aligned}$$

A direct calculation gives $dx/d\tau = (|x_0|/\cos\theta)(-\sin\tau + i\cos\tau)e^{i\tau} = (|x_0|/\cos\theta)(ie^{i\tau})e^{i\tau} = (|x_0|/\cos\theta)ie^{i2\tau}$. Hence, $(d/ds)\,\mathrm{Re}\,x^{-1} = \mathrm{Re}\,(dx^{-1}/ds) = -\mathrm{Re}\,(x^{-2}dx/ds) = -\mathrm{Re}\,(x^{-2}dx/d\tau \cdot d\tau/ds) = -\mathrm{Re}\,\{(\cos\theta/(|x_0|\cos\tau))^2 e^{-i2\tau} \cdot (|x_0|/\cos\theta)ie^{i2\tau} \cdot (-\cos\theta/|x_0|)\} = 0$. This proves that the function $\mathrm{Re}\,(1/x)$ is constant as $x$ travels on the curve $\Gamma''$ and consequently the assertion (12.3) holds.

We consider the case when $x_0 \in D_1$. By virtue of Property (ii) in Proposition B, we have, for example, $|U(x)| \leqq |U(x_0)| = |u^0| < b^0\chi_\alpha(\arg x_0)$ on $\Gamma'$. But, by the definition of $\chi_\alpha(\arg x)$, we have $\chi_\alpha(\arg x) \equiv 1$ constantly when $x \in \Gamma'$. Thus we have

$$(12.4)\qquad |U(x)| < b^0\chi_\alpha(\arg x) \quad \text{on} \quad \Gamma'\ .$$

This proves that the first inequality in (8.11) holds on the curve $\Gamma'$. In quite a similar way we can prove that the second one in (8.11) does on the same curve.

We consider the case when $x_0 \in D_2$. By virtue of (12.3), we have, for example,

$$(12.5)\qquad \begin{aligned} |U(x)| &< |u^0||x/x_0|^\alpha \quad \text{on} \quad \Gamma'' \\ &= |u^0|(\cos\tau/\cos\theta)^\alpha \quad \theta \leqq \tau \leqq \pi/4\ . \end{aligned}$$

But,

$$(12.6)\quad |u^0| < b^0\chi_\alpha(\arg x_0) = \begin{cases} b^0 & \text{if} \quad |\theta - \pi/2| \leqq \pi/2\,, \\ b^0|\cos\theta|^\alpha & \text{if} \quad \pi/2 < |\theta - \pi/2| < \pi - \varepsilon\,. \end{cases}$$

Suppose that $0 \leqq \theta < \pi/4$. Then, $|U(x)|$ satisfies the inequality $|U(x)| < b^0(\cos\tau/\cos\theta)^\alpha$ for a $\tau$-interval $[\theta, \pi/4]$. Since $x$ is on $\Gamma''$, we have $0 \leqq \tau < \pi/4$, and

$$(12.7)\qquad 0 < \cos\tau/\cos\theta \leqq 1$$

for the $\tau$-interval $[\theta, \pi/4]$. Hence, it holds that $|U(x)| < b^0\chi_\alpha(\arg x)$ on $\Gamma''$, because, in the $\tau$-interval under consideration, we have $\chi_\alpha(\arg x) \equiv 1$ identically.

Suppose that $-\pi/2 + \varepsilon < \theta < 0$. Then, owing to (12.5) and (12.6), we have the inequality $|U(x)| < b^0|\cos\tau|^\alpha$ for $\theta \leqq \tau \leqq \pi/4$. Thus, by the definition of $\chi_\alpha(\arg x)$, we have $|U(x)| < b^0\chi_\alpha(\arg x)$ on $\Gamma''$.

In any case, we obtain $|U(x)| < b^0\chi_\alpha(\arg x)$ on $\Gamma''$. At the end point of $\Gamma''$, say $x_1$, we have $|U(x_1)| < b^0\chi_\alpha(\arg x_1) = b^0$. Hence, by virtue of (12.2), it follows that $|U(x)| < b^0 = b^0\chi_\alpha(\arg x)$ on $\Gamma'$, because $\chi_\alpha(\arg x) \equiv 1$ for $x \in \Gamma'$. Thus we have proved that $|U(x)| < b^0\chi_\alpha(\arg x)$ on $\Gamma_{x_0}$. Similarly, we can prove that $|V(x)| < b^0\chi_\beta(\arg x)$ on $\Gamma_{x_0}$.

The case when $x_0 \in D_3$ can be treated in almost the same way. Thus the inequality (8.11) has now been verified.

**13. Verification of the inequalities** (8.14). By virtue of (12.1) and (12.3), we have $|U(x)| \leqq |u^0||x/x_0|^\alpha$, $|V(x)| \leqq |v^0||x/x_0|^\beta$ on $\Gamma''$. The relations (8.6) and (8.7) imply that $|x/x_0| \leqq (\sin\varepsilon)^{-1}$. Hence, we have $|U(x)| \leqq |u^0|(\sin\varepsilon)^{-\alpha}$, $|V(x)| \leqq |v^0|(\sin\varepsilon)^{-\beta}$ on $\Gamma''$. But, the inequalities (8.12) and (8.13) show that $|U(x)| \leqq |U(x_1)|$, $|V(x)| \leqq |V(x_1)|$, where $x_1$ is the start point of the curve $\Gamma'$ or the end point of the curve $\Gamma''$. Thus we have the inequalities

$$|U(x)| \leqq \max\{|u^0|, |u^0|(\sin\varepsilon)^{-\alpha}\} = |u^0|(\sin\varepsilon)^{-\alpha} \quad \text{on} \quad \Gamma_{x_0}\,,$$
$$|V(x)| \leqq |v^0|(\sin\varepsilon)^{-\beta} \quad \text{on} \quad \Gamma_{x_0}\,,$$

because, $\varepsilon$ is supposed to be sufficiently small. Thus we have the inequalities (8.14).

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DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
TOKYO, 158 JAPAN