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CLOSED DERIVATIONS IN $C(I)$

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Introduction. Closed derivations in C^*-algebras have been studied by many authors motivated by mathematical physics. In commutative case closed derivations are also of great interest in connection with differentiations. In this paper we will discuss closed derivations in $C(I)$, where $C(I)$ is the algebra of all real valued continuous functions on the unit interval $I = [0, 1]$.

Let δ be a derivations in $C(I)$. Throughout this paper the domain $\mathscr{D}(\delta)$ of δ will be always assumed to be a dense subalgebra in $C(I)$ and we put $W_\delta = \{x \in I; \delta(f)(x) = 0$ for every f in $\mathscr{D}(\delta)$ with $\|f\| = |f(x)|\}$. δ is said to be *quasi well-behaved* iff the interior W_δ° of W_δ is dense in I. Batty [2], Goodman [3], and Sakai [6] have shown that a closed derivation δ in $C(I)$ is quasi well-behaved if and only if there exist $\lambda \in C(I)$ and an automorphism α of $C(I)$ such that $\delta \supset \lambda\alpha(d/dx)\alpha^{-1}$. But in [4] it has been shown that there exist non quasi well-behaved closed derivations in $C(I)$, those induced by non-atomic signed measures on I.

Let δ be a closed derivation in $C(I)$ and put $A_\delta = \{x \in I, \delta(f)(x) \neq 0$ for some f in $\mathscr{D}(\delta)\}$. In this paper we shall show that there exists an open dense set U in A_δ and a continuous function μ on U such that the restriction δ_E of δ to any closed interval E contained in U is the derivation induced by a non-atomic signed measure $\mu|_E$ on E.

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Closed derivation in $C(I)$. We first present several lemmas before stating our main theorem. Throughout this section δ will always denote a closed derivation in $C(I)$.

LEMMA 1. *Let f be a function in $\mathscr{D}(\delta)$ with $f(x_0) = \delta(f)(x_0) = 0$ for some x_0 in $(0, 1)$ and define $\tilde{f}$ and $\widetilde{\delta(f)}$ by the following:*

$$\tilde{f} = \begin{cases} f & on \quad [0, x_0] \\ 0 & on \quad [x_0, 1] \end{cases} \quad and \quad \widetilde{\delta(f)} = \begin{cases} \delta(f) & on \quad [0, x_0] \\ 0 & on \quad [x_0, 1] . \end{cases}$$

Then $\tilde{f}$ belongs to $\mathscr{D}(\delta)$ and satisfies $\delta(\tilde{f}) = \widetilde{\delta(f)}$.

PROOF. If $f = 0$ on $[0, x_0]$, by [3, Lemma 1.1.5], the lemma is clear. Thus we assume $\max_{0\leqq x\leqq x_0}|f(x)| \neq 0$. Let ε be a positive number with $\varepsilon \leqq \max_{0\leqq x\leqq x_0}|f(x)|/2$ and set $\alpha = \max\{0 \leqq x \leqq x_0;\ |f(x)| = 2\varepsilon\}$, $\beta = \min\{\alpha \leqq x \leqq x_0;\ |f(x)| = \varepsilon\}$, and $\gamma = \min(\{1\} \cup \{x \geqq x_0;\ |f(x)| = \varepsilon/3\})$. Taking $-f$ instead of f if necessary, we may assume $f(\alpha) = 2\varepsilon$ and $f(\beta) = \varepsilon$. Let η be an arbitrary positive number. Then, by [3, Lemma 1.1.5] and the continuity of f and $\delta(f)$, $\alpha \leqq x \leqq x_0$ implies $|\delta(f)(x)| \leqq \eta$ for ε sufficiently small. Since $\mathscr{D}(\delta)$ is a Silov algebra, we can find g_1 and g_2 in $\mathscr{D}(\delta)$ in such a way that $0 \leqq g_1 \leqq 1$, $-1 \leqq g_2 \leqq 0$,

$$g_1 = \begin{cases} 0 & \text{on} \quad [\beta, 1] \\ 1 & \text{on} \quad [0, \alpha] , \end{cases} \quad \text{and} \quad g_2 = \begin{cases} 0 & \text{on} \quad [0, x_0] \\ -1 & \text{on} \quad [\gamma, 1] . \end{cases}$$

