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# CLASSICAL SCHOTTKY GROUPS OF REAL TYPE OF GENUS TWO, I

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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**0. Introduction.** With respect to the boundaries of the Schottky spaces many results were obtained by Chuckrow, Marden, Bers, Sato, Rodriguez, and others. Chuckrow [2] studied the boundary groups of the space as limits of sequences of Schottky groups, and Marden [6] showed by studying the boundary groups of the Schottky space and the classical Schottky space that not every Schottky group is classical. On the other hand, Bers [1] and Sato [12], [13] studied the augmented Schottky space obtained by adding points representing compact Riemann surfaces with nodes to the Schottky space. Sato [14] obtained a uniformization theorem of compact Riemann surfaces with nodes. The boundary of the Schottky space consists of the boundary points due to Chuckrow, namely some discrete groups, and the boundary points due to Bers-Sato, namely points representing compact Riemann surfaces with nodes (Rodriguez [11]).

In this paper we will consider a subspace of the classical Schottky space of genus two, which is called the classical Schottky space of real type of genus two, and we will determine the structure of the boundary of the subspace. The space consists of all equivalence classes of marked classical Schottky groups generated by the following Möbius transformations $A_1$ and $A_2$: $A_j(z) = (a_j z + b_j)/(c_j z + d_j)$ $(a_j, b_j, c_j, d_j \in \boldsymbol{R}, a_j d_j - b_j c_j \neq 0; j = 1, 2)$. We will divide the space into eight subspaces according to type (see §1). The space of the fourth type corresponds to the Teichmüller space for tori with a hole, with respect to which beautiful results were obtained by Keen [3].

This is the first part of a series of papers entitled "Classical Schottky groups of real type of genus two". In the first part, we will only consider the spaces of the first and fourth types. The first part has the following three aims: (1) to represent the shape of the spaces of the first and fourth types by using the coordinates introduced in Sato [12], [13] (Theorems 1 and 4); (2) to determine fundamental regions for the Schottky

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modular group of genus two acting on the above spaces (Theorems 2 and 5); (3) to consider which Riemann surface a point on the closure of the spaces represents. By combining our results (Theorems 1 and 4) with Purzitsky [9, Theorems 2 and 3], we see that the Schottky spaces of the first and fourth types coincide with the classical Schottky spaces of the same types, respectively.

In §1 we will state definitions and divide the Schottky space of real type into eight subspaces according to type. In §2 we will consider automorphisms of the free group on two generators and list properties of the automorphisms in a series of lemmas. In §3 we will represent the shape of the classical Schottky space of the first type by using the coordinates introduced in Sato [12], [13]. In §4 we will determine a fundamental region for the Schottky modular group acting on the space. In §5 we will consider which Riemann surface a point on the closure of the space represents. In §6 we will treat the classical Schottky space of the fourth type. In §7 we will consider the relationship between the Teichmüller space in Keen [3] for tori with a hole and the space of the fourth type.

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## 1. Definitions.

1.1. DEFINITION 1.1. Let $C_1, C_{g+1}; \cdots; C_g, C_{2g}$ be a set of $2g$, $g \geqq 1$, mutually disjoint Jordan curves on the Riemann sphere which comprise the boundary of a $2g$-ply connected region $\omega$. Suppose there are $g$ Möbius transformations $A_1, \cdots, A_g$ which have the property that $A_j$ maps $C_j$ onto $C_{g+j}$ and $A_j(\omega) \cap \omega = \emptyset$, $1 \leqq j \leqq g$. Then the $g$ necessarily loxodromic transformations $A_j$ generate a *marked Schottky group* $G = \langle A_1, \cdots, A_g \rangle$ of genus $g$ with $\omega$ as a fundamental region. In particular, if all $C_j$ ($j = 1, 2, \cdots, 2g$) are circles, then we call $A_1, \cdots, A_g$ a *set of classical generator* of $G$. A *classical Schottky group* is a Schottky group for which there exists some set of classical generators.

DEFINITION 1.2. We say two marked Schottky groups $G = \langle A_1, \cdots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1, \cdots, \hat{A}_g \rangle$ to be *equivalent* and denote the fact by $G \sim \hat{G}$ if there exists a Möbius transformation $T$ such that $\hat{A}_j = TA_jT^{-1}$ for all $j = 1, 2, \cdots, g$. The *Schottky space* of genus $g$, denoted by $\mathfrak{S}_g$, is the set of all equivalence classes of marked Schottky groups of genus $g \geqq 1$.

DEFINITION 1.3. The *classical Schottky space*, denoted by $\mathfrak{S}_g^0$, is defined to consist of all elements of $\mathfrak{S}_g$ for which there exists some set of classical generators. We denote by $\mathfrak{S}_g^{00}$ the set of all equivalence classes of marked Schottky groups $G = \langle A_1, \cdots, A_g \rangle$ of genus $g$ such that $A_1, \cdots, A_g$ is a set of classical generators.

1.2. In this paper, we only consider the spaces $\mathfrak{S}_2$, $\mathfrak{S}_2^0$, and $\mathfrak{S}_2^{00}$ of genus $g = 2$. Let $G = \langle A_1, A_2 \rangle$ be a marked Schottky group. Let $\lambda_j$ ($|\lambda_j| > 1$), $p_j$ and $p_{2+j}$ ($j = 1, 2$) be the multiplier, the repelling and the attracting fixed points of $A_j$, respectively. We define $t_j$ by setting $t_j = 1/\lambda_j$ ($j = 1, 2$). Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We determine a Möbius transformation $T$ by $T(p_1) = 0$, $T(p_3) = \infty$ and $T(p_2) = 1$ and define $\rho$ by $\rho = T(p_4)$. Thus $\rho \in \boldsymbol{C} - \{0, 1\}$.

REMARKS. (1) Let $G_0 = \langle A_{10}, A_{20} \rangle$ be a fixed marked Schottky group. Let $C_{10}$, $C_{30}$; $C_{20}$, $C_{40}$ be mutually disjoint Jordan curves and $\omega_0$ a fundamental region as in Definition 1. Let $C_{50}$ be a Jordan curve in $\omega_0$ such that $C_{10}$ and $C_{30}$ (resp. $C_{20}$ and $C_{40}$) are in the interior (resp. the exterior) of $C_{50}$. Then $\Sigma_0 = \{C_{10}, C_{20}, C_{30}, C_{40}; C_{50}\}$ is a standard system of Jordan curves (see [14, p. 558] for the definition).

