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INEQUALITIES OF FEJÉR-RIESZ TYPE FOR HOLOMORPHIC FUNCTIONS ON CERTAIN PRODUCT DOMAINS

Dedicated to the memory of Professor Teishirô Saitô

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1. Introduction. Let f be a holomorphic function in a neighborhood of the closed unit disc in the complex plane $\boldsymbol{C}$ and l be a chord of the boundary circle C. Then the following inequality holds for every p, $0 < p < \infty$:

$$\int_l |f(z)|^p |dz| \leqq K_l \int_C |f(z)|^p |dz| \tag{1}$$

in which K_l is a constant depending only on l, and $K_l < 1$ ([1], [5]). If l coincides with a diameter of the disc, then $K_l = 1/2$ and the Fejér-Riesz inequality follows ([2]), and this is extended to the H^p-functions on the unit ball of $\boldsymbol{C}^n$, $n \geqq 2$ ([4]).

The purpose of the present note is to obtain an inequality similar to (1) for the H^p-functions on a domain in $\boldsymbol{C}^N$ which is a product of balls in $\boldsymbol{C}^{n_j}$, $j = 1, \cdots, m$. This inequality gives, as a special case, an extension of (1) to H^p-functions on the unit polydisc in $\boldsymbol{C}^n$ which is not treated in [4], and we note that the constant appearing in the inequality exhibits a remarkable contrast to that for the unit ball.

2. Statements of results. Let $\boldsymbol{C}^N = \boldsymbol{C}^{n_1} \times \cdots \times \boldsymbol{C}^{n_m}$ and let $Z = (Z^1, \cdots, Z^m) \in \boldsymbol{C}^N$, where we shall use the notations $Z^j = (z_1^j, \cdots, z_{n_j}^j) \in \boldsymbol{C}^{n_j}$ and $X^j = (x_1^j, x_2^j, \cdots, x_{2n_j-1}^j, x_{2n_j}^j) \in \boldsymbol{R}^{2n_j}$ with $z_k^j = x_{2k-1}^j + i x_{2k}^j$, $k = 1, \cdots, n_j$; $j = 1, \cdots, m$. We shall write $\|Z^j\|^2 = |z_1^j|^2 + \cdots + |z_{n_j}^j|^2$ and $\|X^j\|^2 = (x_1^j)^2 + \cdots + (x_{2n_j}^j)^2$. If $A^j = (a_1^j, \cdots, a_{2n_j}^j) \in \boldsymbol{R}^{2n_j}$, we write $A^j X^j = a_1^j x_1^j + \cdots + a_{2n_j}^j x_{2n_j}^j$. We consider a domain $\boldsymbol{B} = \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_m$ in $\boldsymbol{C}^N$, where $\boldsymbol{B}_j$ is the unit ball in $\boldsymbol{C}^{n_j}$ centered at the origin, i.e., $\boldsymbol{B}_j$ is the set of points Z^j such that $\|Z^j\| < 1$. We let $\partial \boldsymbol{B}$ stand for the Bergman-Šilov boundary of $\boldsymbol{B}$, $\partial \boldsymbol{B} = \partial \boldsymbol{B}_1 \times \cdots \times \partial \boldsymbol{B}_m$, where $\partial \boldsymbol{B}_j$ is the boundary of $\boldsymbol{B}_j$. We denote the Lebesgue measure on $\partial \boldsymbol{B}$ by $d\tau$; more precisely, this means that $d\tau$ is the product measure of elements of the surface area of spheres $\partial \boldsymbol{B}_j$, $j = 1, \cdots, m$. The Hardy space $H^p(\boldsymbol{B})$, $0 < p < \infty$, is defined and properties we need can be derived as in the case of polydiscs ([6]); especially, if $f \in H^p(\boldsymbol{B})$, then the radial limit $f^*(Z)$ exists for almost all $Z \in \partial\boldsymbol{B}$, and $f^* \in L^p(\partial\boldsymbol{B})$. We denote by L a hyperplane in $\boldsymbol{R}^{2N}$ and by $d\sigma$ the Lebesgue measure on it.

