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PRODUCTS OF DISTRIBUTIONS IN H^p SPACES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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0. Introduction. The purpose of the present paper is to extend the following.

THEOREM A (Coifman-Rochberg-Weiss [4], Uchiyama [11]). *If K is a singular integral operator of Calderón-Zygmund type on $\boldsymbol{R}^n$ and if $p, q, r > 0$ satisfy $1/p = 1/q + 1/r < 1 + 1/n$, then*

$$\| h\cdot Kg - K'h\cdot g \|_{H^p(\boldsymbol{R}^n)} \leqq C\| h \|_{H^q(\boldsymbol{R}^n)} \| g \|_{H^r(\boldsymbol{R}^n)} ,$$

where K' denotes the operator conjugate to K.

As for the definition of $H^p(\boldsymbol{R}^n)$, see Fefferman-Stein [5]; as for K and K', see the definitions in the next section.

In Theorem A, the restriction $1/p < 1 + 1/n$ cannot be relaxed since, for $f = h\cdot Kg - K'h\cdot g$, we cannot expect $\mathscr{F}f(\xi) = o(|\xi|^{n/p-n})$ as $\xi \to 0$ ($\mathscr{F}$ denotes the Fourier transform) if $1/p \geqq 1 + 1/n$, whereas any distribution f in $H^p(\boldsymbol{R}^n)$ has this property.

In this paper, we shall extend Theorem A to the case $1/p \geqq 1 + 1/n$ by showing that we can obtain a well defined bilinear map $H^q \times H^r \to H^p$, $1/p = 1/q + 1/r$, for all $q, r > 0$ if we form the following "products":

$$h\cdot K^2 g - 2K'h\cdot Kg + K'^2 h\cdot g,$$
$$h\cdot K^3 g - 3K'h\cdot K^2 g + 3K'^2 h\cdot Kg - K'^3 h\cdot g, \cdots , \quad \text{etc.}$$

(the theorem in the next section will give a slightly more general "product").

The argument of this paper is a refinement of that given by Uchiyama [11]; we shall refine the calculations in [11] so that we can use the inequality

$$\left| \int fg \right| \leqq C\| f \|_{H^p} \| g \|_{\mathrm{Lip}(n/p-n)} , \quad 0 < p < 1 ,$$

as well as Hölder's inequality.

Throughout this paper, we use the following.

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NOTATION. ϕ and ψ denote fixed smooth functions on $\boldsymbol{R}^n$ which have the following properties: $\phi(x)=1$ for $|x|\leqq 1$, $\phi(x)=0$ for $|x|\geqq 2$, $\psi(x)=1-\phi(x)$. $B(x, t)$ denotes the closed ball with center $x\in\boldsymbol{R}^n$ and radius t with respect to the usual metric. The mark $\cdot$ is sometimes used in compensation for parentheses; no operator operates beyond this mark. The letter C denotes a constant which may be different in different places. $f*g$ denotes the convolution of f and g. k^{*j} is defined by $k^{*j}=k^{*(j-1)}*k$, $j=1, 2, \cdots$, and $k^{*0}=\delta=$ the Dirac measure. For $\alpha=(\alpha_1, \cdots, \alpha_n)$ with α_j nonnegative integers, differential operator ∂^α and its order $|\alpha|$ are defined by $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$, where ∂_j is defined by $\partial_j f(x)=(\partial/\partial x_j)f(x)$. We shall also use the notation $(\partial/\partial x)^\alpha f(x)=\partial^\alpha f(x)$.

1. The result.

The result of this paper is the following.

THEOREM. *If $K_1, \cdots, K_N$ are singular integral operators of Calderón-Zygmund type on $\boldsymbol{R}^n$ and if $p, q, r>0$ satisfy*

$$1/p=1/q+1/r<1+N/n\ ,$$

then there is a constant C depending only on $p, q, r, K_1, \cdots, K_N$ and n such that, for all $h\in\mathscr{S}\cap H^q(\boldsymbol{R}^n)$ and all $g\in\mathscr{S}\cap H^r(\boldsymbol{R}^n)$,

$$\Big\|\sum_J(-1)^{|J|}\big(\prod_{j\in J}K_j'\big)h\cdot\big(\prod_{j\in J^c}K_j\big)g\Big\|_{H^p}\leqq C\|h\|_{H^q}\|g\|_{H^r}\ ,$$

where the summation ranges over all the subsets J of $\{1, \cdots, N\}$. Here K_j' denotes the operator conjugate to K_j, $|J|$ the cardinality of J, J^c the complement of J, $\prod$ the product of operators; if J or J^c is the empty set, the corresponding product $\prod$ means the identity operator.

To explain the singular integral operators in this theorem, we need the following

PROPOSITION 1.1. (i) *Suppose that k is a smooth function in $\boldsymbol{R}^n\setminus\{0\}$ satisfying*

$$|(\partial/\partial x)^\alpha k(x)|\leqq C_\alpha|x|^{-n-|\alpha|} \tag{1.1}$$

and

$$\sup_{0<a<b<\infty}\left|\int_{a<|x|<b}k(x)dx\right|<\infty\ . \tag{1.2}$$

Then there are sequences $\{a_j\}$ and $\{b_j\}$ such that $a_j\to 0$, $b_j\to\infty$ and

$$\lim k(\cdot)\psi(\cdot/a_j)\phi(\cdot/b_j)=\tilde{k}$$

exists in $\mathscr{S}'(\boldsymbol{R}^n)$.

(ii) *If $\tilde{k}$ is the distribution in* (i) *and c is a complex number, then the Fourier transform of $\tilde{k} + c\delta$, $\delta =$ the Dirac measure, is a bounded function m which is smooth in $\boldsymbol{R}^n\backslash\{0\}$ and satisfies*

$$|(\partial/\partial\xi)^\alpha m(\xi)| \leqq C'_\alpha |\xi|^{-|\alpha|} . \tag{1.3}$$

(iii) *Conversely, if m is a bounded function satisfying the conditions of* (ii), *then there are a distribution $\tilde{k}$ arising in* (i) *and a complex number c such that m is the Fourier transform of $\tilde{k} + c\delta$.*

PROOF. (i) can be seen if we choose $\{a_j\}$ so that

$$\lim \int k(x)\psi(x/a_j)\phi(x)dx$$

exists. (1.3) for $\alpha \neq 0$ and (iii) are proved by means of the technique in [6; Proof of Theorem 2.5]. Proof of (1.3) for $\alpha = 0$ can be found in [1; Theorem 3].