Then $h = f + 2\|f\|(g_1 + g_2)$ belongs to $\mathscr{D}(\delta)$ and we have $h(x) \geqq \varepsilon$ for $x \in [0, \beta]$ and $h(x) \leqq \varepsilon/3$ for $x \in [x_0, 1]$. Let p be a C^1-function with $0 \leqq p' \leqq 2$ and

$$p(x) = \begin{cases} x & \text{if} \quad x \geqq \varepsilon \\ 0 & \text{if} \quad x \leqq \varepsilon/3 , \end{cases}$$

where p' is the usual derivative of p. Then we have $(p(h) - 2\|f\|g_1 - \tilde{f})(x) = 0$ for $x \in [0, \beta] \cup [x_0, 1]$ and $|(p(h) - 2\|f\|g_1 - \tilde{f})(x)| = |p(h)(x)| + |f(x)| \leqq 4\varepsilon$ for $x \in [\beta, x_0]$. By [6, Theorem 3.8] and [3, Lemma 1.1.5], we also have $p(h) \in \mathscr{D}(\delta)$,

$$\begin{aligned} &(\delta(p(h) - 2\|f\|g_1) - \widetilde{\delta(f)})(x) \\ &\quad = (p'(h)\delta(h) - 2\|f\|\delta(g_1) - \widetilde{\delta(f)})(x) \\ &\quad = 0 \quad \text{for} \quad x \in [0, \beta] \cup [x_0, 1] , \end{aligned}$$

and

$$\begin{aligned} &|(\delta(p(h) - 2\|f\|g_1) - \widetilde{\delta(f)})(x)| \\ &\quad \leqq 2|\delta(h)(x)| + 2\|f\|\ |\delta(g_1)(x)| + |\delta(f)(x)| \\ &\quad = 3|\delta(f)(x)| \leqq 3\eta \quad \text{for} \quad x \in [\beta, x_0] . \end{aligned}$$

It follows that $\|p(h) - 2\|f\|g_1 - \tilde{f}\| \leqq 4\varepsilon$ and $\|\delta(p(h) - 2\|f\|g_1) - \widetilde{g(f)}\| \leqq 3\eta$. Since we can take ε and η arbitrarily small, the closedness of δ implies that $\tilde{f} \in \mathscr{D}(\delta)$ and $\delta(\tilde{f}) = \widetilde{\delta(f)}$. This completes the proof.

LEMMA 2. *Let f_1 and f_2 be functions in $\mathscr{D}(\delta)$ such that $f_1(x_0) = f_2(x_0)$ and $\delta(f_1)(x_0) = \delta(f_2)(x_0)$ for some x_0 in $(0, 1)$. We define functions f and F in $C(I)$ by the following*:

$$f = \begin{cases} f_1 & \text{on} \quad [0, x_0] \\ f_2 & \text{on} \quad [x_0, 1] \end{cases} \quad \text{and} \quad F = \begin{cases} \delta(f_1) & \text{on} \quad [0, x_0] \\ \delta(f_2) & \text{on} \quad [x_0, 1] . \end{cases}$$

Then f belongs to $\mathscr{D}(\delta)$ and satisfies $\delta(f) = F$.

PROOF. By assumption we have $(f_1 - f_2)(x_0) = \delta(f_1 - f_2)(x_0) = 0$. Put

$$g = \begin{cases} f_1 - f_2 & \text{on} \quad [0, x_0] \\ 0 & \text{on} \quad [x_0, 1] \end{cases} \quad \text{and} \quad G = \begin{cases} \delta(f_1 - f_2) & \text{on} \quad [0, x_0] \\ 0 & \text{on} \quad [x_0, 1]\,. \end{cases}$$

Then Lemma 1 shows that $g \in \mathscr{D}(\delta)$ and $\delta(g) = G$, so that we have $f = g + f_2 \in \mathscr{D}(\delta)$ and $\delta(f) = G + \delta(f_2) = F$. This completes the proof.