Our main result is the following, in which, if $m = 1$ and L passes through the origin, then the inequality holds for the constant 1/2 by [4, Theorem 1].

THEOREM. *Every function $f \in H^p(\boldsymbol{B})$ satisfies the following inequality for any p, $0 < p < \infty$, and for any hyperplane L in $\boldsymbol{R}^{2N}$:*

$$\int_{L\cap\boldsymbol{B}} |f(Z)|^p d\sigma(Z) \leqq 2^{-(m-1)} m^{1/2} \int_{\partial\boldsymbol{B}} |f^*(Z)|^p d\tau(Z) . \tag{2}$$

When a single space $\boldsymbol{C}^n$ is considered, a point in it is denoted by $z = (z_1, \cdots, z_n)$ with $z_k = x_{2k-1} + ix_{2k}$, $x_j \in \boldsymbol{R}$, $j = 1, \cdots, 2n$. We shall denote by $\varDelta$ and T the unit polydisc in $\boldsymbol{C}^n$ centered at the origin and the Bergman-Šilov boundary of $\varDelta$, respectively, i.e., $\varDelta = \{z \in \boldsymbol{C}^n \mid |z_j| < 1, j = 1, \cdots, n\}$ and $T = \{z \in \boldsymbol{C}^n \mid |z_j| = 1, j = 1, \cdots, n\}$.

COROLLARY. *Every $f \in H^p(\varDelta)$ satisfies the inequality for any p, $0 < p < \infty$, and any L in $\boldsymbol{R}^{2n}$:*

$$\int_{L\cap\varDelta} |f(z)|^p d\sigma(z) \leqq 2^{-(n-1)} n^{1/2} \int_T |f^*(z)|^p d\tau(z) .$$

3. A lemma. We shall need the following. Although this can be proved in the same way as [4, Lemma 1], slight modifications should be made.

LEMMA. *Let $\boldsymbol{B}$ be the unit ball in $\boldsymbol{C}^n$ centered at 0 and let L be a hyperplane in $\boldsymbol{R}^{2n}$. Then there exists a constant K for which every holomorphic function f in a neighborhood of $\bar{\boldsymbol{B}}$ satisfies the inequality for every p, $0 < p < \infty$,*

$$\int_{L\cap\boldsymbol{B}} |f(z)|^p d\sigma(z) \leqq K \int_{\partial\boldsymbol{B}} |f(z)|^p d\tau(z) .$$

$K \leqq 1$ in general, and $K = 1/2$ for any hyperplane L passing through the origin.

PROOF. First, we parametrize the unit sphere $\partial\boldsymbol{B}$ by the mapping $\varPhi(\theta_1, \cdots, \theta_{2n-1}) = (x_1, \cdots, x_{2n-1}, x_{2n})$, where

$$\begin{aligned} x_1 &= \cos\theta_1 , \\ x_j &= \sin\theta_1 \cdot\cdots\cdot \sin\theta_{j-1} \cos\theta_j , \quad j = 2, \cdots, 2n-1 , \\ x_{2n} &= \sin\theta_1 \cdots\cdots\cdots\cdots \sin\theta_{2n-2} \sin\theta_{2n-1} , \end{aligned} \tag{3}$$