The operators in the theorem are the ones in the following definitions.

DEFINITION 1.1. We shall call K a *singular integral operator of Calderón-Zygmund type* if it is defined by

$$Kf = \tilde{k} * f + cf = \mathscr{F}^{-1}(m\mathscr{F}f) \tag{1.4}$$

with $\tilde{k}$, c and m in Proposition 1.1.

By Proposition 1.1, these operators form a commutative algebra. It is well-known that such operators can be extended to H^p for all $p > 0$ as bounded operators; see [5; § 12].

DEFINITION 1.2. If K is an operator defined by (1.4), the *conjugate operator* K' is defined by $K'f = \tilde{k}(-\cdot) * f + cf = \mathscr{F}^{-1}(m(-\cdot)\mathscr{F}f)$.

The rest of the paper will be devoted to the proof of the theorem.

2. Preliminary lemmas.

DEFINITION 2.1. For a nonnegative integer m, $x \in \boldsymbol{R}^n$ and $t > 0$, we define

$$\mathscr{T}_m(x, t) = \{\varphi \in \mathscr{S} \,|\, \operatorname{supp} \varphi \subset B(x, t),\ \|\partial^\alpha \varphi\|_{L^\infty} \leqq t^{-n-|\alpha|} \text{ for } |\alpha| \leqq m\} ,$$
$$\tilde{\mathscr{T}}_m(x, t) = \{\varphi \in \mathscr{S} \,|\, |\partial^\alpha \varphi| \leqq t(t + |\cdot - x|)^{-n-1-|\alpha|} \text{ for } |\alpha| \leqq m\} ;$$

and for $f \in \mathscr{S}'$,

$$M_{m,t}f(x) = \sup \{|\langle f, \varphi\rangle| \,|\, \varphi \in \mathscr{T}_m(x, s),\ 0 < s \leqq t\} ,$$
$$M_m f(x) = \sup \{|\langle f, \varphi\rangle| \,|\, \varphi \in \mathscr{T}_m(x, s),\ 0 < s < \infty\} ,$$
$$\tilde{M}_m f(x) = \sup \{|\langle f, \varphi\rangle| \,|\, \varphi \in \tilde{\mathscr{T}}_m(x, s),\ 0 < s < \infty\} .$$

DEFINITION 2.2. For a nonnegative integer m, we define $\mathscr{K}_m$ as the set of all $k\in\mathscr{S}$ which satisfy

$$|(\partial/\partial x)^\alpha k(x)|\leqq|x|^{-n-|\alpha|}\quad\text{for}\quad|\alpha|\leqq m$$

and

$$\sup_{0<a<\infty}\left|\int_{|x|<a}k(x)dx\right|\leqq 1\,.$$

We shall list up the lemmas which will be used later.

LEMMA 2.1. *If* $\infty>p>q>0$, *then*

$$\|\{\tilde{M}_0(|f|^q)\}^{1/q}\|_{L^p}\leqq C_{p,q}\|f\|_{L^p}\,.$$

If $0<p\leqq 1$ *and* $m>n/p-n$, *then*

$$\|\tilde{M}_m f\|_{L^p}\leqq C_p\|f\|_{H^p}\,.$$

Conversely, if $M_m f\in L^p$, $0<p<\infty$, *for some* $m\geqq 0$, *then* $f\in H^p$ *and*

$$\|f\|_{H^p}\leqq C_{p,m}\|M_m f\|_{L^p}\,.$$

LEMMA 2.2. *Let* $0<p\leqq 1$ *and* $m\geqq 0$. *If* $\operatorname{supp} f\subset B(x_0,t)$ *and* $M_{m,4t}f\in L^p(B(x_0,2t))$, *then* f *can be decomposed as* $f=g+\lambda\theta$, *where* $g\in H^p$, $\theta\in L^\infty$, $\|\theta\|_{L^\infty}\leqq t^{-n/p}$, λ *is a complex number, supports of* g *and* θ *are contained in* $B(x_0,t)$, *and*

$$\|g\|_{H^p}+|\lambda|\leqq C_{p,m}\left(\int_{B(x_0,2t)}(M_{m,4t}f)^p\right)^{1/p}\,.$$

LEMMA 2.3. *Let* $0<p\leqq 1$ *and* $m\geqq 0$. *If* $\operatorname{supp} f\subset B(x_0,t)$, $M_{m,4t}f\in L^p(B(x_0,2t))$ *and*

$$\int f(x)x^\alpha dx=0\quad\textit{for}\quad|\alpha|\leqq[n/p-n]\,,$$

then $f\in H^p$ *and*

$$\|f\|_{H^p}\leqq C_{p,m}\left(\int_{B(x_0,2t)}(M_{m,4t}f)^p\right)^{1/p}\,.$$

LEMMA 2.4. *For* $|x-x_0|<2t$,

$$M_m(f(\cdot)\phi(2(\cdot-x_0)/t))(x)\leqq C_m M_{m,3t}f(x)\,.$$

LEMMA 2.5. *If* $f\in H^p$, $0<p\leqq 1$ *and* $\operatorname{supp} f\subset B(x_0,t)$, *then* f *can be decomposed as follows*: $f=\sum_{j=1}^\infty\lambda_j a_j$, *where* λ_j *are complex numbers,* a_j *are bounded functions,* $\|a_j\|_{L^\infty}\leqq\rho_j^{-n/p}$, $\operatorname{supp} a_j\subset B(x_j,\rho_j)\subset B(x_0,2t)$, $\int a_j(x)x^\alpha dx=0$ *for* $|\alpha|\leqq[n/p-n]$, *and* $(\sum_{j=1}^\infty|\lambda_j|^p)^{1/p}\leqq C_p\|f\|_{H^p}$.