(2) We can define a mapping $\phi$ of $\mathfrak{S}_2$ into $D^{*2} \times (\boldsymbol{C} - \{0, 1\})$ by setting $\phi([G]) = (t_1, t_2, \rho)$, where $[G]$ denotes the equivalence class of $G$, that is, a point in $\mathfrak{S}_2$. We denote by $\mathfrak{S}_2(\Sigma_0)$ the image of $\mathfrak{S}_2$ under the mapping $\phi$. We call $\mathfrak{S}_2(\Sigma_0)$ the Schottky space associated with $\Sigma_0$. We similarly define the classical Schottky space associated with $\Sigma_0$, which is denoted by $\mathfrak{S}_2^0(\Sigma_0)$, and we can similarly define the space $\mathfrak{S}_2^{00}(\Sigma_0)$. Conversely $\lambda_1$, $\lambda_2$ and $p_4$ are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in D^{*2} \times (\boldsymbol{C} - \{0, 1\})$ under the normalization condition $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$; we define $\lambda_j$ ($j = 1, 2$) and $p_4$ by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$, respectively. We determine $A_1(\tau, z)$, $A_2(\tau, z) \in$ Möb from $\tau$ as follows: The multiplier, the repelling and the attracting fixed points of $A_j(\tau, z)$ are $\lambda_j$, $p_j$ and $p_{2+j}$, respectively. Thus we obtain a mapping $\psi$ of $D^{*2} \times (\boldsymbol{C} - \{0, 1\})$ into Möb by setting $\psi(\tau) = \langle A_1(\tau, z), A_2(\tau, z) \rangle$. Then we note that $\psi\phi = \mathrm{id}$. (see [13, pp. 28–29] for the detail).

DEFINITION 1.4. A Möbius transformation $A(z) = (az + b)/(cz + d)$ is called a *real Möbius transformation* if $a, b, c, d \in \boldsymbol{R}$ *and* $ad - bc \neq 0$. If $G = \langle A_1, \cdots, A_g \rangle$ is a marked (classical) Schottky group such that $A_j$ is a real Möbius transformation for each $j = 1, 2, \cdots, g$, (that is, $\phi([G]) \in \boldsymbol{R}^3$ if $g = 2$), then we call $G$ a marked (classical) Schottky group of *real type*.

In the case of $g = 2$, there are eight kinds of marked (classical) Schottky groups of real type as follows.

DEFINITION 1.5. Let $(t_1, t_2, \rho)$ be the point in $\mathfrak{S}_2(\Sigma_0)$ corresponding to the equivalence class $[G]$ of a marked Schottky group $G = \langle A_1, A_2 \rangle$.

(1) $G$ is of the *first type* (Type I) if $t_1 > 0$, $t_2 > 0$ and $\rho > 0$.
(2) $G$ is of the *second type* (Type II) if $t_1 > 0$, $t_2 < 0$ and $\rho > 0$.
(3) $G$ is of the *third type* (Type III) if $t_1 > 0$, $t_2 < 0$ and $\rho < 0$.
(4) $G$ is of the *fourth type* (Type IV) if $t_1 > 0$, $t_2 > 0$ and $\rho < 0$.
(5) $G$ is of the *fifth type* (Type V) if $t_1 < 0$, $t_2 > 0$ and $\rho > 0$.
(6) $G$ is of the *sixth type* (Type VI) if $t_1 < 0$, $t_2 < 0$ and $\rho > 0$.
(7) $G$ is of the *seventh type* (Type VII) if $t_1 < 0$, $t_2 < 0$ and $\rho < 0$.
(8) $G$ is of the *eighth type* (Type VIII) if $t_1 < 0$, $t_2 > 0$ and $\rho < 0$.

DEFINITION 1.6. For each $k = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VIII}$, we call the set of all equivalence classes of marked Schottky groups of Type $k$ or the set of all points $(t_1, t_2, \rho)$ of Type $k$ the *real Schottky space of Type* $k$, and denoted it by $R_k\mathfrak{S}_2$.

DEFINITION 1.7. For each $k = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VIII}$, the intersection of $R_k\mathfrak{S}_2$ with $\mathfrak{S}_2^0(\Sigma_0)$ is called the *real classical Schottky space of Type* $k$, and is denoted by $R_k\mathfrak{S}_2^0$. We denote by $R_k\mathfrak{S}_2^{00}$ the intersection of $R_k\mathfrak{S}_2$ with $\mathfrak{S}_2^{00}(\Sigma_0)$ for each $k = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VIII}$.

## 2. Automorphisms of a free group on two generators.

2.1. Let $G = \langle A_1, A_2 \rangle$ be a free group on two generators. We will use the following theorem in §§4 and 6.

THEOREM A (Neumann [6]). *Then group* $\Phi_2$ *of automorphisms of* $G$ *has the following presentation*:

$$\Phi_2 = \langle N_1, N_2, N_3 \,|\, (N_2N_1N_2N_3)^2 = 1,\ N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1 = 1,\ N_1N_3N_1N_3 = N_3N_1N_3N_1 \rangle\ ,$$

*where*

$$N_1\colon (A_1, A_2) \mapsto (A_1, A_2^{-1})\ , \quad N_2\colon (A_1, A_2) \mapsto (A_2, A_1)$$

*and*

$$N_3\colon (A_1, A_2) \mapsto (A_1, A_1A_2)\ .$$

Let $G = \langle A_1, A_2 \rangle$ be a marked Schottky group. Then $S = \Omega(G)/G$ is a compact Riemann surface of genus 2, where $\Omega(G)$ is the region of discontinuity of $G$. The homeomorphisms of the Riemann surface $S$ induced by the elements of $\Phi_2$ include orientation preserving as well as orientation reversing ones. The mapping $N_3$ preserves orientation, while $N_1$ and $N_2$ reverse orientation.

2.2. Let $E$ be the space of marked free groups on two generators. By defining $N_j$ ($j = 1, 2, 3$) as in Theorem A for each $G = \langle A_1, A_2\rangle \in E$, we can regard $N_j$ as automorphisms of $E$. If $G = \langle A_1, A_2\rangle$ and $\hat{G} = \langle \hat{A}_1, \hat{A}_2\rangle$ are equivalent marked free groups on two generators, then $N_j(\langle A_1, A_2\rangle)$ is equivalent to $N_j(\langle \hat{A}_1, \hat{A}_2\rangle)$ for each $j = 1, 2, 3$. Therefore we can regard $N_j$ ($j = 1, 2, 3$) as automorphisms of the space of all equivalence classes of marked free groups on two generators.