$0 \leqq \theta_1, \cdots, \theta_{2n-2} \leqq \pi$, $0 \leqq \theta_{2n-1} < 2\pi$. With respect to this parametrization, we have $d\tau = \prod_{j=1}^{2n-2}(\sin\theta_j)^{2n-j-1}d\theta_1 \cdots d\theta_{2n-1}$. Next we begin with the hyperplane L, $L \cap \boldsymbol{B} \neq \varnothing$, defined by the equation $x_{2n} = a$, $0 \leqq a < 1$. Functions $x_1, \cdots, x_{2n-1}$ in (3) and $x_{2n} = a$ can be used as a parametrization Ψ for $L \cap \boldsymbol{B}$, $\Psi\colon G \to L \cap \boldsymbol{B}$, where G is defined by $G = \{(\theta_1, \cdots, \theta_{2n-1}) \in \boldsymbol{R}^{2n-1} \mid \|\Psi(\theta_1, \cdots, \theta_{2n-1})\| < 1, 0 < \theta_j < \pi, j = 1, \cdots, 2n-1\}$, and we have $d\sigma = \prod_{j=1}^{2n-1}(\sin\theta_j)^{2n-j}d\theta_1 \cdots d\theta_{2n-1}$. Writing $\theta' = (\theta_1, \cdots, \theta_{2n-2})$ and $Q = (0, \pi) \times \cdots \times (0, \pi) \subset \boldsymbol{R}^{2n-2}$, we define $D = \{\theta' \in Q \mid (\theta', \theta_{2n-1}) \in G$ for some $\theta_{2n-1} \in (0, \pi)\}$. Take an arbitrary point $\theta' \in D$, and let $(\theta', \theta_{2n-1}) \in G$. Then θ' determines a point $z' = (z_1, \cdots, z_{n-1})$, and θ_{2n-1} satisfies the inequality

$$(\sin\theta_1 \cdots \sin\theta_{2n-2})^2(\cos\theta_{2n-1})^2 + a^2 < 1 - \|z'\|^2\,,$$

where $\|z'\|^2 = |z_1|^2 + \cdots + |z_{n-1}|^2$, hence runs through an interval $(\alpha, \beta) \subset (0, \pi)$. The corresponding point $z_n = x_{2n-1} + ia$ lies on a chord l of a circle C in $\boldsymbol{C}$ of radius $(1 - \|z'\|^2)^{1/2}$. On the other hand, for a point $z_n \in C$, we can write $z_n = (1 - \|z'\|^2)^{1/2}e^{it}$, $0 \leqq t < 2\pi$, and $(z', z_n) = \Phi(\theta', t)$. Now the inequality (1) implies that

$$\begin{aligned} J := &\int_l |f(z', z_n)|^p |dz_n| \\ &\leqq K_l(1 - \|z'\|^2)^{1/2} \int_0^{2\pi} |f(z', (1 - \|z'\|^2)^{1/2}e^{it})|^p dt\,; \end{aligned}$$

here, since $|dz_n| = \sin\theta_1 \cdots \sin\theta_{2n-1}d\theta_{2n-1}$, $\theta_{2n-1} \in (\alpha, \beta)$, on the left-hand side, we get

$$J = \int_\alpha^\beta |(f\circ\Psi)(\theta', \theta_{2n-1})|^p \sin\theta_1 \cdots \sin\theta_{2n-1}d\theta_{2n-1}\,.$$

It follows that

$$\begin{aligned} &\int_{L\cap B} |f(z)|^p d\sigma(z) \\ &= \int_G |(f\circ\Psi)(\theta_1, \cdots, \theta_{2n-2}, \theta_{2n-1})|^p \prod_{j=1}^{2n-1}(\sin\theta_j)^{2n-j}d\theta_1 \cdots d\theta_{2n-1} \\ &= \int_D d\theta_1 \cdots d\theta_{2n-2} \int_\alpha^\beta |(f\circ\Psi)(\theta', \theta_{2n-1})|^p \prod_{j=1}^{2n-2}(\sin\theta_j)^{2n-j-1} \sin\theta_1 \cdots \sin\theta_{2n-1}d\theta_{2n-1} \\ &\leqq \int_D \prod_{j=1}^{2n-2}(\sin\theta_j)^{2n-j-1}d\theta_1 \cdots d\theta_{2n-2} \int_0^{2\pi} |f(z', (1 - \|z'\|^2)^{1/2}e^{it})|^p (1 - \|z'\|^2)^{1/2}dt \\ &\leqq \int_Q d\theta_1 \cdots d\theta_{2n-2} \int_0^{2\pi} |f(z', z_n)|^p |z_n| \prod_{j=1}^{2n-2}(\sin\theta_j)^{2n-j-1}dt \\ &= \int_{\partial B} |f(z)|^p |z_n| d\tau(z) \leqq \int_{\partial B} |f(z)|^p d\tau(z)\,. \end{aligned}$$