LEMMA 2.6. *If* $k\in\mathscr{K}_0$ *and* $v\in\mathscr{T}_m(x,t)$ *with* $m\geqq 1$, *then, for* $|\beta|+$

$|\gamma| \leqq m-1$ and $|w-x| < 1000t$,

$$\left|\int \partial^\beta v(z)\partial^\gamma k(w-z)dz\right| \leqq C_n t^{-n-|\beta|-|\gamma|}\,.$$

LEMMA 2.7. *If k_1 and k_2 are functions in $\mathscr{K}_m$ with $m \geqq 1$, then $C^{-1}k_1 * k_2$ belongs to $\mathscr{K}_{m-1}$, where C depends only on n and m.*

LEMMA 2.8. *There is a constant C depending only on n such that $\|\mathscr{F}k\|_{L^\infty} \leqq C$ for all $k \in \mathscr{K}_1$.*

LEMMA 2.9. *Let θ be a smooth function with* $\operatorname{supp}\theta \subset B(0, 1)$ *and $\int \theta = 1$; set $\theta_t = t^{-n}\theta(\cdot/t)$, $t > 0$. Then, for all $k \in \mathscr{K}_m$ with $m \geqq 1$,*

$$|(k(\cdot)\psi(-\cdot/10t)) * f(x) - \theta_t * k * f(x)| \leqq C\tilde{M}_{m-1}f(x)\,,$$

where the constant C depends only on n and m.

PROOF OF LEMMA 2.1. The first part is derived from the Hardy-Littlewood maximal theorem; see [10; Chap. II, § 3]. The second part is proved by means of the atomic decomposition of H^p; cf. [11; Lemma 7]. The third part is a result of Fefferman and Stein [5; Theorem 11].

PROOF OF LEMMA 2.2. We can take $\lambda\theta$ which has the estimates in the lemma and satisfies

$$\int (f - \lambda\theta)(x)x^\alpha dx = 0 \quad \text{for} \quad |\alpha| \leqq [n/p - n]\,.$$

Hence the lemma is reduced to Lemma 2.3.

PROOF OF LEMMA 2.3. We shall estimate $M_{m'}f(x)$ for $m' = m + [n/p - n] + 1$. Take $\varphi \in \mathscr{T}_{m'}(x, s)$. If P denotes the Taylor series of φ expanded about x_0 up to the terms of degree $[n/p - n]$, then

$$\int f\varphi = \int f(\varphi - P)\phi((\cdot - x_0)/t)\,.$$

(i) If $|x - x_0| < 2t$ and $s \leqq 4t$, then $\left|\int f\varphi\right| \leqq M_{m',4t}f(x) \leqq M_{m,4t}f(x)$;

(ii) if $|x - x_0| < 2t$ and $s > 4t$, then $C^{-1}(\varphi - P)\phi((\cdot - x_0)/t) \in \mathscr{T}_m(x, 4t)$, and hence

$$\left|\int f\varphi\right| \leqq CM_{m,4t}f(x)\,;$$

(iii) if $|x - x_0| > 2t$ and $s \leqq |x - x_0|/2$, then $\int f\varphi = 0$;

(iv) if $|x - x_0| > 2t$ and $s > |x - x_0|/2$, then, for every $y \in B(x_0, 2t)$,

$$C^{-1}(t/|x - x_0|)^{-n-[n/p-n]-1}(\varphi - P)\phi((\cdot - x_0)/t) \in \mathscr{T}_m(y, 4t),$$

and hence

$$\left|\int f\varphi\right| \leqq C(t/|x-x_0|)^{n+[n/p-n]+1}\left(t^{-n}\int_{B(x_0,2t)}(M_{m,4t}f)^p\right)^{1/p} .$$

(i)-(iv) give the pointwise estimate for the function $M_{m'}f(x)$ in terms of $M_{m,4t}f$; by integration, we obtain the desired estimate.

PROOF OF LEMMA 2.4. Let $|x-x_0|<2t$ and $\varphi\in\mathscr{T}_m(x, s)$. Then the function $C^{-1}\phi(2(\cdot - x_0)/t)\varphi(\cdot)$ belongs to $\mathscr{T}_m(x, 3t)$ if $s>3t$ and to $\mathscr{T}_m(x, s)$ if $s\leqq 3t$ (the constant C depends only on n and m). Thus the claim follows from the definition of the maximal functions.

PROOF OF LEMMA 2.5. Modify the proof of Latter [7]. If f has compact support and belongs to H^p, $p\leqq 1$, then it is orthogonal to polynomials of degree less than or equal to $[n/p-n]$; hence g_k constructed in the proof of [7] can be, after the multiplication of some constant, a p-atom by itself.

PROOF OF LEMMA 2.6. By integration by parts, the integral in the lemma can be rewritten as

$$\int_{|w-z|<1001t}(\partial^{\beta+\gamma}v(z)-\partial^{\beta+\gamma}v(w))k(w-z)dz+\partial^{\beta+\gamma}v(w)\int_{|w-z|<1001t}k(w-z)dz ,$$

which is majorized in absolute value by

$$\int_{|w-z|<1001t}\sqrt{n}\,t^{-n-|\beta+\gamma|-1}|w-z|^{1-n}dz+t^{-n-|\beta+\gamma|}=C_n t^{-n-|\beta+\gamma|} .$$

PROOF OF LEMMA 2.7. First we shall estimate

$$(\partial/\partial x)^{\alpha}(k_1*k_2)(x)=\int\partial^{\alpha}k_1(x-y)k_2(y)dy .$$

Decompose this integral as follows:

$$\begin{aligned}\int\{\cdots\}\phi(10y/|x|)dy+\int\{\cdots\}\phi(10(y-x)/|x|)dy&\\+\int\{\cdots\}(1-\phi(10y/|x|)-\phi(10(y-x)/|x|))dy&\\=\mathrm{I}+\mathrm{II}+\mathrm{III} .&\end{aligned}$$