DEFINITION 2.1. Let $\phi_1$ and $\phi_2$ be automorphisms of $G = \langle A_1, A_2\rangle$. We say $\phi_1$ and $\phi_2$ are equivalent if $\phi_1(G)$ is equivalent to $\phi_2(G)$, and denote this by $\phi_1 \sim \phi_2$.

DEFINITION 2.2. Let $G = \langle A_1, A_2\rangle$ be a marked Schottky group and $\Phi_2$ the group of automorphisms of $G$. The *modular group* of the Schottky space of genus 2 or the *Schottky modular group* of genus 2, which is denoted by $\mathrm{Mod}(\mathfrak{S}_2)$, is the set of all equivalence classes of orientation preserving automorphisms in $\Phi_2$. We denote by $[\Phi_2]$ the set of all equivalence classes of automorphisms in $\Phi_2$.

Let $(t_1, t_2, \rho)$ be the point in $\mathfrak{S}_2(\Sigma_0)$ corresponding to a marked Schottky group $G = \langle A_1, A_2\rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of $(t_1, t_2, \rho)$ under the mappings $N_j$ ($j = 1, 2, 3$), that is, $(t_1(1), t_2(1), \rho(1))$, $(t_1(2), t_2(2), \rho(2))$ and $(t_1(3), t_2(3), \rho(3))$ are the points in $\mathfrak{S}_2(\Sigma_0)$ corresponding to marked Schottky groups $\langle A_1, A_2^{-1}\rangle$, $\langle A_2, A_1\rangle$ and $\langle A_1, A_1A_2\rangle$, respectively. Let $p$ and $q$ be two solutions of the equation

$$t_1(1 - t_2)z^2 - (\rho - t_2 - \rho t_1t_2 + t_1)z + \rho(1 - t_2) = 0\ .$$

We set $X = \rho - t_2 - \rho t_1 t_2 + t_1$ and $Y = \rho - t_2 + \rho t_1 t_2 - t_1$. Then by easy calculation, we have the following.

LEMMA 2.1. (1) $t_1(1) = t_1$, $t_2(1) = t_2$ *and* $\rho(1) = 1/\rho$.
(2) $t_1(2) = t_2$, $t_2(2) = t_1$ *and* $\rho(2) = \rho$.
(3) $t_1(3) = t_1$, $t_2(3)^{1/2} + 1/t_2(3)^{1/2} = Y/(t_1^{1/2}t_2^{1/2}(\rho - 1))$,

*and for Type* I

$$\rho(3)^{1/2} + 1/\rho(3)^{1/2} = \begin{cases} X/(\rho^{1/2}t_1^{1/2}(1 - t_2)) & \text{if} \quad p > 0 \quad \text{and} \quad q > 0 \\ -X/(\rho^{1/2}t_1^{1/2}(1 - t_2)) & \text{if} \quad p < 0 \quad \text{and} \quad q < 0\ , \end{cases}$$

*and for Type* IV,

$$\begin{aligned} &(-\rho(3))^{1/2} + 1/(-\rho(3))^{1/2} \\ &\qquad = (X^2 - 4t_1\rho(1 - t_2))^{1/2}/(-\rho)^{1/2}t_1^{1/2}(1 - t_2)\ . \end{aligned}$$

2.3. Let $A(z)=(az+b)/(cz+d)$, $ad-bc=1$, be a loxodromic transformation. We denote by $\lambda(A)$ ($|\lambda(A)|>1$), $p(A)$ and $q(A)$ the multiplier, the repelling and the attracting fixed points of $A$, respectively. We define $t(A)$ and $\rho(A)$ by $t(A)=1/\lambda(A)$ and $\rho(A)=q(A)/p(A)$, respectively, if $p(A)\neq 0$. Let $(t_1, t_2, \rho)$ be the point in $\mathfrak{S}_2(\Sigma_0)$ corresponding to a marked Schottky group $G=\langle A_1, A_2\rangle$. We can set

$$A_1(z)=z/t_1$$

and

$$A_2(z)=((\rho-t_2)z+\rho(t_2-1))/((1-t_2)z+(t_2\rho-1))\ .$$

We note that $t_j=t(A_j)$ $(j=1,2)$ and $\rho=\rho(A_2)$. We denote by $\boldsymbol{Z}$ the set of all integers. We easily see the following lemmas.

LEMMA 2.2. *$\langle A_1^{-1}, A_2^{-1}\rangle$ is equivalent to $\langle A_1, A_2\rangle$.*

PROOF. We set $T(z)=\rho/z$. Then $TA_1^{-1}T^{-1}=A_1$ and $TA_2^{-1}T^{-1}=A_2$. q.e.d.

LEMMA 2.3. *If $l+m=n$, $l, m, n\in\boldsymbol{Z}$, then $\langle A_1, A_1^nA_2\rangle$ is equivalent to $\langle A_1, A_1^mA_2A_1^l\rangle$.*

LEMMA 2.4. *Let $t_1$ be a real number with $0<t_1<1$. Assume that $1/t_1^k<\rho<1/t_1^{k+1}$ for an integer $k$. Let $m, n\in\boldsymbol{Z}$. Then*

(1) $\rho(A_1^nA_2)>\rho(A_1^mA_2)$ *if* $n>m$;

(2) $t(A_1^nA_2)<t(A_1^mA_2)$ *if* $|n+k|>|m+k|$ *and* $t(A_1^nA_2)=t(A_1^mA_2)$ *if* $|n+k|=|m+k|$.

LEMMA 2.5. *Let $t_1$ be a real number with $0<t_1<1$. Assume that $-1/t_1^{k+1}<\rho<-1/t_1^k$ for an integer $k$. Let $m, n\in\boldsymbol{Z}$. Then*

(1) $\rho(A_1^nA_2)<\rho(A_1^mA_2)<0$ *if* $n>m$;

(2) $t(A_1^nA_2)<t(A_1^mA_2)$ *if* $|n+k|>|m+k|$ *and* $t(A_1^nA_2)=t(A_1^mA_2)$ *if* $|n+k|=|m+k|$.

