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A CHARACTERIZATION OF DECOMPOSABLE OPERATORS

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Abstract. An operator T means a bounded linear transformation on a complex Banach space X. For an operator T and for a closed subset F of the complex plane C, we let $X_T(F)=\{x\in X$: there exists an analytic function $f|C\setminus F\to X$ such that $(z-T)f(z)\equiv x\}$, and if E is an arbitrary subset of C, we let $X_T(E)=\bigcup\{X_T(F): F\subset E$ and F is closed$\}$. If $X_T(F)$ is closed for all closed subsets F of C, we say that T satisfies the closure condition (C). In this paper, we show that an operator T is decomposable if and only if (1) T satisfies the closure condition (C) and (2) $X_T(G_1\cup G_2)=X_T(G_1)+X_T(G_2)$ for any pair of open subsets G_1 and G_2 of C. This is a generalization of Plafker's result in [5] for strongly decomposable operators. And we show some applications of this result.

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1. Preliminaries. For an operator T, we denote by $\sigma(T)$ its spectrum and by $\rho(T)$ its resolvent set. $\mathrm{Lat}(T)$ is the lattice of all invariant subspaces of T, and $T|Y$ denotes the restriction of T to $Y\in\mathrm{Lat}(T)$. An invariant subspace Y of T is called a spectral maximal space of T if $Z\in\mathrm{Lat}(T)$ and $\sigma(T|Z)\subset\sigma(T|Y)$ imply $Z\subset Y$. We denote by $\mathrm{SM}(T)$ the family of all spectral maximal spaces of T.

For $n\geqq 2$, an operator T is called strongly n-decomposable (resp. strongly n-quasidecomposable) if for any open covering $\{G_1,\cdots,G_n\}$ of $\sigma(T)$ there exists a system $\{Y_1,\cdots,Y_n\}\subset\mathrm{SM}(T)$ such that

(1) $Y=Y\cap Y_1+\cdots+Y\cap Y_n$ (resp. (1)′ Y is the closure of $\{Y\cap Y_1+\cdots+Y\cap Y_n\}$ and $T|Y$ satisfies the closure condition (C)) for all $Y\in\mathrm{SM}(T)$ and that

(2) $\sigma(T|Y_i)\subset G_i$ for every $i=1,\cdots,n$. An operator T is called n-decomposable (resp. n-quasidecomposable) if we postulate (1) (resp. (1)′) only for $Y=X$. An operator T is called strongly decomposable (resp. decomposable) if T is strongly n-decomposable (resp. n-decomposable) for all $n\geqq 2$. An operator T is called strongly quasidecomposable (resp- quasidecomposable) if T is strongly n-quasidecomposable (resp. n-quasi-decomposable) for all $n\geqq 2$.

An operator T is said to have the single-valued extension property or the property (A) if there exists no non-zero X-valued analytic function f such that $(z - T)f(z) \equiv 0$. If an operator T has the property (A), then for any fixed $x \in X$ there exists a maximal analytic extension $f_x(z)$ of $(z - T)^{-1}x$ such that $(z - T)f_x(z) \equiv x$. We denote by $\rho_T(x)$ the domain of $f_x(z)$ and $\sigma_T(x) = \boldsymbol{C}\setminus\rho_T(x)$. If an operator T has the property (A), then it is easy to show that $X_T(E) = \{x \in X: \sigma_T(x) \subset E)$ for an arbitrary subset E of $\boldsymbol{C}$.

2. **Main results.** The proof of the following lemma is a modification of Radjabalipour [6, Theorem 1], and this plays an important role in our discussions.

LEMMA 1. *If an operator T is 2-decomposable, then $X_T(F) \subset X_T(\bar{G}_1) + X_T(\bar{G}_2)$ for any closed subset F of $\boldsymbol{C}$ and for any open covering $\{G_1, G_2\}$ of F, where $\bar{G}$ denotes the closure of G in the complex plane $\boldsymbol{C}$.*

PROOF. Let D_1 and D_2 be open subsets such that $\bar{D}_i \subset G_i$, $i = 1, 2$. Then $\{G_1 \cap G_2, \boldsymbol{C}\setminus(D_1 \cap D_2)^-\}$ is an open covering of $\sigma(T)$, where $(D_1 \cap D_2)^-$ denotes the closure of $D_1 \cap D_2$. Hence there exist Y_1 and Y_2 in SM(T) such that (1) $X = Y_1 + Y_2$ and that (2) $\sigma(T|Y_1) \subset G_1 \cap G_2$, $\sigma(T|Y_2) \subset \boldsymbol{C}\setminus(D_1 \cap D_2)^-$. Hence we have $Y_1 \subset X_T(\sigma(T|Y_1)) \subset X_T(J)$, where $J = (G_1 \cap G_2)^-$ and $D_1 \cap D_2 \subset \rho(T|Y_2)$. Since T is 2-decomposable, T has the property (A) and satisfies the closure condition (C) and $X_T(J) \in$ SM(T) by [2, Chap. 2, Corollary 1.4 and Theorem 1.5]. For any $x \in X$, there exist $x_i \in Y_i$, $i = 1, 2$, such that $x = x_1 + x_2$ by (1). And since