We can estimate I and II by using Lemma 2.6; $|\mathrm{I}|, |\mathrm{II}|\leqq C|x|^{-n-|\alpha|}$ for $|\alpha|\leqq m-1$. As for III, we have, for $|\alpha|\leqq m$,

$$|\mathrm{III}|\leqq C\int_{|y|>|x|/10}|y|^{-2n-|\alpha|}dy\leqq C|x|^{-n-|\alpha|} .$$

Thus

$$|(\partial/\partial x)^\alpha(k_1 * k_2)(x)| \leqq C|x|^{-n-|\alpha|}, \quad |\alpha| \leqq m-1 . \tag{2.1}$$

Next we shall show

$$\sup_{0<a<\infty}\left|\int (k_1 * k_2)(x)\phi(x/a)dx\right| \leqq C \tag{2.2}$$

with C depending only on n, which combined with (2.1) proves the lemma. Decompose the integral in (2.2) as follows:

$$\begin{aligned}\int\left(\int k_1(x-y)\phi(x/a)dx\right)k_2(y)\phi(y/10a)dy + \iint k_1(x-y)k_2(y)\phi(x/a)\psi(y/10a)dxdy \\ = \mathrm{I} + \mathrm{II} .\end{aligned}$$

We can estimate I by using Lemma 2.6 twice; $|\mathrm{I}| \leqq C$. As for II, we have

$$|\mathrm{II}| \leqq C\iint_{|x|<2a;\,|y|>10a}|y|^{-2n}dxdy \leqq C .$$

This proves (2.2).

Proof of Lemma 2.8. See [1; Theorem 3].

Proof of Lemma 2.9. This lemma follows from the estimate

$$C^{-1}(k(\cdot)\psi(-\cdot/10t) - (\theta_t * k)(\cdot)) \in \widetilde{\mathscr{T}}_{m-1}(0, t) ,$$

which is proved in a way similar to the proof of Lemma 2.6.

3. Proof of the theorem. We shall prove the theorem in the special case $K_1 = \cdots = K_N = K$. The general case requires little modification; see the remark at the end of this paper. In the special case, the "product" of the theorem can be rewritten as

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}K'^jh\cdot K^{N-j}g ,$$

where $K'^0 = K^0 =$ the identity operator.

Take two sufficiently large integers m and m' (they can be determined depending only on p, q, r, N and n). We shall prove that there are positive numbers $u < q$, $v < r$ and C such that the following pointwise inequality holds throughout $\boldsymbol{R}^n$:

$$\begin{aligned}M_{m'}&\left(\sum_{j=0}^{N}(-1)^j\binom{N}{j}K'^jh\cdot K^{N-j}g\right) \\ &\leqq C\sum_{j=0}^{N}(M_m(K'^jh) + \widetilde{M}_mh)(M_m(K^{N-j}g) + \widetilde{M}_mg) \\ &\quad + C\{M_0((M_mh)^u)\}^{1/u}\{M_0((M_mg)^v)\}^{1/v}\end{aligned} \tag{3.1}$$

(u and v depend only on p, q, r, N and n; C depends only on p, q, r, N, n and K). Since K and K' are bounded operators in H^s for all $s>0$, (3.1) gives, via Hölder's inequality and Lemma 2.1, the inequality of the theorem.

We assume that $K=k*\cdot$, $k\in\mathscr{K}_{m''}$ with an m'' sufficiently large and show (3.1) with C depending only on p, q, r, N and n. Proof for the general K requires an easy limiting argument, which will be omitted.

From now on we simply denote $\mathscr{T}_m$, $\tilde{\mathscr{T}}_m$, M_m and $\tilde{M}_m$ by $\mathscr{T}$, $\tilde{\mathscr{T}}$, M and $\tilde{M}$, respectively. We use the following notation: if $Af=a*f$, then, for $t>0$,

$$A^{(t)}f=(a(\cdot)\psi(-\cdot/10t))*f\,.$$

Take $\varphi\in\mathscr{T}_{m'}(x_0,t)$, $x_0\in\boldsymbol{R}^n$, $t>0$ arbitrarily. What we must do is to estimate

$$\int\varphi(h\cdot K^Ng-NK'h\cdot K^{N-1}g\pm\cdots+(-1)^NK'^Nh\cdot g)\,.$$

The first step is to rewrite this integral as

$$\begin{aligned}\int\cdots\int&\left(\varphi(y)-N\varphi(z_1)+\binom{N}{2}\varphi(z_2)\mp\cdots+(-1)^N\varphi(w)\right)\\&\times h(y)k(y-z_1)k(z_1-z_2)\cdots k(z_{N-1}-w)g(w)dydz_1\cdots dz_{N-1}dw\end{aligned}$$

and decompose it into the following four terms:

$$\begin{aligned}\int\cdots\int&\{\cdots\}\phi((y-x_0)/10t)\phi((w-x_0)/10t)dydz_1\cdots dz_{N-1}dw\\&+\int\cdots\int\{\cdots\}\psi((w-x_0)/10t)dydz_1\cdots dz_{N-1}dw\\&+\int\cdots\int\{\cdots\}\psi((y-x_0)/10t)dydz_1\cdots dz_{N-1}dw\\&-\int\cdots\int\{\cdots\}\psi((y-x_0)/10t)\psi((w-x_0)/10t)dydz_1\cdots dz_{N-1}dw\\&=\mathrm{I}+\mathrm{II}+\mathrm{III}-\mathrm{IV}\,.\end{aligned}$$

We shall estimate these terms separately.

Estimate for III. III can be written as a finite linear combination of the following terms:

$$\mathrm{III}'=\int\varphi(z)\left(\int h(y)k_1(y-z)\psi((y-x_0)/10t)dy\right)\left(\int k_2(z-w)g(w)dw\right)dz\,,$$

where $k_1=k^{*j}$ and $k_2=k^{*(N-j)}$, $j=1,2,\cdots,N$. By Lemma 2.7, we can assume that k_1 and k_2 belong to $\mathscr{K}_{m'''}$ with an m''' sufficiently large (or $k_2=\delta$ when $j=N$).