We define $\tau(A)^2$ by $\tau(A)^2=t(A)+(1/t(A))+2$ for a Möbius transformation $A(z)$. By noting that

$$\tau(A_1^nA_2)^2-\tau(A_1^nA_2^{-1})=\frac{\rho+1}{\rho-1}\,\frac{(1-t_1^{2n})(1-t_2^n)}{t_1^nt_2}\ ,$$

we have the following lemmas:

LEMMA 2.6. (1) *Let $\rho>1$ and let $n$ be an integer.*

(i) *If $n>0$, then $t(A_1^nA_2)<t(A_1^nA_2^{-1})$ and $\rho(A_1^nA_2)>\rho(A_1^nA_2^{-1})$.*

(ii) *If $n<0$, then $t(A_1^nA_2)>t(A_1^nA_2^{-1})$ and $\rho(A_1^nA_2)>\rho(A_1^nA_2^{-1})$.*

(2) *Let $0<\rho<1$ and let $n$ be an integer.*

(i) *If* $n > 0$, *then* $t(A_1^n A_2) > t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) < \rho(A_1^n A_2^{-1})$.
(ii) *If* $n < 0$, *then* $t(A_1^n A_2) < t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) < \rho(A_1^n A_2^{-1})$.

LEMMA 2.7. (1) *Let* $\rho < -1$ *and let* $n$ *be an integer.*
(i) *If* $n > 0$, *then* $t(A_1^n A_2) < t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) < \rho(A_1^n A_2^{-1})$.
(ii) *If* $n < 0$, *then* $t(A_1^n A_2) > t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) < \rho(A_1^n A_2^{-1})$.
(2) *Let* $-1 < \rho < 0$ *and let* $n$ *be an integer.*
(i) *If* $n > 0$, *then* $t(A_1^n A_2) > t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) > \rho(A_1^n A_2^{-1})$.
(ii) *If* $n < 0$, *then* $t(A_1^n A_2) < t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) > \rho(A_1^n A_2^{-1})$.
(3) *Let* $\rho = -1$ *and let* $n$ *be an integer. Then* $t(A_1^n A_2) = t(A_1^n A_2^{-1})$ *and* $\rho(A_1^n A_2) = \rho(A_1^n A_2^{-1})$.

LEMMA 2.8. *Let* $l, m, n \in \boldsymbol{Z}$. *Then*
(1) $t(A_1^n A_2^{-1}) = t(A_1^{-n} A_2)$ *and* $\rho(A_1^n A_2^{-1})\rho(A_1^{-n} A_2) = 1$.
(2) $t(A_1^n A_2) = t(A_1^{-n} A_2^{-1})$ *and* $\rho(A_1^n A_2)\rho(A_1^{-n} A_2^{-1}) = 1$.
(3) $t(A_1^n A_2^m) = t(A_2^m A_1^n)$ *and* $\rho(A_1^n A_2^m) = \rho(A_2^m A_1^n)$.
(4) $t(A_1^n A_2^m A_1^l) = t(A_1^{n+l} A_2^m)$ *and* $\rho(A_1^n A_2^m A_1^l) = \rho(A_1^{n+l} A_2^m)$.

## 3. The first type——The domain of existence.

3.1. In this section we will determine the shape of the real classical Schottky space $R_{\mathrm{I}}\mathfrak{S}_2^0$ of type I in $\boldsymbol{R}^3$. Throughout this section let

$$A_1(z) = z/t_1$$

and

$$A_2(z) = ((\rho - t_2)z + \rho(t_2 - 1))/((1 - t_2)z + (\rho t_2 - 1)) ,$$

where $0 < t_1 < 1$, $0 < t_2 < 1$ and $\rho > 0$. The following proposition is fundamental in this section.

PROPOSITION 3.1. *Fix* $t_1$ *with* $0 < t_1 < 1$.
(1) *Let* $1 < \rho < 1/t_1$. *If* $t_2^{1/2} = (1 - t_1^{1/2}\rho^{1/2})/(\rho^{1/2} - t_1^{1/2})$, *then* $A_1 A_2^{-1}$ *is a parabolic transformation whose fixed point is* $\rho^{1/2}t_1^{-1/2}$. *Furthermore* $G = \langle A_1, A_2\rangle$ *is a discontinuous group and the region bounded by the following four circles* $C_1$, $C_2$, $C_3$ *and* $C_4$ *is a fundamental region for* $G$:
$C_1$: $|z| = \rho^{1/2}t_1^{1/2}$,
$C_2$: $|z - \{(1 + \rho)/2 + \rho^{1/2}t_1^{1/2}\}/2| = \{(1 + \rho)/2 - \rho^{1/2}t_1^{1/2}\}/2$,
$C_3$: $|z| = \rho^{1/2}t_1^{-1/2}$,
$C_4$: $|z - \{\rho^{1/2}t_1^{-1/2} + (\rho + t_2)/(t_2 + 1)\}/2| = \{\rho^{1/2}t_1^{-1/2} - (\rho + t_2)/(t_2 + 1)\}/2$.
(2) *Let* $t_1 < \rho < 1$. *If*

$$t_1^{1/2} = (\rho^{1/2} - t_1^{1/2})/(1 - t_1^{1/2}\rho^{1/2}) ,$$

*then* $A_1 A_2$ *is a parabolic transformation whose fixed point is* $\rho^{1/2}t_1^{-1/2}$. *Furthermore* $G = \langle A_1, A_2\rangle$ *is a discontinuous group and the region* *bounded by the following four circles $C_1$, $C_2$, $C_3$ and $C_4$ is a fundamental region for $G$:*

$C_1$: $|z| = \rho^{1/2}t_1^{1/2}$,

$C_2$: $|z - \{\rho^{1/2}t_1^{-1/2} + (1+\rho)/2\}/2| = \{\rho^{1/2}t_1^{-1/2} - (1+\rho)/2\}/2$,

$C_3$: $|z| = \rho^{1/2}t_1^{-1/2}$,

$C_4$: $|z - \{(\rho + t_2)/(t_2+1) + \rho^{1/2}t_1^{1/2}\}/2| = \{(\rho + t_2)/(t_2+1) - \rho^{1/2}t_1^{1/2}\}/2$.

This proposition is proved by straightforward computations.