Finally, let L be an arbitrary hyperplane in $\boldsymbol{R}^{2n}$, $L \cap \boldsymbol{B} \neq \varnothing$. Take a unit vector w in $\boldsymbol{C}^n$ orthogonal to L with respect to the real inner product $\operatorname{Re}\langle u, v\rangle$ of $\boldsymbol{R}^{2n}$, where $\langle u, v\rangle = \sum_{j=1}^{n} u_j \bar{v}_j$ for $u = (u_1, \cdots, u_n)$, $v = (v_1, \cdots, v_n) \in \boldsymbol{C}^n$. Choose a unitary transformation U in $\boldsymbol{C}^n$ so that $Uw = (0, \cdots, 0, i)$. Then $L' := U(L)$ is a hyperplane defined by the equation $\operatorname{Im} z'_n = \text{const.}$, $z' = (z'_1, \cdots, z'_n) \in L'$. Hence, denoting by $d\sigma'$ the measure on L', we get

$$\int_{L' \cap B} |(f \circ U^{-1})(z')|^p d\sigma'(z') \leqq \int_{\partial B} |(f \circ U^{-1})(z')|^p d\tau(z') .$$

It follows that

$$\int_{L \cap B} |f(z)|^p d\sigma(z) \leqq \int_{\partial B} |f(z)|^p d\tau(z) .$$

REMARK. Let $Uz = z'$. Then $|z'_n| = |\langle z', (0, \cdots, 0, i)\rangle| = |\langle z', w'\rangle|$; hence we have

$$\int_{L \cap B} |f(z)|^p d\sigma(z) \leqq K \int_{\partial B} |f(z)|^p |\langle z, w'\rangle| d\tau(z) .$$

4. **Proof of Theorem.** Let L be a hyperplane in $\boldsymbol{R}^{2N}$, $L \cap \boldsymbol{B} \neq \varnothing$, defined by the equation $\sum_{j=1}^{m} A^j X^j + a = 0$, where $A^j = (a_1^j, \cdots, a_{2n_j}^j) \in \boldsymbol{R}^{2n_j}$, $j = 1, \cdots, m$, and $a \in \boldsymbol{R}$. We may suppose that $\|A^m\| \prod_{j=1}^{m-1} (2n_j)$ is the maximum among the values $\|A^k\| \prod_{j \neq k} (2n_j)$, $k = 1, \cdots, m$, and that $a_{2n_m}^m \neq 0$. We shall derive the inequality:

$$\int_{L \cap B} |f(Z)|^p d\sigma(Z) \leqq \left(\sum_{j=1}^{m} \|A^j\|^2\right)^{1/2} \|A^m\|^{-1} \prod_{j=1}^{m-1} (2n_j)^{-1} \int_{\partial B} |f^*(Z)|^p d\tau(Z) .$$

This is sufficient, because, letting $\|A^k\| = \max\{\|A^j\| \mid j = 1, \cdots, m\}$, we have

$$\begin{aligned}(\sum \|A^j\|^2)^{1/2} \|A^m\|^{-1} \prod_{j=1}^{m-1} (2n_j)^{-1} &\leqq (\sum \|A^j\|^2)^{1/2} \|A^k\|^{-1} \prod_{j \neq k} (2n_j)^{-1} \\ &\leqq m^{1/2} 2^{-(m-1)} .\end{aligned}$$