Consider the following function in the region $|z - x_0| < t$:

$$f(z) = \int h(y)k_1(y - z)\psi((y - x_0)/10t)dy .$$

Decompose this as follows:

$$\begin{aligned} f(z) &= \int h(y)k_1(y - x_0)\psi((y - x_0)/10t)dy \\ &\quad + \int h(y)(k_1(y - z) - k_1(y - x_0))\psi((y - x_0)/10t)dy \\ &= J_1 + J_2 . \end{aligned}$$

In our notation, $J_1 = K_1'^{(t)}h(x_0)$, where K_1 is the convolution operator with the kernel k_1. Since, for $|z - x_0| < t$, the function $(k_1(\cdot - z) - k_1(\cdot - x_0))\psi((\cdot - x_0)/10t)$ belongs to $\tilde{\mathscr{T}}(x_0, t)$ when it is multiplied by some constant C depending only on n and m, it holds that $|J_2| \leqq C\tilde{M}h(x_0)$. Thus we obtain, for $|z - x_0| < t$,

$$|f(z)| \leqq |K_1'^{(t)}h(x_0)| + C\tilde{M}h(x_0) . \tag{3.2}$$

Moreover, if $1 \leqq |\alpha| \leqq m$, then

$$C^{-1}t^{|\alpha|}(\partial/\partial z)^\alpha k_1(\cdot - z)\psi((\cdot - x_0)/10t) \in \tilde{\mathscr{T}}(x_0, t)$$

with C depending only on n and m, and hence

$$|(\partial/\partial z)^\alpha f(z)| \leqq Ct^{-|\alpha|}\tilde{M}h(x_0) . \tag{3.3}$$

(3.2) and (3.3) mean that

$$(|K_1'^{(t)}h(x_0)| + C\tilde{M}h(x_0))^{-1}\varphi f \in \mathscr{T}(x_0, t) .$$

Hence

$$|\mathrm{III}'| \leqq (|K_1'^{(t)}h(x_0)| + C\tilde{M}h(x_0))M(K_2 g)(x_0) .$$

This estimate combined with Lemma 2.9 gives the desired estimate for III.

Estimate for II is similar to that for III.

Estimate for IV. IV can be written as a finite linear combination of the following terms:

$$\begin{aligned} \mathrm{IV}' = \iiint &\varphi(z)h(y)k_1(y - z)k_2(z - w)g(w)\psi((y - x_0)/10t) \\ &\times \psi((w - x_0)/10t)dydzdw , \end{aligned}$$

where $k_1 = k^{*j}$ and $k_2 = k^{*(N-j)}$, $j = 1, 2, \cdots, N - 1$. By Lemma 2.7, we can assume that k_1 and k_2 belong to $\mathscr{K}_{m'''}$ with an m''' sufficiently large. In the same way as in the estimate for III, we have, for $|z - x_0| < t$,

$$\left|\int h(y)k_1(y-z)\psi((y-x_0)/10t)\,dy\right| \leqq |K_1'^{(t)}h(x_0)| + C\tilde{M}h(x_0)\ ,$$

$$\left|\int k_2(z-w)g(w)\psi((w-x_0)/10t)dw\right| \leqq |K_2^{(t)}g(x_0)| + C\tilde{M}g(x_0)\ .$$

From these estimate, we see that

$$|\mathrm{IV}'| \leqq C(|K_1'^{(t)}h(x_0)| + \tilde{M}h(x_0))(|K_2^{(t)}g(x_0)| + \tilde{M}g(x_0))\ ,$$

which combined with Lemma 2.9 gives the desired estimate for IV.

Estimate for I. This is the essential part of the proof.

For $\theta \in \mathscr{S}$, we set

$$\zeta_N(\theta; y, w) = \int\cdots\int(\theta(y) - N\theta(z_1) \pm \cdots + (-1)^N\theta(w)) \times k(y-z_1)\cdots k(z_{N-1}-w)dz_1\cdots dz_{N-1}\ .$$

We define an operator $A_{N,\varphi}$ by

$$A_{N,\varphi}f(w) = \int f(y)\zeta_N(\varphi; y, w)dy\cdot\phi((w-x_0)/20t)\ .$$

Mapping properties of this operator are given in the following lemmas.

LEMMA 3.1. *Suppose that* $0<\varepsilon<1$ *and* $(1-\varepsilon)/n < 1/s < 1+(N-\varepsilon)/n$. *Then, for* $f \in H^s$ *with* $\operatorname{supp} f \subset B(x_0, 20t)$, *we have*

$$\|A_{N,\varphi}f\|_{X(\varepsilon,s)} \leqq C_{\varepsilon,s}t^{-n-N+\varepsilon}\|f\|_{H^s}\ ,$$

where $X(\varepsilon, s)$ *is defined as follows*: $X(\varepsilon, s) = H^p$ *with* $1/s - 1/p = (N-\varepsilon)/n$ *if* $1/s > (N-\varepsilon)/n$; $=$ BMO *if* $1/s = (N-\varepsilon)/n$; $= \dot{B}^{\alpha}_{\infty,\infty}$ (*the homogeneous Besov space*; *see* [2; § 6.3]) *with* $1/s + \alpha/n = (N-\varepsilon)/n$ *if* $1/s < (N-\varepsilon)/n$.

LEMMA 3.2. *Let* ε, s *and* $X(\varepsilon, s)$ *be the same as in Lemma* 3.1. *If* $\|f\|_{L^\infty} \leqq t^{-n/s}$ *and* $\operatorname{supp} f \subset B(x_0, 20t)$, *then*

$$\|A_{N,\varphi}f\|_{X(\varepsilon,s)} \leqq C_{\varepsilon,s}t^{-n-N+\varepsilon}\ .$$

LEMMA 3.3. *If* θ *and* θ' *are bounded functions with supports contained in* $B(x_0, 20t)$, *then*

$$\left|\int(A_{N,\varphi}\theta)\theta'\right| \leqq C\|\theta\|_{L^\infty}\|\theta'\|_{L^\infty}\ .$$

Proofs of these lemmas will be given in the next section.