For the sake of simplicity, we introduce the following notation:

$$t_{2,n}(t_1, \rho) = (1 - t_1^{n/2}\rho^{1/2})/(\rho^{1/2} - t_1^{n/2})$$

for each integer $n$.

PROPOSITION 3.2. *Fix $t_1$ with $0 < t_1 < 1$.*

(1) *Let $1/t_1^{n-1} < \rho < 1/t_1^n$ for a positive integer $n$. Then the intersection point of two curves*

$$K^-(n)\colon t_2^{1/2} = -t_{2,n-1}(t_1, \rho) \quad \textit{and} \quad K^+(n)\colon t_2^{1/2} = t_{2,n}(t_1, \rho)$$

*is $P(n) := (t_1, t_{2,n}(t_1), \rho_n(t_1))$, where*

$$t_{2,n}(t_1)^{1/2} = \frac{t_1^{(n-1)/2}\{(1-t_1^n)^{1/2} - t_1^{1/2}(1-t_1^{n-1})^{1/2}\}}{(1-t_1^{n-1})^{1/2} + (1-t_1^n)^{1/2}}$$

*and*

$$\rho_n(t_1)^{1/2} = \frac{t_1^{(2n-1)/2} + 1 + \{(1-t_1^{n-1})(1-t_1^n)\}^{1/2}}{t_1^{(n-1)/2}(t_1^{1/2}+1)} .$$

(2) *Let $t_1^n < \rho < t_1^{n-1}$ for a positive integer $n$. Then the intersection point of two curves*

$$K^-(-n)\colon t_2^{1/2} = -t_{2,n-1}(t_1, \rho)^{-1} \quad \textit{and} \quad K^+(-n)\colon t_2^{1/2} = t_{2,n}(t_1, \rho)^{-1}$$

*is $P(-n) := (t_1, t_{2,-n}(t_1), \rho_{-n}(t_1))$, where*

$$t_{2,-n}(t_1)^{1/2} = \frac{(1-t_1^n)^{1/2} - (1-t_1^{n-1})^{1/2}}{t_1^{(n-1)/2}\{t_1^{1/2}(1-t_1^{n-1})^{1/2} + (1-t_1^n)^{1/2}\}}$$

*and*

$$\rho_{-n}(t_1)^{1/2} = \frac{t_1^{(2n-1)/2} + 1 - \{(1-t_1^{n-1})(1-t_1^n)\}^{1/2}}{t_1^{(n-1)/2}(t_1^{1/2}+1)} .$$

We have this proposition by elementary calculations.

REMARKS. (1) $t_{2,n}(t_1) = t_{2,-n}(t_1)$, $\rho_n(t_1)\rho_{-n}(t_1) = 1$ and $P(1) = P(-1)$ (cf. Lemma 2.8).

(2) $\{\rho_n(t_1)\}$ and $\{t_{2,n}(t_1)\}$ $(n = 1, 2, 3, \cdots)$ are monotone increasing and decreasing sequences, respectively.

(3) Both $\{\rho_{-n}(t_1)\}$ and $\{t_{2,-n}(t_1)\}$ $(n = 1, 2, 3, \cdots)$ are monotone decreasing sequences.

PROPOSITION 3.3. *Fix $t_1$ with $0 < t_1 < 1$.*

(1) *Let $1/t_1^{n-1} < \rho < \rho_n(t_1)$ $(n = 2, 3, \cdots)$. If $t_2^{1/2} = -t_{2,n-1}(t_1, \rho)$, then $A_1^{n-1}A_2^{-1}$ is parabolic.*

(2) *Let $\rho_n(t_1) < \rho < 1/t_1^n$ $(n = 1, 2, \cdots)$. If $t_2^{1/2} = t_{2,n}(t_1, \rho)$, then $A_1^n A_2^{-1}$ is parabolic.*

(3) *Let $\rho_{-n}(t_1) < \rho < t_1^{n-1}$ $(n = 2, 3, \cdots)$. If $t_2^{1/2} = -t_{2,n-1}(t_1, \rho)^{-1}$, then $A_1^{n-1}A_2$ is parabolic.*

(4) *Let $t_1^n < \rho < \rho_{-n}(t_1)$ $(n = 1, 2, \cdots)$. If $t_2^{1/2} = t_{2,n}(t_1, \rho)^{-1}$, then $A_1^n A_2$ is parabolic.*

3.2. We introduce the following sets in $\boldsymbol{R}^3$. For the sake of simplicity, we write $\tau$ for a point $(t_1, t_2, \rho) \in \boldsymbol{R}^3$ in the definitions. We set

$$H(n) = \{\tau \mid t_2 = 0,\ t_1^{-(n-1)} < \rho < t_1^{-n},\ 0 < t_1 < 1\} \quad (n \geqq 1)\ ,$$
$$H(-n) = \{\tau \mid t_2 = 0,\ t_1^{n} < \rho < t_1^{n-1},\ 0 < t_1 < 1\} \quad (n \geqq 1)\ ,$$
$$H(1, n) = \{\tau \mid t_1 = 0,\ t_2^{-(n-1)} < \rho < t_2^{-n},\ 0 < t_2 < 1\} \quad (n \geqq 1)\ ,$$
$$H(-1, -n) = \{\tau \mid t_1 = 0,\ t_2^{n} < \rho < t_2^{n-1},\ 0 < t_2 < 1\} \quad (n \geqq 1)\ ,$$
$$F^-(n) = \{\tau \mid t_2^{1/2} = -t_{2,n-1}(t_1, \rho),\ t_1^{-(n-1)} < \rho < \rho_n(t_1),\ 0 < t_1 < 1\} \quad (n \geqq 2)\ ,$$
$$F^-(-n) = \{\tau \mid t_2^{1/2} = -t_{2,n-1}(t_1, \rho)^{-1},\ \rho_n(t_1) < \rho < t_1^{n-1},\ 0 < t_1 < 1\} \quad (n \geqq 2)\ ,$$
$$F^+(n) = \{\tau \mid t_2^{1/2} = t_{2,n}(t_1, \rho),\ \rho_n(t_1) < \rho < t_1^{-n},\ 0 < t_1 < 1\} \quad (n \geqq 1)\ ,$$
$$F^+(-n) = \{\tau \mid t_2^{1/2} = t_{2,n}(t_1, \rho)^{-1},\ t_1^{n} < \rho < \rho_{-n}(t_1),\ 0 < t_1 < 1\} \quad (n \geqq 1)\ ,$$
$$F_0 = \{\tau \mid 0 < t_1 < 1,\ 0 < t_2 < 1,\ \rho = 1\}\ .$$

3.3. We now inductively introduce many surfaces from $F^+(n)$ and $F^-(n)$. For $n_0 = 1, 2, 3, \cdots$ and $m_0 = 2, 3, \cdots$, we let

$$F^+(1, n_0) = N_2(F^+(n_0)) \quad (\text{resp. } F^+(-1, -n_0) = N_2(F^+(-n_0)))\ ,$$

and

$$F^-(1, m_0) = N_2(F^-(m_0)) \quad (\text{resp. } F^-(-1, -m_0) = N_2(F^-(-n_0)))\ ,$$

respectively, where $N_2$ is the automorphism defined in §2.