$$x = x_1 + x_2 = x_1 + (z - T)(z - T|Y_2)^{-1}x_2$$

for all $z \in \rho(T|Y_2)$, we have

$$\hat{x} = \hat{x}_1 + \hat{x}_2 = \hat{x}_2 = (z - T^J)\{(z - T|Y_2)^{-1}x_2\}^\wedge$$

for all $z \in D_1 \cap D_2$, where $\hat{y}$ or $y^\wedge \in X/X_T(J)$ is the canonical image of $y \in X$ and T^J is the operator on $X/Y_T(J)$ induced by T. Since the canonical map $\hat{}$ is continuous, we have $\hat{x} \in X_{T^J}(\boldsymbol{C}\setminus(D_1 \cap D_2))$ for all $x \in X$. And since T^J has the property (A) by [3, Corollary], we have $\sigma(T^J) \subset \boldsymbol{C}\setminus(D_1 \cap D_2)$. Hence if $x \in X_T(F)$, then we have $\hat{x} \in X_{T^J}(F)$, and so

$$\hat{x} \in X_{T^J}(F \cap (\boldsymbol{C}\setminus(D_1 \cap D_2))) = X_{T^J}((F\setminus D_1) \cup (F\setminus D_2)) .$$

Since $F\setminus D_1$ and $F\setminus D_2$ are disjoint closed subsets, by Riesz's theorem we have $\hat{x} = \hat{y}_1 + \hat{y}_2$, where $\hat{y}_i \in X_{T^J}(F\setminus D_i)$, $i = 1, 2$. Hence we have $x = y_1 + y_2 + u$, where $u \in X_T(J)$ and $y_i \in X_T((F\setminus D_i) \cup (G_1 \cap G_2)^-) \subset X_T(\bar{G}_j)$ for $i \neq j$ by [6, Lemma 1]. Hence we have $x \in X_T(\bar{G}_1) + X_T(\bar{G}_2)$.

LEMMA 2. *For an operator T, the following assertions are equivalent.*

(1) $X_T(F) \subset X_T(\bar{G}_1) + X_T(\bar{G}_2)$ *for any closed subset F of $\boldsymbol{C}$ and for any open covering $\{G_1, G_2\}$ of F.*

(2) $X_T(G_1 \cup G_2) = X_T(G_1) + X_T(G_2)$ *for any pair of open subsets G_1 and G_2 of $\boldsymbol{C}$.*

PROOF. The implication (2) $\Rightarrow$ (1) is obvious by $X_T(F) \subset X_T(G_1 \cup G_2) = X_T(G_1) + X_T(G_2) \subset X_T(\bar{G}_1) + X_T(\bar{G}_2)$. To prove the implication (1) $\Rightarrow$ (2), let G_1 and G_2 be and open subsets of $\boldsymbol{C}$. Then for any closed subset $F \subset G_1 \cup G_2$, we can choose open subsets D_1 and D_2 such that $\bar{D}_i \subset G_i$, $i = 1, 2$, and that $F \subset D_1 \cup D_2$. Hence by (1), we have $X_T(F) \subset X_T(\bar{D}_1) + X_T(\bar{D}_2)$, and so $X_T(F) \subset X_T(G_1) + X_T(G_2)$. Since F is any closed subset contained in $G_1 \cup G_2$, we have $X_T(G_1 \cup G_2) \subset X_T(G_1) + X_T(G_2)$. Conversely since $X_T(G_1 \cup G_2) \supset X_T(G_1), X_T(G_2)$ is clear, we have $X_T(G_1 \cup G_2) \supset X_T(G_1) + X_T(G_2)$.

REMARK 1. We can prove that the following assertions are equivalent for an operator T by the same arguement as above.

(1) $X_T(F)$ is contained in the closure of $\{X_T(\bar{G}_1) + X_T(\bar{G}_2)\}$ for any closed subset F of $\boldsymbol{C}$ and for any open covering $\{G_1, G_2\}$ of F.

(2) The closure of $X_T(G_1 \cup G_2)$ equals the closure of $\{X_T(G_1) + X_T(G_2)\}$ for any pair of open subsets G_1 and G_2 of $\boldsymbol{C}$.