We shall proceed to estimate I. Take ε, u and v such that $0<\varepsilon<1$, $1/p < 1+(N-\varepsilon)/n$, $0<u<q$, $0<v<r$, $1/u + 1/v = 1+(N-\varepsilon)/n$. By Lemmas 2.2 and 2.4, we have

$$h(\cdot)\phi((\cdot - x_0)/10t) = h_1 + \lambda\theta\ ,$$

$$g(\cdot)\phi((\cdot - x_0)/10t) = g_1 + \lambda'\theta' ,$$

where h_1, g_1, θ and θ' have their supports contained in $B(x_0, 20t)$ and

$$\begin{aligned} &\| h_1 \|_{H^u} + |\lambda| \leqq C\Big(\int_{B(x_0,40t)} (Mh)^u\Big)^{1/u} , \\ (3.4) \qquad &\| g_1 \|_{H^v} + |\lambda'| \leqq C\Big(\int_{B(x_0,40t)} (Mg)^v\Big)^{1/v} , \\ &\| \theta \|_{L^\infty} \leqq t^{-n/u} , \qquad \| \theta' \|_{L^\infty} \leqq t^{-n/v} \end{aligned}$$

(if $u > 1$ or $v > 1$, we take $\lambda\theta = 0$ or $\lambda'\theta' = 0$). Corresponding to this, I can be decomposed into four terms:

$$\begin{aligned} \mathrm{I} &= \iint h_1(y)\zeta_N(\varphi; y, w)g_1(w)dydw + \lambda \iint \theta(y)\zeta_N(\varphi; y, w)g_1(w)dydw \\ &\quad + \lambda' \iint h_1(y)\zeta_N(\varphi; y, w)\theta'(w)dydw + \lambda\lambda' \iint \theta(y)\zeta_N(\varphi; y, w)\theta'(w)dydw \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 . \end{aligned}$$

Since $X(\varepsilon, u)$ (notation in Lemma 3.1) is the dual space of H^v (see [5; Theorem 2], [7] and/or [3; Theorems 2.1 and 2.5]), if $1/u > (1 - \varepsilon)/n$, we can use Lemma 3.1 to obtain

$$|\mathrm{I}_1| \leqq C\| A_{N,\varphi}h_1 \|_{X(\varepsilon,u)} \| g_1 \|_{H^v} \leqq Ct^{-n-N+\varepsilon} \| h_1 \|_{H^u} \| g_1 \|_{H^v} .$$

The same estimate holds also in the case $1/u \leqq (1 - \varepsilon)/n$ since in this case we have $1/v > (1 - \varepsilon)/n$ and hence we can argue with the roles of u and v interchanged using a variant of Lemma 3.1 (i.e., a lemma for the operator dual to $A_{N,\varphi}$). Similarly Lemma 3.2 and its variant give

$$|\mathrm{I}_2| \leqq Ct^{-n-N+\varepsilon}|\lambda| \, \| g_1 \|_{H^v} , \qquad |\mathrm{I}_3| \leqq Ct^{-n-N+\varepsilon} \| h_1 \|_{H^u} |\lambda'| .$$

(I_2 and I_3 appear only in the case $u \leqq 1$ or $v \leqq 1$; hence we can always apply Lemma 3.2 or its variant.) As for I_4, we use Lemma 3.3 to obtain

$$|\mathrm{I}_4| \leqq Ct^{-n-N+\varepsilon}|\lambda| \, |\lambda'| .$$

These estimates and (3.4) give

$$\begin{aligned} |\mathrm{I}| &\leqq C\Big(t^{-n} \int_{B(x_0,40t)} (Mh)^u\Big)^{1/u} \Big(t^{-n} \int_{B(x_0,40t)} (Mg)^v\Big)^{1/v} \\ &\leqq C\{M_0((Mh)^u)(x_0)\}^{1/u}\{M_0((Mg)^v)(x_0)\}^{1/v} . \end{aligned}$$

Thus we have reduced the proof of the theorem to that of Lemmas 3.1–3.3.

4. Proof of Lemmas 3.1–3.3. In order to prove Lemmas 3.1 and 3.2, we use the following Lemmas.

LEMMA 4.1. $(\partial/\partial y_j)\zeta_N(\theta; y, w) = -(\partial/\partial w_j)\zeta_N(\theta; y, w) + \zeta_N(\partial_j\theta; y, w)$.

LEMMA 4.2. *If* $|y - x_0| < 1000t$, $|w - x_0| < 1000t$ *and* $|\alpha + \beta + \gamma| \leqq \tilde{m}$ *($\tilde{m}$ sufficiently large), then, for any* $\varepsilon > 0$,

$$|(\partial/\partial y)^\beta(\partial/\partial w)^\alpha\zeta_N(\partial^\gamma\varphi; y, w)| \\ \leqq C_\varepsilon t^{-n-N+\varepsilon-|\gamma|}|y - w|^{-\varepsilon}\operatorname{Max}\{t^{-n+N-|\alpha|-|\beta|}, |y - w|^{-n+N-|\alpha|-|\beta|}\} .$$

PROOF OF LEMMA 4.1. Repeated application of integration by parts.

In order to prove Lemma 4.2, we need the following

LEMMA 4.3. *If P is a polynomial of degree less than or equal to $N-1$, then*

$$\int\cdots\int\left(P(y) - NP(z_1) + \binom{N}{2}P(z_2) \mp \cdots + (-1)^N P(w)\right) \\ \times k(y - z_1)k(z_1 - z_2)\cdots k(z_{N-1} - w)dz_1\cdots dz_{N-1} = 0 .$$