For $n = 2, 3, \cdots$, $n_0 = 1, 2, \cdots$ and $m_0 = 2, 3, \cdots$, we let

$$F^+(n, n_0) = N_3^{n-1}(F^+(1, n_0))\ , \quad F^+(-n, -n_0) = N_3^{-(n-1)}(F^+(-1, -n_0)),$$
$$F^-(n, m_0) = N_3^{n-1}(F^-(1, m_0)) \text{ and } F^-(-n, -m_0) = N_3^{-(n-1)}(F^-(-1, -m_0))\ ,$$

where $N_3$ is the automorphism defined in §2.

Inductively, we now define: For $n = 2, 3, \cdots$, we let

$$F^+(1, n_k, \cdots, n_1, n_0) = N_2(F^+(n_k, \cdots, n_1, n_0)) ,$$
$$F^+(n, n_k, \cdots, n_1, n_0) = N_3^{n-1}(F^+(1, n_k, \cdots, n_1, n_0))$$
$$\text{for} \quad n_j \geqq 1 \ (0 \leqq j \leqq k) ;$$
$$F^-(1, n_k, \cdots, n_1, n_0) = N_2(F^-(n_k, \cdots, n_1, n_0)) ,$$
$$F^-(n, n_k, \cdots, n_1, n_0) = N_3^{n-1}(F^-(1, n_k, \cdots, n_1, n_0))$$
$$\text{for} \quad n_0 \geqq 2 \quad \text{and} \quad n_j \geqq 1 \ (1 \leqq j \leqq k) ;$$
$$F^+(-1, -n_k, \cdots, -n_1, -n_0) = N_2(F^+(-n_k, \cdots, -n_1, -n_0))$$
$$F^+(-n, -n_k, \cdots, -n_1, -n_0) = N_3^{-(n-1)}(F^+(-1, -n_k, \cdots, -n_1, -n_0))$$
$$\text{for} \quad n_j \geqq 1 \ (0 \leqq j \leqq k) ;$$
$$F^-(-1, -n_k, \cdots, -n_1, -n_0) = N_2(F^-(-n_k, \cdots, -n_1, -n_0))$$
$$F^-(-n, -n_k, \cdots, -n_1, -n_0) = N_3^{-(n-1)}(F^-(-1, -n_k, \cdots, -n_1, -n_0))$$
$$\text{for} \quad n_0 \geqq 2 \quad \text{and} \quad n_j \geqq 1 \ (1 \leqq j \leqq k) .$$

REMARKS. (1) $F^+(n, 1) = F^+(n)$ and $F^+(-n, -1) = F^+(-n)$ for $n = 1, 2, \cdots$.

(2) $F^-(n, 1) = F^-(n)$ and $F^-(-n, -1) = F^-(n)$ for $n = 2, 3, \cdots$.

3.4. We have the following Lemmas 3.1, 3.2, 3.3 and 3.4 by Proposition 2.1 and straightforward calculations.

LEMMA 3.1. *Let $N_1$ be the automorphism defined in* §2.

(1) $N_1(F^+(n_k, \cdots, n_1, n_0)) = F^+(-n_k, \cdots, -n_1, -n_0)$ *for* $n_j = \pm 1, \pm 2, \cdots$ $(0 \leqq j \leqq k)$.

(2) $N_1(F^-(n_k, \cdots, n_1, n_0)) = F^-(-n_k, \cdots, -n_1, -n_0)$ *for* $n_0 = \pm 2, \pm 3, \cdots$ *and* $n_j = \pm 1, \pm 2, \cdots$ $(1 \leqq j \leqq k)$.

(3) $N_1(F_0) = F_0$.

LEMMA 3.2. *Let $N_3$ be the automorphism defined in* §2.

(1) $N_3(F^+(n_0)) = F^+(n_0 + 1)$ *for* $n_0 \geqq 1$,

$N_3(F^+(n_k, n_{k-1}, \cdots, n_0)) = F^+(n_k + 1, n_{k-1}, \cdots, n_0)$ *for* $n_k \geqq 1$, $n_j \geqq 2$ $(0 \leqq j \leqq k-1)$, $k \geqq 1$.

(2) $N_3(F^+(-n_k, -n_{k-1}, \cdots, -n_0)) = F^+(-n_k + 1, -n_{k-1}, \cdots, -n_0)$ *for* $n_j \geqq 2$ $(0 \leqq j \leqq k)$.

(3) $N_3(F^-(n_0)) = F^-(n_0 + 1)$ *for* $n_0 \geqq 2$,

$N_3(F^-(n_k, n_{k-1}, \cdots, n_0)) = F^-(n_k + 1, n_{k-1}, \cdots, n_0)$ *for* $n_k \geqq 1$, $n_j \geqq 2$ $(0 \leqq j \leqq k-1)$, $k \geqq 1$.

(4) $N_3(F^-(-n_0)) = F^-(-n_0 + 1)$ *for* $n_0 \geqq 3$,

$N_3(F^-(-n_k, -n_{k-1}, \cdots, -n_0)) = F^-(-n_k + 1, -n_{k-1}, \cdots, -n_0)$ *for* $n_j \geqq 2$ $(0 \leqq j \leqq k)$, $k \geqq 1$.

LEMMA 3.3. *Let $N_3$ be automorphism defined in* §2.