We put $A_\delta = \{x \in I;\ \delta(f)(x) \neq 0 \text{ for some } f \text{ in } \mathscr{D}(\delta)\}$. Note that A_δ is an open set in I.

LEMMA 3. *Let x_0 be in A_δ and U an arbitrary neighborhood of x_0. Then there exists f in $\mathscr{D}(\delta)$ satisfying $\delta(f)(x_0) = 1$, $0 \leqq \delta(f) \leqq 2$, and* $\operatorname{supp} \delta(f) \subset U$, *where* $\operatorname{supp} \delta(f)$ *is the support of $\delta(f)$.*

PROOF. $x_0 \in A_\delta$ implies that there exists a function g in $\mathscr{D}(\delta)$ with $\delta(g)(x_0) = 1$. We assume $x_0 \in (0, 1)$. If x_0 is zero or one, we can also prove this lemma in a similar way. Take α and β in I in such a way that $[\alpha, \beta] \subset U$, $x_0 \in (\alpha, \beta)$, and $0 < \delta(g) \leqq 2$ on $[\alpha, \beta]$. By [3, Lemma 1.1.5], $\delta(g) \neq 0$ on $[\alpha, \beta]$ implies that there exist $x_1 \in (\alpha, x_0)$ and $x_2 \in (x_0, \beta)$ with $g(x_1) \neq g(x_0)$ and $g(x_2) \neq g(x_0)$. We shall consider only the case where $g(x_1) < g(x_0) < g(x_2)$. In the other case the proof is the same. Take a number k with $0 < k < \min\{g(x_0) - g(x_1),\ g(x_2) - g(x_0)\}$ and put $\alpha' = \max(\{\alpha\} \cup \{x \in [\alpha, x_1];\ g(x) = g(x_0) - k\})$ and $\beta' = \min(\{\beta\} \cup \{x \in [x_2, \beta];\ g(x) = g(x_0) + k\})$. Since $\mathscr{D}(\delta)$ is a Silov algebra, there exists a function h in $\mathscr{D}(\delta)$ satisfying $-1 \leqq h \leqq 0$ on $[\alpha', x_1]$, $0 \leqq h \leqq 1$ on $[x_2, \beta']$, and

$$h = \begin{cases} -1 & \text{on} \quad [0, \alpha'] \\ 0 & \text{on} \quad [x_1, x_2] \\ 1 & \text{on} \quad [\beta', 1]\,. \end{cases}$$

Then $e = g + 2\|g\|h$ is an element in $\mathscr{D}(\delta)$ such that $e(x) \notin [e(x_0) - k, e(x_0) + k]$ for $x \in [0, x_1] \cup [x_2, 1]$ and, by [3, Lemma 1.1.5], $\delta(e) = \delta(g)$ on $[x_1, x_2]$. Take a function p in $C^1(\boldsymbol{R})$ such that $0 \leqq p' \leqq 1$, $p' = 0$ on $\boldsymbol{R} \setminus [e(x_0) - k, e(x_0) + k]$, and $p'(e(x_0)) = 1$. Then, by [6, Theorem 3.8], $p(e)$ is a function in $\mathscr{D}(\delta)$ with $\delta(p(e)) = p'(e)\delta(e)$, so that we have $\delta(p(e))(x_0) = 1$, $0 \leqq \delta(p(e)) \leqq 2$, and $\operatorname{supp} \delta(p(e)) \subset [x_1, x_2] \subset U$. Setting $f = p(e)$, this completes the proof.

Let E be an arbitrary closed subinterval of I and denote the restriction of a function g in $C(I)$ to E by $g|_E$. We define the restriction δ_E of δ to E by $\delta_E(f|_E) = \delta(f)|_E$ for f in $\mathscr{D}(\delta)$. Then, by [3, Lemma 1.1.5], δ_E is well defined and becomes a derivation in $C(E)$ whose domain $\mathscr{D}(\delta_E)$ is $\{f|_E;\ f \in \mathscr{D}(\delta)\}$.