PROOF. Let $\mathfrak{S}_N$ denote the symmetric group over $\{1, \cdots, N\}$. For each $\sigma \in \mathfrak{S}_N$, define a linear transformation $(z_j)_{j=1}^{N-1} \mapsto (z_j^\sigma)_{j=1}^{N-1}$ by the following rule: if $y - z_1 = a_1$, $z_1 - z_2 = a_2$, $\cdots$, $z_{N-1} - w = a_N$, then $y - z_1^\sigma = a_{\sigma(1)}$, $z_1^\sigma - z_2^\sigma = a_{\sigma(2)}$, $\cdots$, $z_{N-1}^\sigma - w = a_{\sigma(N)}$. The Jacobian of this transformation is $+1$ or -1 and the function $k(y - z_1)\, k(z_1 - z_2)\cdots k(z_{N-1} - w)$ is invariant under each of the transformation. On the other hand, if P is a polynomial of degree less than N, then

$$\sum_{\sigma\in\mathfrak{S}_N}\left(P(y) - NP(z_1^\sigma) + \binom{N}{2}P(z_2^\sigma) \mp \cdots + (-1)^N P(w)\right) = 0 .$$

(This equality reduces, by an elementary calculation, to the following equalities for polynomials in $a_1, \cdots, a_N$:

$$\sum_{j=1}^{N}(-1)^{N-j}\binom{N}{j}\sum_{\sigma\in\mathfrak{S}_N}(a_{\sigma(1)} + \cdots + a_{\sigma(j)})^m = 0 , \quad 1 \leqq m \leqq N-1 .$$

Hence, if we denote the integrand of the lemma by $I(z_1, \cdots, z_{N-1}) = I(z)$, then

$$\int I(z)dz = (N!)^{-1}\sum_{\sigma\in\mathfrak{S}_N}\int I(z^\sigma)dz = 0 .$$

This proves Lemma 4.3.

PROOF OF LEMMA 4.2. We may assume $\gamma = 0$. Applying Taylor's formula to φ and using Lemma 4.3, we can write $\zeta_N(\varphi; y, w)$ as a finite linear combination of the following terms:

$$J_1 = \int_0^1 (1 - s)^{N'-1}\partial^\sigma\varphi(x_1 + s(y - x_1))ds\cdot(y - x_1)^\sigma k_1(y - w) ,$$

$$J_2 = \int_0^1 (1-s)^{N'-1}\partial^\sigma\varphi(x_1 + s(w-x_1))ds\cdot(w-x_1)^\sigma k_1(y-w)\ ,$$

$$J_3 = \iint_{0<s<1;z\in R^n} (1-s)^{N'-1}\partial^\sigma\varphi(x_1+s(z-x_1))(z-x_1)^\sigma k_2(y-z)k_3(z-w)dzds\ ,$$

where $|\sigma| = N'$, $1 \leqq N' \leqq N$, $x_1 \in \boldsymbol{R}^n$ (we can fix N' and x_1 arbitrarily), $k_1 = k^{*N}$, $k_2 = k^{*j}$ and $k_3 = k^{*(N-j)}$, $j = 1, 2, \cdots, N-1$. If we take x_1 which satisfies $|y - x_1| < |y - w|$ and $|w - x_1| < |y - w|$, then

$$|(\partial/\partial y)^\beta(\partial/\partial w)^\alpha J_1| \leqq Ct^{-n-N'}|y-w|^{-n+N'-|\alpha|-|\beta|}\ ,$$
$$|(\partial/\partial y)^\beta(\partial/\partial w)^\alpha J_2| \leqq Ct^{-n-N'}|y-w|^{-n+N'-|\alpha|-|\beta|}\ .$$

If x_1 is as above and if $-n + N' - |\alpha| - |\beta| \leqq 0$, then, for any $\varepsilon > 0$,

$$|(\partial/\partial y)^\beta(\partial/\partial w)^\alpha J_3| \leqq C_\varepsilon t^{-n-N'+\varepsilon}|y-w|^{-n+N'-|\alpha|-|\beta|-\varepsilon}$$

(this estimate can be obtained in the same way as in the proof of Lemma 2.7). Thus we obtain the desired estimate by taking $N' = N$ if $-n + N - |\alpha| - |\beta| < 0$ and $N' = n + |\alpha| + |\beta|$ if $-n + N - |\alpha| - |\beta| \geqq 0$. This completes the proof of Lemma 4.2.

PROOF OF LEMMA 3.1. We shall prove that

$$\|(-\Delta)^{(N-1)/2}A_{N,\varphi}f\|_{H^v} \leqq C_{\varepsilon,s}t^{-n-N+\varepsilon}\|f\|_{H^s}\ , \tag{4.1}$$

where $1/s - 1/v = (1-\varepsilon)/n$ and Δ denotes the Laplacian. This estimate together with the well known mapping properties of the operator $(-\Delta)^{-(N-1)/2}$ (fractional integration; see [3; Theorem 4.1] and [8; Theorem 3.2]) gives the desired result. In order to prove (4.1), it is sufficient to show the same estimate for $\partial^\alpha A_{N,\varphi}f$, $|\alpha| = N-1$.

First consider the case $s > 1$. We can write

$$\partial^\alpha A_{N,\varphi}f(w) = \int f(y)(\partial/\partial w)^\alpha(\zeta_N(\varphi; y, w)\phi((w-x_0)/20t))dy\ .$$

Lemma 4.2 shows that the integrand is majorized in absolute value by $C_\varepsilon|f(y)|t^{-n-N+\varepsilon}|y-w|^{-n+1-\varepsilon}$. Hence the fractional integration theorem in L^p-spaces, $p > 1$ ([9; Chap. V, § 1]), gives the desired estimate.