THEOREM 1 (cf. [6]). *For an operator T, the following assertions are equivalent.*

(1) *T is 2-decomposable.*

(2) *T satisfies the closure condition* (C) *and $X_T(G_1 \cup G_2) = X_T(G_1) + X_T(G_2)$ for any pair of open subsets G_1 and G_2 of $\boldsymbol{C}$.*

(3) *T is decomposable.*

PROOF. The implication (1) $\Rightarrow$ (2) is clear by Lemmas 1 and 2. To prove the implication (2) $\Rightarrow$ (3), for any $n \geqq 2$, let $\{G_1, \cdots, G_n\}$ be any open covering of $\sigma(T)$. Then we can choose open subsets D_i such that $\bar{D}_i \subset G_i$, $i = 1, \cdots, n$, and that $\sigma(T) \subset D_1 \cup \cdots \cup D_n$. Hence by (2), we have $X = X_T(\sigma(T)) \subset X_T(D_1 \cup \cdots \cup D_n) = X_T(D_1) + \cdots + X_T(D_n) \subset X_T(\bar{D}_1) + \cdots + X_T(\bar{D}_n)$, and so $X = X_T(\bar{D}_1) + \cdots + X_T(\bar{D}_n)$. Since T satisfies the closure condition (C), T has the property (A) by [7, Theorem 2.13]. Hence we have $X_T(\bar{D}_i) \in \mathrm{SM}(T)$ and $\sigma(T \mid X_T(\bar{D}_i)) \subset \bar{D}_i \subset G_i$ for every $i = 1, \cdots, n$ by [2, Chap. 1, Proposition 3.8]. Hence T is decomposable. The implication (3) $\Rightarrow$ (1) is clear.

Since an operator T is strongly n-decomposable if and only if $T \mid Y$ is n-decomposable for all $Y \in \mathrm{SM}(T)$ (see [1, Theorem 1.7]), we have the following:

COROLLARY (cf. Plafker [5]). *For an operator T, the following assertions are equivalent.*

(1) *T is strongly 2-decomposable.*

(2) *$T|Y$ satisfies the closure condition* (C) *and $X_{T|Y}(G_1 \cup G_2) = X_{T|Y}(G_1) + X_{T|Y}(G_2)$ for any $Y \in \mathrm{SM}(T)$ and for any pair of open subsets G_1 and G_2 of $\boldsymbol{C}$.*

(3) *T is strongly decomposable.*

THEOREM 2 (cf. Jafarian [4, Corollary 8.4]). *For an operator T, the following assertions are equivalent.*

(1) *T is strongly 2-quasidecomposable.*

(2) *$T|Y$ satisfies the closure condition* (C) *and the closure of $X_{T|Y}(G_1 \cup G_2)$ equals the closure of $\{X_{T|Y}(G_1) + X_{T|Y}(G_2)\}$ for any $Y \in \mathrm{SM}(T)$ and for any pair of open subsets G_1 and G_2 of $\boldsymbol{C}$.*

(3) *T is strongly quasidecomposable.*

PROOF. To prove the implication (1) $\Rightarrow$ (2), we show first that $X_T(F)$ is contained in the closure of $\{X_T(\bar{G}_1) + X_T(\bar{G}_2)\}$ for any closed subset F of $\boldsymbol{C}$ and for any open covering $\{G_1, G_2\}$ of F. Since T has the closure condition (C), we have that $X_T(F)$ is closed and belongs to $\mathrm{SM}(T)$. Let D_1 and D_2 be open subsets such that $\bar{D}_i \subset G_i$, $i = 1, 2$, and that $F \subset D_1 \cup D_2$. Then $\{G_1 \cap G_2, \boldsymbol{C} \diagdown (D_1 \cap D_2)^-\}$ is an open covering of $\sigma(T)$, and so there exist Y_1 and Y_2 in $\mathrm{SM}(T)$ such that (1) $X_T(F)$ is the closure of $\{X_T(F) \cap Y_1 + X_T(F) \cap Y_2\}$ and that (2) $\sigma(T|Y_1) \subset G_1 \cap G_2$, $\sigma(T|Y_2) \subset \boldsymbol{C} \diagdown (D_1 \cap D_2)^-$. Then for any $x \in X_T(F)$, there exist $x_1^n \in X_T(F) \cap Y_1$ and $x_2^n \in X_T(F) \cap Y_2$, for every n, such that $x_n \equiv x_1^n + x_2^n \to x$ as $n \to \infty$. Then by the same arguement as in the proof of Lemma 1, we have $\hat{x}_n \in X_{T^J}(F \cap (\boldsymbol{C} \diagdown (D_1 \cap D_2)))$ and so $x_n \in X_T(\bar{G}_1) + X_T(\bar{G}_2)$. Hence x belongs to the closure of $\{X_T(\bar{G}_1) + X_T(\bar{G}_2)\}$. Since if an operator T is strongly n-quasidecomposable, then so is $T|Y$ for all $Y \in \mathrm{SM}(T)$ (see [4, Proposition 5.2]), the implication (1) $\Rightarrow$ (2) is proved by Remark 1.