(1) $N_3^{-1}(F^+(1)) = \{(t_1, 1, 1) | 0 < t_1 < 1\}$.
(2) $N_3^{-1}(F^-(2)) = \{(t_1, 1, 1) | 0 < t_1 < 1\}$.
(3) $N_3(F^+(-1)) = \{(t_1, 1, 1) | 0 < t_1 < 1\}$.
(4) $N_3(F^-(-2)) = \{(t_1, 1, 1) | 0 < t_1 < 1\}$.
(5) $N_3(F_0) = \{(t_1, 0, 1/t_1) | 0 < t_1 < 1\}$.
(6) $N_3^{-1}(F_0) = \{(t_1, 0, t_1) | 0 < t_1 < 1\}$.

LEMMA 3.4. *Let $N_3$ be the automorphism defined in* §2.
(1) $N_3(F^+(-1, -2)) = F^+(1, 2)$,
$N_3(F^+(-1, -n_0)) = F^+(1, 2, n_0 - 1)$ $(n_0 \geqq 3)$.
(2) $N_3(F^+(-1, -2, -n_{k-2}, \cdots, -n_0)) = F^+(1, n_{k-2} + 1, n_{k-3}, \cdots, n_0)$,
$N_3(F^+(-1, -n_{k-1}, -n_{k-2}, \cdots, -n_0)) = F^+(1, 2, n_{k-1} - 1, n_{k-2}, \cdots, n_0)$
$(n_{k-1} \geqq 3)$
(3) $N_3(F^-(-1, -2)) = \{(t_1, 1, 1) | 0 < t_1 < 1\}$,
$N_3(F^-(-1, -n_0)) = F^-(1, 2, n_0 - 1)$ $(n_0 \geqq 3)$.
(4) $N_3(F^-(-1, -2, -n_{k-2}, \cdots, -n_0)) = F^-(1, n_{k-2} + 1, n_{k-3}, \cdots, n_0)$,
$N_3(F^-(-1, -n_{k-1}, -n_{k-2}, \cdots, -n_0)) = F^-(1, 2, n_{k-1} - 1, n_{k-2}, \cdots, n_0)$
*for* $n_{k-1} \geqq 3$ $(k \geqq 2)$.

3.5. We denote by $M(1)$ (resp. $M(-1)$) the three-dimensional manifolds bounded by $H(1)$, $H(1, 1)$, $F^+(1)$ and $F_0$ (resp. $H(-1)$, $H(-1, -1)$, $F^+(-1)$ and $F_0$), and denote by $M(n)$ $(n = \pm 2, \pm 3, \cdots)$ the three-dimensional manifolds bounded by three surfaces $H(n)$, $F^+(n)$ and $F^-(n)$. Then we easily see the following lemmas.

LEMMA 3.5. *If* $(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2^{00}$, *then* $1 < \rho < 1/t_1$ *or* $t_1 < \rho < 1$ *for each* $0 < t_1 < 1$.

LEMMA 3.6. *Let* $(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2^{00}$. *If* $0 < t_1' \leqq t_1$ *and* $0 < t_2' \leqq t_2$, *then* $(t_1', t_2', \rho) \in R_{\mathrm{I}}\mathfrak{S}_2^{00}$.

From Proposition 3.1 and Lemmas 3.5 and 3.6, we have the following.

PROPOSITION 3.4. $R_{\mathrm{I}}\mathfrak{S}_2^{00} = M(1) \cup M(-1)$.

3.6. We denote by $\boldsymbol{R}_{\mathrm{I}}^3$ the set $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 | 0 < t_1 < 1,\ 0 < t_2 < 1,\ \rho > 0\}$. We denote by $M(n_k, n_{k-1}, \cdots, n_0)$ (resp. $M(-n_k, -n_{k-1}, -n_0)$) the three-dimensional manifolds in $\boldsymbol{R}_{\mathrm{I}}^3$ bounded by $F^+(n_k, n_{k-1}, \cdots, n_0)$ and $F^-(n_k, n_{k-1}, \cdots, n_0)$ (resp. $F^+(-n_k, -n_{k-1}, \cdots, -n_0)$ and $F^-(-n_k, -n_{k-1}, \cdots, -n_0)$) for $n_k \geqq 1$ and $n_j \geqq 2$ $(0 \leqq j \leqq k-1)$.

REMARK. $M(n) = M(n, 1)$ *and* $M(-n) = M(-n, -1)$ *for* $n \geqq 2$.

The following proposition follows from Lemmas 3.1, 3.2, 3.3 and 3.4.

PROPOSITION 3.5. *Let $N_j$ $(j = 1, 2, 3)$ be the automorphisms defined* *in* §2. *Then for either* $n_k \geqq 1$, $n_j \geqq 2$ $(0 \leqq j \leqq k-1)$, *or* $n_k \leqq -1$, $n_j \leqq -2$ $(0 \leqq j \leqq k-1)$,

(1) (i) $N_1(M(1)) = M(-1)$, $N_1(M(-1)) = M(1)$,

(ii) $N_1(M(n_k, n_{k-1}, \cdots, n_0)) = M(-n_k, -n_{k-1}, \cdots, -n_0)$ $(k \geqq 1)$;

(2) $N_2(M(n_k, n_{k-1}, \cdots, n_0)) = \begin{cases} M(\pm 1, n_k, n_{k-1}, \cdots, n_0) & \text{if } n_k \neq \pm 1 \\ M(n_{k-1}, n_{k-2}, \cdots, n_0) & \text{if } n_k = \pm 1, \end{cases}$

(3) (i) $N_3(M(n_k, n_{k-1}, \cdots, n_0)) = M(n_k + 1, n_{k-1}, \cdots, n_0)$ *if* $n_k \geqq 1$ *or* $n_k \leqq -2$,

(ii) $N_3(M(-1, -2)) = M(1, 2)$,

$N_3(M(-1, -n_0)) = M(1, 2, n_0 - 1)$ *for* $n_0 \geqq 3$,

(iii) $N_3(M(-1, -2, -n_{k-2}, \cdots, -n_0)) = M(1, n_{k-2} + 1, n_{k-3}, \cdots, n_0)$ *for* $n_j \geqq 2$ $(0 \leqq j \leqq k-2)$, $k \geqq 2$,

$N_3(M(-1, -n_{k-1}, -n_{k-2}, \cdots, -n_0)) = M(1, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0)$ *for* $n_{k-1} \geqq 3$, $n_j \geqq 2$ $(0 \leqq j \leqq k-2)$, $k \geqq 2$.