PROPOSITION 4. *Let E be a closed subinterval of I. Then δ_E is a closed derivation in $C(E)$.*

PROOF. Set $E=[x_0, x_1]$ $(x_0<x_1)$ and let f_n be a sequence in $\mathscr{D}(\delta_E)$ such that $f_n \to f$ and $\delta_E(f_n) \to F$ as $n \to \infty$ in $C(E)$. If $x_i \in A_\delta (i=0, 1)$, by Lemma 3, there exists h_i in $\mathscr{D}(\delta)$ such that $\delta(h_i)(x_i)=1$ and $\delta(h_i)(x_{1-i})=0$. If $x_i \in A_\delta$, we put $h_i=0$. Setting $g_n=f_n-\sum_{i=0,1}\delta_E(f_n)(x_i)h_i|_E$, we have $g_n \in \mathscr{D}(\delta_E)$, $\delta_E(g_n)(x_i)=0$, $\lim_{n\to\infty} g_n = f-\sum_{i=0,1}F(x_i)h_i|_E$, and $\lim_{n\to\infty}\delta_E(g_n)=F-\sum_{i=0,1}F(x_i)\delta(h_i)|_E$ in $C(E)$. We put

$$\tilde{g}_n(x)=\begin{cases} g_n(x_1) & \text{if} \quad x \geqq x_1 \\ g_n(x) & \text{if} \quad x_1 \geqq x \geqq x_0 \\ g_n(x) & \text{if} \quad x \leqq x_0 \,. \end{cases}$$

Since $\delta_E(g_n)(x_i)=0$ for all n and $i=0, 1$, by Lemma 2, $\tilde{g}_n$ belongs to $\mathscr{D}(\delta)$ and satisfies $\delta(\tilde{g}_n)|_E=\delta_E(g_n)$. Furthermore $\tilde{g}_n$ and $\delta(\tilde{g}_n)$ are Cauchy sequences in $C(I)$. From the closedness of δ, we have $\lim_{n\to\infty}\tilde{g}_n \in \mathscr{D}(\delta)$ and $\delta(\lim_{n\to\infty}\tilde{g}_n)=\lim_{n\to\infty}\delta(\tilde{g}_n)$, and it follows that $f-\sum_{i=0,1}F(x_i)h_i|_E \in \mathscr{D}(\delta_E)$ and $\delta_E(f-\sum_{i=0,1}F(x_i)h_i|_E)=F-\sum_{i=0,1}F(x_i)\delta(h_i)|_E$. Thus we have $f \in \mathscr{D}(\delta_E)$ and $\delta_E(f)=F$, so that δ_E is closed, this completes the proof.

We set $\mathscr{D}_x=\{f \in \mathscr{D}(\delta);\, f(x)=0\}$ for x in I and $B_\delta=\{x \in I;$ there exists a positive number K and an open interval U which contains x such that $\|f\|_U \leqq K\|\delta(f)\|_U$ for all $f \in \mathscr{D}_x\}$, where $\|\ \ \|_U$ is the uniform norm on U. Note that B_δ is an open subset of I.

LEMMA 5. *Let x_0 be in $I \setminus B_\delta$, ε an arbitrary positive number, and $J=(\alpha, \beta)$ an arbitrary open subinterval of I which contains x_0. Then there exists an element f in $\mathscr{D}(\delta)$ such that $0 \leqq f \leqq 1$, $f=1$ on $[\beta, 1]$, $f=0$ on $[0, \alpha]$, and $\|\delta(f)\| \leqq \varepsilon$.*