It is easy to show that (2) implies that the closure of $X_{T|Y}(G_1 \cup \cdots \cup G_n)$ equals the closure of $\{X_{T|Y}(G_1) + \cdots + X_{T|Y}(G_n)\}$ for any $Y \in \mathrm{SM}(T)$ and for any pair of open subsets $G_1, \cdots, G_n$ of $\boldsymbol{C}$. Hence $T|Y$ is quasidecomposable for any $Y \in \mathrm{SM}(T)$ by the same argument as in the proof of Theorem 1. Hence T is strongly quasidecomposable by [4, Theorem 5.3]. The implication (3) $\Rightarrow$ (1) is clear.

REMARK 2. It is an interesting problem to know whether for a 2-quasidecomposable operator, (♯) $X_T(F)$ is contained in the closure of $\{X_T(\bar{G}_1) + X_T(\bar{G}_2)\}$ for any closed subset F of $\boldsymbol{C}$ and for any open covering

$\{G_1, G_2\}$ of F. If the answer is affirmative, then we can get a characterization of quasidecomposable operators similar to Theorem 1 by Remark 1.

The following lemma and theorem are inspired by the proof of Lemma 1.

LEMMA 3. *If an operator T is 2-quasidecomposable and if the operator $T^{\bar{G}}$ on $X/X_T(\bar{G})$ induced by T satisfies the closure condition* (C) *for some open subset G of $\boldsymbol{C}$, then we have $\sigma(T^{\bar{G}}) \subset \boldsymbol{C} \setminus G$.*

PROOF. Let $F \subset G$ be any closed subset. Then $\{G, \boldsymbol{C} \setminus F\}$ is an open covering of $\sigma(T)$. Hence there exist Y_1 and Y_2 in $\mathrm{SM}(T)$ such that (1) X is the closure of $Y_1 + Y_2$ and that (2) $\sigma(T | Y_1) \subset G$, $\sigma(T | Y_2) \subset \boldsymbol{C} \setminus F$. Therefore we have $Y_1 \subset X_T(\bar{G})$ and $Y_2 \subset X_T(\sigma(T | Y_2))$. Then for any vector $x = x_1 + x_2$ where $x_i \in Y_i$, $i = 1, 2$, we have

$$\hat{x} = \hat{x}_1 + \hat{x}_2 = \hat{x}_2 \in X_{T^{\bar{G}}}(\sigma(T | Y_2))$$

where $\hat{x} \in X/X_T(\bar{G})$ is the canonical image of x. Since $X_{T^{\bar{G}}}(\sigma(T | Y_2))$ is closed by assumption and since the canonical map $\hat{}$ is continuous, we have $\hat{x} \in X_{T^{\bar{G}}}(\sigma(T | Y_2))$ for all x in the closure of $Y_1 + Y_2$. Hence by (1), we have $X/X_T(G) \subset X_{T^{\bar{G}}}(\sigma(T | Y_2))$, and so $X/X_T(G) = X_{T^{\bar{G}}}(\sigma(T | Y_2))$. Hence we have

$$\sigma(T^{\bar{G}}) = \sigma(T^{\bar{G}} | X_{T^{\bar{G}}}(\sigma(T | Y_2)) \subset \sigma(T | Y_2) \subset \boldsymbol{C} \setminus F .$$

Since F is any closed subset contained in G, we have $\sigma(T^{\bar{G}}) \subset \boldsymbol{C} \setminus G$.

THEOREM 3. *If an operator T is 2-quasidecomposable and if the operator $T^{\bar{G}}$ on $X/X_T(\bar{G})$ induced by T satisfies the closure condition* (C) *for all open subsets G of $\boldsymbol{C}$, then T is decomposable.*

PROOF. By Theorem 1 and Lemma 2, we have only to show that $X_T(F) \subset X_T(\bar{G}_1) + X_T(\bar{G}_2)$ for any closed subset F of $\boldsymbol{C}$ and for any open covering $\{G_1, G_2\}$ of F. By Lemma 3, we have $\sigma(T^J) \subset \boldsymbol{C} \setminus (G_1 \cap G_2)$, where $J = (G_1 \cap G_2)^-$ and T^J is the operator on $X/X_T(J)$ induced by T. Let $x \in X_T(F)$ be given. Then we have

$$\hat{x} \in X_{T^J}(F \cap (\boldsymbol{C} \setminus (G_1 \cap G_2))) = X_{T^J}((F \setminus G_1) \cup (F \setminus G_2)) ,$$

and the rest of the proof is the same as that of Lemma 1.

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