Now, the defining equation of L becomes

$$x_{2n_m}^m = \sum_{j=1}^{m-1} B^j X^j + B^{m\prime} X^{m\prime} + b , \tag{4}$$

where $B^j = (b_1^j, \cdots, b_{2n_j}^j)$, $j = 1, \cdots, m-1$, $B^{m\prime} = (b_1^m, \cdots, b_{2n_m-1}^m)$ with $b_k^j = -a_k^j (a_{2n_m}^m)^{-1}$, and $X^{m\prime} = (x_1^m, \cdots, x_{2n_m-1}^m)$. First, we shall prove the above inequality for functions f holomorphic in a neighborhood of $\bar{\boldsymbol{B}}$. It suffices to show

$$\int_{L \cap B} |f(Z)|^p d\sigma(Z) \leqq c_m c_1^{-1} \prod_{j=1}^{m-1} (2n_j)^{-1} \int_{\partial B} |f(Z)|^p d\tau(Z) , \tag{5}$$

where $c_m = (\|B^1\|^2 + \cdots + \|B^m\|^2)^{1/2}$, $c_1 = \|B^m\|$ with $B^m = (b_1^m, \cdots, b_{2n_m-1}^m, -1)$. Since the case $m = 1$ is proved in Lemma, we have only to verify the inequality (5) under the assumption that the case $m - 1$ is valid. Let G be the open subset of $\boldsymbol{R}^{2N-1}$ consisting of points $(X^1, \cdots, X^{m-1}, X^{m\prime}) \in \boldsymbol{R}^{2N-1}$ such that $\|X^j\| < 1$, $j = 1, \cdots, m-1$, and $\|X^{m\prime}\|^2 + (x_{2n_m}^m)^2 < 1$, where $x_{2n_m}^m$ is the function of $X^1, \cdots, X^{m-1}, X^{m\prime}$ defined by the equation (4). Let $\Psi: G \to L \cap \boldsymbol{B}$ be the transformation defined by $X^j = X^j$, $j = 1, \cdots, m-1$, and $X^m = (X^{m\prime}, x_{2n_m}^m)$. The measure $d\sigma$ on $L \cap \boldsymbol{B}$ with respect to this parametrization is $d\sigma = c_m dX^1 \cdots dX^{m-1} dX^{m\prime}$, where we write $dX^j = dx_1^j \cdots dx_{2n_j}^j$, $j = 1, \cdots, m-1$, and $dX^{m\prime} = dx_1^m \cdots dx_{2n_m-1}^m$. Let $D = \{X^1 \in \boldsymbol{B}_1 \mid (X^1, X^2, \cdots, X^{m\prime}) \in G$ for some $(X^2, \cdots, X^{m\prime}) \in \boldsymbol{R}^{2(N-n_1)-1}\}$. Take an arbitrary point $X^1 \in D$. Then a hyperplane $L' := L'(X^1)$ in $\boldsymbol{R}^{2(N-n_1)}$ is determined by the equation

$$x_{2n_m}^m = \sum_{j=2}^{m-1} B^j X^j + B^{m\prime} X^{m\prime} + (B^1 X^1 + b)\ .$$

Let $G'(X^1)$ be the open subset of $\boldsymbol{R}^{(N-n_1)-1}$ such that $\|X^j\| < 1$, $j = 2, \cdots, m-1$, and $\|X^{m\prime}\|^2 + (x_{2n_m}^m)^2 < 1$. Then a parametrization $\Psi': G'(X^1) \to L' \cap \boldsymbol{B}'$, where $\boldsymbol{B}' = \boldsymbol{B}_2 \times \cdots \times \boldsymbol{B}_m$, is defined by $X^j = X^j$, $j = 2, \cdots, m-1$, and $X^m = (X^{m\prime}, x_{2n_m}^m)$. The measure $d\sigma'$ on $L' \cap \boldsymbol{B}'$ is given by $d\sigma' = c_{m-1} dX^2 \cdots dX^{m\prime}$, $c_{m-1} = (\|B^2\|^2 + \cdots + \|B^m\|^2)^{1/2}$. Note that the set $G'(X^1)$ consists of points $(X^2, \cdots, X^{m\prime})$ such that $(X^1, X^2, \cdots, X^{m\prime}) \in G$. For an arbitrary function f holomorphic in a neighborhood of $\bar{\boldsymbol{B}}$ and p, $0 < p < \infty$, we have