PROOF. By the definition of B_δ, $x_0 \in I \setminus B_\delta$ implies that there exists g in $\mathscr{D}_{x_0}$ with $\|g\|_J=4$, $\|\delta(g)\|_J \leqq \varepsilon$. Let x_1 be an element in $\bar{J}$ with $|g(x_1)|=4$. We may assume that $g(x_1)=4$ and $x_0<x_1$. Otherwise, the proof is the same. Put $\gamma=\min\{x>x_0;\, g(x)=1\}$ and $\sigma=\max\{x<x_1;\, g(x)=3\}$. Then we can find h in $\mathscr{D}(\delta)$ such that $-1 \leqq h \leqq 0$ on $[x_0, \gamma]$, $0 \leqq h \leqq 1$ on $[\sigma, x_1]$, and

$$h=\begin{cases} \ \ 1 & \text{on} \quad [x_1, 1] \\ \ \ 0 & \text{on} \quad [\gamma, \sigma] \\ -1 & \text{on} \quad [0, x_0] \,. \end{cases}$$

Let p be a C^1-function satisfying $0 \leqq p \leqq 1$, $0 \leqq p' \leqq 1$, $p(x)=0$ if $x \leqq 1$, and $p(x)=1$ if $x \geqq 3$. Putting $f=p(g+2\|g\|h)$, by [6, Theorem 3.8], we have $f \in \mathscr{D}(\delta)$, $\|\delta(f)\| = \|p'(g + 2\|g\|h)\delta(g + 2\|g\|h)\| \leqq \varepsilon$, $f = 0$ on $[0, \gamma]$, and $f = 1$ on $[\sigma, 1]$. This completes the proof.

We recell the closed derivations induced by non-atmic signed measures (cf. [4]). Let $E = [x_0, x_1]$ be a closed interval and μ a non-atomic measures on E with the support E. We define a linear mapping δ_μ in $C(E)$ by the following:

$$\delta_\mu\left(\lambda \mathbf{1}_E + \int_{x_0}^{\cdot} f d\mu\right) = f \quad \text{for} \quad f \text{ in } C(E) \quad \text{and} \quad \lambda \text{ in } \boldsymbol{R}\,,$$

where $\mathbf{1}_E$ is the unit element of $C(E)$. [4, Theorem 2.2] has shown that δ_μ is well defined and becomes a closed derivation in $C(E)$ whose domain is

$$\left\{\lambda \mathbf{1}_E + \int_{x_0}^{\cdot} f d\mu;\ f \in C(E) \quad \text{and} \quad \lambda \in \boldsymbol{R}\right\}.$$

Now we state our main theorem.

THEOREM 6. *Let δ be a closed derivation in $C(I)$. Then the following conditions are satisfied:*

(i) *$A_\delta \cap B_\delta$ is a dense open subset in A_δ.*

(ii) *There exists a continuous real-valued function μ on $A_\delta \cap B_\delta$ such that, for any closed interval E contained in $A_\delta \cap B_\delta$, the restriction μ_E of μ to E is a non-atomic signed measure on E and satisfies $\delta_E = \delta_{\mu_E}$.*

PROOF. We suppose that $A_\delta \cap B_\delta$ is not dense in A_δ. Then we can take a closed interval $J = [\alpha, \beta]$ with $\alpha < \beta$ and $J \subset A_\delta \cap B_\delta^c$, where B_δ^c is the complement of B_δ. By Proposition 4, the restriction δ_J of δ to J is a closed derivation in $C(J)$.