Next suppose that $s \leqq 1$. By Lemmas 2.3 and 2.5, it is sufficient to show the estimate

$$\left(\int_{B(x_0,80t)} (M_{b,160t}(\partial^\alpha A_{N,\varphi}f))^v\right)^{1/v} \leqq C_{\varepsilon,s}t^{-n-N+\varepsilon} \tag{4.2}$$

for a sufficiently large b and for f's which satisfy

$$\begin{gathered}\operatorname{supp} f \subset B(y_0, \rho) \subset B(x_0, 40t)\ , \qquad \|f\|_{L^\infty} \leqq \rho^{-n/s}\ ,\\ \int f(y)y^\beta dy = 0 \quad \text{for} \quad |\beta| \leqq [n/s - n]\ .\end{gathered} \tag{4.3}$$

In order to prove (4.2)-(4.3), we take $\theta \in \mathscr{T}_b(x, a)$, $|x - x_0| < 80t$, $a \leqq 160t$ and study the function

$$\eta_{\theta,\alpha}(y) = \int (\partial/\partial w)^\alpha(\zeta_N(\varphi; y, w)\phi((w - x_0)/20t))\cdot\theta(w)dw ,$$

where $|\alpha| = N - 1$. We shall estimate

$$(4.4) \qquad (\partial/\partial y)^\beta \eta_{\theta,\alpha}(y) = \int (\partial/\partial y)^\beta(\partial/\partial w)^\alpha(\zeta_N(\varphi; y, w)\phi((w - x_0)/20t))\cdot\theta(w)dw$$

in the region $|y - x_0| < 40t$. If $|y - x| > 2a$, in order to estimate this, we replace the integrand by its absolute value and use Lemma 4.2; we obtain

$$(4.5) \qquad |(\partial/\partial y)^\beta \eta_{\theta,\alpha}(y)| \leqq C_\varepsilon t^{-n-N+\varepsilon}|y - x|^{-n+1-\varepsilon-|\beta|} .$$

If $|y - x| < 2a$, we estimate (4.4) in the following way: we rewrite the integral by using Lemma 4.1 repeatedly and by integration by parts to obtain

$$(\partial/\partial y)^\beta \eta_{\theta,\alpha}(y) = \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \int (\partial/\partial w)^{\alpha'}\zeta_N(\partial^{\beta''}\varphi; y, w)\cdot(\partial/\partial w)^{\beta'}$$
$$\times \{(\partial/\partial w)^{\alpha''}\phi((w - x_0)/20t)\cdot\theta(w)\}dw ;$$

then we replace the integrand by its absolute value and use Lemma 4.2 to obtain

$$(4.6) \qquad |(\partial/\partial y)^\beta \eta_{\theta,\alpha}(y)| \leqq C_\varepsilon t^{-n-N+\varepsilon} a^{-n+1-\varepsilon-|\beta|} .$$

Now let f be a function satisfying (4.3) and θ be as before. If P is the Taylor series of $\eta_{\theta,\alpha}$ expanded about y_0 up to the terms of degree $[n/s - n]$, then

$$\int (\partial^\alpha A_{N,\varphi} f)\theta = \int f\eta_{\theta,\alpha} = \int f(\eta_{\theta,\alpha} - P) .$$

If $|x - y_0| < 2\rho$, we use (4.5) and (4.6) with $\beta = 0$ to obtain

$$(4.7) \qquad \left|\int (\partial^\alpha A_{N,\varphi} f)\theta\right| \leqq \int |f\eta_{\theta,\alpha}| \leqq C_\varepsilon t^{-n-N+\varepsilon}\rho^{-n/s+1-\varepsilon} ;$$

if $|x - y_0| > 2\rho$, we use (4.5) and (4.6) with $|\beta| = [n/s - n] + 1$ to obtain

$$(4.8) \quad \left|\int (\partial^\alpha A_{N,\varphi} f)\theta\right| \leqq \int |f(\eta_{\theta,\alpha} - P)| \leqq C_\varepsilon t^{-n-N+\varepsilon}\rho^{-n/s+c+1+n}|x - y_0|^{-n-c-\varepsilon} ,$$

where $c = [n/s - n]$. (4.7) and (4.8) give the pointwise estimate for the function $M_{b,160t}(\partial^\alpha A_{N,\varphi} f)(x)$ in the region $|x - x_0| < 80t$. Now we obtain (4.2) by simple computation. This completes the proof of Lemma 3.1.

PROOF OF LEMMA 3.2. For the same reason as in the proof of Lemma 3.1, we can reduce the proof to the estimate for $\partial^\alpha A_{N,\varphi}f$ with $|\alpha| = N-1$. If f is the function in Lemma 3.2 and θ is as in the proof of Lemma 3.1, then (4.5) and (4.6) for $\beta = 0$ give

$$\left|\int(\partial^\alpha A_{N,\varphi}f)\theta\right| \leqq \int|f\eta_{\theta,\alpha}| \leqq Ct^{-n/s-n-N+1}\ ;$$

hence, for $|x - x_0| < 80t$,

$$M_{b,160t}(\partial^\alpha A_{N,\varphi}f)(x) \leqq Ct^{-n/s-n-N+1}\ .$$

Integrating this and using Lemma 2.3, we obtain the desired estimate. This completes the proof of Lemma 3.2.

PROOF OF LEMMA 3.3. The integral in the lemma is a finite linear combination of the following terms:

$$J = \iiint\varphi(z)\theta(y)k_1(y-z)k_2(z-w)\theta'(w)dydzdw\ ,$$

where $k_1 = k^{*j}$ and $k_2 = k^{*(N-j)}$, $j = 0, 1, \cdots, N$. By Lemma 2.8, the convolution operators with kernels k_1 or k_2 have bounded norms as operators in L^2. Hence

$$|J| \leqq C\|\theta\|_{L^2}\|\varphi\|_{L^\infty}\|\theta'\|_{L^2} \leqq C\|\theta\|_{L^\infty}\|\theta'\|_{L^\infty}\ .$$

This proves Lemma 3.3.

Thus we have completed the proof of the theorem.

REMARK. We shall indicate the modification of the proof necessary in the general case. The general "product" in the theorem can be rewritten as

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}(N!)^{-1}\sum_{\sigma\in\mathfrak{S}_N}\left(\prod_{i=1}^{j}K'_{\sigma(i)}\right)h\cdot\left(\prod_{i=j+1}^{N}K_{\sigma(i)}\right)g\ ,$$

where $\mathfrak{S}_N$ denotes the symmetric group over $\{1, \cdots, N\}$. Hence we can obtain the necessary modification by replacing a function of the form $k(x_1)\cdots k(x_N)$ which appears in the proof for the special case by

$$(N!)^{-1}\sum_{\sigma\in\mathfrak{S}_N}k_{\sigma(1)}(x_1)\cdots k_{\sigma(N)}(x_N)\ ;$$

the proof (of the theorem) with this replacement is just the same as that in the special case.

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