$$\begin{aligned} I := \int_{L \cap \boldsymbol{B}} |f(Z)|^p d\sigma(Z) &= c_m \int_G |(f \circ \Psi)(X^1, X^2, \cdots, X^{m\prime})|^p dX^1 dX^2 \cdots dX^{m\prime} \\ &= c_m \int_D dX^1 \int_{G'(X^1)} |f(X^1, \Psi'(X^2, \cdots, X^{m\prime}))|^p dX^2 \cdots dX^{m\prime} \\ &= c_m c_{m-1}^{-1} \int_D dX^1 \int_{L' \cap \boldsymbol{B}'} |f(Z^1, Z')|^p d\sigma'(Z')\ , \end{aligned}$$

where we write $Z' = (Z^2, \cdots, Z^m)$. The induction hypothesis implies that

$$\int_{L' \cap \boldsymbol{B}'} |f(Z^1, Z')|^p d\sigma'(Z') \leqq c_{m-1} c_1^{-1} \prod_{j=2}^{m-1} (2n_j)^{-1} \int_{\partial \boldsymbol{B}'} |f(Z^1, Z')|^p d\tau'(Z')\ ,$$

where $d\tau'$ denotes the measure on $\partial \boldsymbol{B}'$, hence

$$\begin{aligned} I &\leqq c_m c_1^{-1} \prod_{j=2}^{m-1} (2n_j)^{-1} \int_D dX^1 \int_{\partial \boldsymbol{B}'} |f(Z^1, Z')|^p d\tau'(Z') \\ &\leqq c_m c_1^{-1} \prod_{j=2}^{m-1} (2n_j)^{-1} \int_{\boldsymbol{B}_1} dX^1 \int_{\partial \boldsymbol{B}'} |f(Z^1, Z')|^p d\tau'(Z') \\ &= c_m c_1^{-1} \prod_{j=2}^{m-1} (2n_j)^{-1} \int_0^1 r^{2n_1 - 1} dr \int_{\partial \boldsymbol{B}_1} d\tau_1(Z^1) \int_{\partial \boldsymbol{B}'} |f(rZ^1, Z')|^p d\tau'(Z')\ , \end{aligned}$$

where $d\tau_1$ denotes the measure on $\partial \boldsymbol{B}_1$. Here, since $f(Z^1, Z')$ is a holomorphic function of Z^1 in a neighborhood of $\bar{\boldsymbol{B}}_1$ for every $Z' \in \partial \boldsymbol{B}'$, we see by [3, Lemma 2] that

$$\int_{\partial B_1} |f(rZ^1, Z')|^p d\tau_1(Z^1) \leqq \int_{\partial B_1} |f(Z^1, Z')|^p d\tau_1(Z^1) ,$$

hence

$$\begin{aligned} I &\leqq c_m c_1^{-1} \prod_{j=2}^{m-1} (2n_j)^{-1} (2n_1)^{-1} \int_{\partial B'} d\tau'(Z') \int_{\partial B_1} |f(Z^1, Z')|^p d\tau_1(Z^1) \\ &= c_m c_1^{-1} \prod_{j=1}^{m-1} (2n_j)^{-1} \int_{\partial B} |f(Z)|^p d\tau(Z) . \end{aligned}$$

Now take an arbitrary function $f \in H^p(\boldsymbol{B})$, $0 < p < \infty$. Then, from Fatou's lemma and the inequality (2) which is valid for functions $f(rZ)$, $0 < r < 1$, holomorphic in neighborhoods of $\bar{\boldsymbol{B}}$, it follows that

$$\begin{aligned} \int_{L \cap B} |f(Z)|^p d\sigma(Z) &\leqq \liminf_{r \to 1} \int_{L \cap B} |f(rZ)|^p d\sigma(Z) \\ &\leqq \lim_{r \to 1} 2^{-(m-1)} m^{1/2} \int_{\partial B} |f(rZ)|^p d\tau(Z) \\ &= 2^{-(m-1)} m^{1/2} \int_{\partial B} |f^*(Z)|^p d\tau(Z) . \end{aligned}$$

The proof is completed.

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