For an element g in $C(J)$ and a positive number ε, there exists a $C^1(J)$-function h with $\|h - g\|_J \leqq \varepsilon/2$. Furthermore we can find an integer n such that $n^{-1}(\beta - \alpha)\|h'\|_J \leqq 1$ and $|g(x) - g(y)| \leqq \varepsilon/2$ for every x and y in J with $|x - y| \leqq n^{-1}(\beta - \alpha)$. Put $x_k = \alpha + kn^{-1}(\beta - \alpha)$ for $k = 0, 1, \cdots, n$. Since $J \subset B_\delta^c$, Lemma 5 shows that, for $k = 1, 2, \cdots, n$, there exists an element f_k in $\mathscr{D}(\delta_J)$ satisfying $0 \leqq f_k \leqq 1$, $\|\delta_J(f_k)\|_J \leqq \varepsilon$, $f_k = 0$ on $[\alpha, x_{k-1}]$, and $f_k = 1$ on $[x_k, \beta]$. Note that $\delta_J(f_k) = 0$ on $[\alpha, x_{k-1}] \cup [x_k, \beta]$. If we set $f = \alpha \mathbf{1}_J + n^{-1}(\beta - \alpha)\sum_{k=1}^n f_k$, where $\mathbf{1}_J$ is the unit element in $C(J)$, we have $f \in \mathscr{D}(\delta_J)$, $\|\delta_J(f)\|_J \leqq n^{-1}\varepsilon(\beta - \alpha)$, $f(x_k) = x_k (k = 0, 1, \cdots, n)$, and $x_{k-1} \leqq f(x) \leqq x_k$ for $x_{k-1} \leqq x \leqq x_k$ $(k = 1, 2, \cdots, n)$. By [6, Theorem 3.8], we have $h(f) \in \mathscr{D}(\delta_J)$ and $\|\delta_J(h(f))\|_J = \|h'(f)\delta_J(f)\|_J \leqq n^{-1}\varepsilon(\beta - \alpha)\|h'\|_J \leqq \varepsilon$. On the other hand, if $x_{k-1} \leqq x \leqq x_k$, we also have $|(h(f) - g)(x)| \leqq |h(f(x)) - g(f(x))| + |g(f(x)) - g(x)| \leqq \varepsilon$, so that $\|h(f) - g\|_J \leqq \varepsilon$. Since ε is arbitrary and δ_J is closed, it follows that $g \in \mathscr{D}(\delta_J)$ and $\delta_J(g) = 0$. This is a contradiction, so that $A_\delta \cap B_\delta$ is dense in A_δ.

Now we prove the second part of the theorem. Let E be a closed interval in $A_\delta \cap B_\delta$. By $E \subset B_\delta$ and the compactness of E, there exist points x_i in E and open subintervals V_i of I and positive numbers K_i ($i = 1, 2, \cdots, n$) such that $x_i \in V_i$, $E \subset \bigcup_{i=1}^n V_i$, and $\| f \|_{V_i} \leqq K_i \| \delta(f) \|_{V_i}$ for f in $\mathscr{D}_{x_i}$. Then we have $\| f \|_V \leqq 2n(\max_i K_i) \| \delta(f) \|_V$ for f in $\mathscr{D}_\gamma$, where we put $E = [\gamma, \sigma]$ and $V = \bigcup_{i=1}^n V_i$. Let g be an arbitrary element in $\mathscr{D}(\delta_E)$ with $g(\gamma) = 0$. Lemma 3 shows that there exists h_1 in $\mathscr{D}(\delta)$ such that $\delta(h_1)(\gamma) = \delta_E(g)(\gamma)$, $\delta(h_1)(\sigma) = \delta_E(g)(\sigma)$, and $\| \delta(h_1) \| \leqq 2 \| \delta_E(g) \|_E$. Since $\mathscr{D}(\delta)$ is a Silov algebra, we can find h_2 in $\mathscr{D}(\delta)$ with $h_2 = g(\gamma) - h_1(\gamma) = -h_1(\gamma)$ on $[0, \gamma]$ and $h_2 = g(\sigma) - h_1(\sigma)$ on $[\sigma, 1]$. Note that $\delta(h_2) = 0$ on $[0, \gamma] \cup [\sigma, 1]$. We put

$$\tilde{g} = \begin{cases} g & \text{on} \quad [\gamma, \sigma] \\ h_1 + h_2 & \text{on} \quad [0, \gamma] \cup [\sigma, 1] . \end{cases}$$

Lemma 2 implies that $\tilde{g} \in \mathscr{D}(\delta)$ and $\| \delta(\tilde{g}) \| \leqq 2 \| \delta_E(g) \|_E$, so that we have

$$\begin{aligned} \| g \|_E \leqq \| \tilde{g} \|_V &\leqq 2n\Big(\max_i K_i\Big) \| \delta(\tilde{g}) \|_V \\ &\leqq 2n\Big(\max_i K_i\Big) \| \delta(\tilde{g}) \| \leqq 4n\Big(\max_i K_i\Big) \| \delta_E(g) \|_E . \end{aligned}$$

It follows that the kernel $K(\delta_E)$ of δ_E is $\{\lambda \mathbf{1}_E, \lambda \in \boldsymbol{R}\}$ and the range $R(\delta_E)$ of δ_E is a closed linear subspace in $C(E)$.

Now we show that $R(\delta_E) = C(E)$. By Lemma 3 and the compactness of E, there is an element ν in $\mathscr{D}(\delta)$ with $\delta(\nu)(x) \neq 0$ on E. Taking $(\delta(\nu)|_E)^{-1}\delta_E$ instead of δ_E if necessary, we may assume that $R(\delta_E)$ contains $\mathbf{1}_E$, where $\mathbf{1}_E$ is the unit element of $C(E)$. Let K be an arbitrary subinterval of E and $\chi(K)$ a characteristic function of K. Then, by Lemmas 2 and 3, the same argument as above implies that there exists a sequence g_n in $\mathscr{D}(\delta_E)$ such that $\delta_E(g_n)$ pointwise converges to $\chi(K)$ and $\| \delta_E(g_n) \|_E \leqq 2 \| \mathbf{1}_E \|_E = 2$. Suppose that ϕ is a continuous linear functional on $C(E)$ such that $\phi(R(\delta_E)) = 0$. Then we have $\int_E \chi(K) d\phi = \lim_{n\to\infty} \phi(\delta_E(g_n)) = 0$ so that $\phi = 0$. Since $R(\delta_E)$ is a closed subspace of $C(E)$, by the Hahn-Banach theorem, we have $R(\delta_E) = C(E)$. It follows from [4, Theorem 2.3] that there exists a unique non-atomic signed measure μ_E on E such that $\delta_E = \delta_{\mu_E}$.

Let G be a connected component of $A_\delta \cap B_\delta$ and G_n a sequence of closed subintervals of G such that $G_n \subset G_{n+1}$ and $\bigcup_n G_n = G$. By the above argument, for each G_n, there is a non-atomic signed measure μ_{G_n} with $\delta_{G_n} = \delta_{\mu_{G_n}}$. The uniqueness of μ_{G_n} implies that μ_{G_n} is the restriction of $\mu_{G_{n+1}}$ to G_n. Considering μ_{G_n} as a function of bounded variation on

G_n which is the restriction of $\mu_{G_{n+1}}$ to G_n, we put $\mu(x) = \lim_{n\to\infty} \mu_{G_n}(x)$ for x in G. Since μ_{G_n} is non-atomic, μ is continuous. Thus we get a continuous function μ on $A_\delta \cap B_\delta$ and our assertion follows from the uniqueness of measure.

REMARK 7. Let δ_0 be the closed extension of the usual derivative d/dx whose kernel is the closed subalgebra of $C(I)$ generated by the Cantor function and the unit element of $C(I)$ (cf. [3]). Then we have $A_{\delta_0} = I$ and B_{δ_0} is the complement of the Cantor set.

REMARK 8. It follows from the proof of [4, Theorem 2.2] that there exists a dense subset U of $A_\delta \cap B_\delta$ such that

$$\delta(f)(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{\mu(x+h) - \mu(x)} \quad \text{for} \quad f \text{ in } \mathscr{D}(\delta) \quad \text{and} \quad x \text{ in } U\,.$$

We set $M_\mu = \{x \in A_\delta \cap B_\delta$; there exists a neighborhood of x on which μ is monotone$\}$. The following corollary is clearly verified by [4, Theorem 3.1].

COROLLARY 9. *Let the notation be the same as in Theorem* 6. *Then δ is quasi well-bebaved if and only if M_μ is dense in $A_\delta \cap B_\delta$.*

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