3.7. Noting that $R_{\mathrm{I}}\mathfrak{S}_2^0 = \cup \phi(\mathfrak{S}_2^{00}) = \cup \phi(M(1) \cup M(-1))$, where $\phi$ runs through $\mathrm{Mod}(\mathfrak{S}_2)$, we have the following theorem from Proposition 3.5 and Theorem A.

THEOREM 1.

$$R_{\mathrm{I}}\mathfrak{S}_2^0 = \bigcup_{k=0}^{\infty} \bigcup_{\substack{n_k \geqq 1 \\ n_j \geqq 2 (0 \leqq j \leqq k-1)}} (M(n_k, n_{k-1}, \cdots, n_0) \cup M(-n_k, -n_{k-1}, \cdots, -n_0)) .$$

PROBLEM. *It is well-known that the classical Schottky space does not coincide with the Schottky space (Marden* [6], *Zarrow* [16]). *Thus we could ask: Does the real classical Stottky space $R_k\mathfrak{S}_2^0$ of the k-th type* ($k = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VIII}$) *coincide with the real Schottky space $R_k\mathfrak{S}_2$ of the k-th type?*

From the above theorem and Purzitsky [9, Theorem 3], the answer to the problem is affirmative for the first type.

COROLLARY 1. $R_{\mathrm{I}}\mathfrak{S}_2^0 = R_{\mathrm{I}}\mathfrak{S}_2$.

REMARK. An example due to Zarrow [16, p. 721] shows that the answer to the problem is negative for the second type. However, recently, we showed that the example constructed by Zarrow turns out to be a classical Schottky group (cf. [16]). We have reasons to believe that $R_{\mathrm{II}}\mathfrak{S}_2^0$, $R_{\mathrm{V}}\mathfrak{S}_2^0$ and $R_{\mathrm{VII}}\mathfrak{S}_2^0$ coincide with $R_{\mathrm{II}}\mathfrak{S}_2$, $R_{\mathrm{V}}\mathfrak{S}_2$ and $R_{\mathrm{VII}}\mathfrak{S}_2$, respectively (cf. [15]).

From Theorem 1, Corollary 1 to Theorem 1 and remarks after Proposition 3.2 we have the following.

COROLLARY 2. *Let $G = \langle A_1, A_2 \rangle$ be a purely loxodromic discrete group* *such that* $t_1 > 0$, $t_2 > 0$ *and* $\rho > 0$, *where* $(t_1, t_2, \rho)$ *is the point in* $D^{*2}\times(\boldsymbol{C} - \{0, 1\})$ *corresponding to* $\langle A_1, A_2\rangle$. *Then the following valids.*

(1) *If* $t_1 \geqq c_1$ *and* $t_2 \geqq c_2$ *for some positive constants* $c_1$ *and* $c_2$, *then* $1/c \leqq \rho \leqq c$ *for a positive constant* $c$ *with* $c > 1$ *depending on* $c_1$ *and* $c_2$.

(2) *If* $t_1 \geqq c_1$ *for some positive constant* $c_1$ *and if* $\rho \geqq c$ *or* $\rho \leqq 1/c$ *for some constant* $c > 1$, *then* $0 < t_2 \leqq c_2$ *for a constant* $c_2$ $(0 < c_2 < 1)$ *depending on* $c_1$ *and* $c$.

## 4. The first type—Fundamental regions.

4.1. In this section we will determine fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{I}}\mathfrak{S}_2^0$. Throughout this section let $N_j$ $(j = 1, 2, 3)$ be the automorphisms defined in §2.

We set

$$T_{2,n}(t_1, \rho) = (1 - \rho t_1^{(2n-1)/2})/(\rho - t_1^{(2n-1)/2})$$
$$\text{for} \quad 1/t_1^{n-1} \leqq \rho \leqq 1/t_1^n\,, \quad 0 < t_1 < 1 \ (n = 1, 2, \cdots)$$

and

$$T_{2,-n}(t_1, \rho) = (\rho - t_1^{(2n-1)/2})/(1 - \rho t_1^{(2n-1)/2})$$
$$\text{for} \quad t_1^n \leqq \rho \leqq t_1^{n-1}\,, \quad 0 < t_1 < 1 \ (n = 1, 2, \cdots)\,.$$

We have the following proposition by straightforward computations.

PROPOSITION 4.1. *Fix* $t_1$ *with* $0 < t_1 < 1$.

(1) *Let* $1/t_1^{n-1} \leqq \rho \leqq 1/t_1^n$ $(n = 1, 2, \cdots)$. *If* $t_2 = T_{2,n}(t_1, \rho)$, then $\langle A_1, A_2\rangle$ *is equal to* $\langle A_1, A_1^{2n-1}A_2\rangle$ *as marked groups. The points* $P(n) = (t_1, t_{2,n}(t_1), \rho_n(t_1))$ *dfined in* §3 *and* $(t_1, 0, t_1^{-(2n-1)/2})$ *lie on the curve* $t_2 = T_{2,n}(t_1, \rho)$.

(2) *Let* $t_1^n \leqq \rho \leqq t_1^{n-1}$ $(n = 1, 2, \cdots)$. *If* $t_2 = T_{2,-n}(t_1, \rho)$, *then* $\langle A_1, A_2\rangle$ *is equal to* $\langle A_1, A_1^{2n-1}A_2\rangle$ *as marked groups. The points* $P(-n) = (t_1, t_{2,-n}(t_1), \rho_{-n}(t_1))$ *defined in* §3 *and* $(t_1, 0, t_1^{2n-1})$ *lie on the curve* $t_2 = T_{2,-n}(t_1, \rho)$.

By Lemma 3.1 (3), the set of invariant points in $R_{\mathrm{I}}\mathfrak{S}_2$ under the mapping $N_1$ is the surface $F_0$ defined in §2. By Lemma 2.1 (2) we easily see the following.

PROPOSITION 4.2. *The set of invariant points in* $R_{\mathrm{I}}\mathfrak{S}_2$ *under the mapping* $N_2$ *is the surface*

$$\{(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2 \,|\, t_1 = t_2,\ 0 < t_1 < 1,\ \rho > 0\}\,.$$

We introduce the following sets in $\boldsymbol{R}_{\mathrm{I}}^3$: For $n = 1, 2, \cdots$, we set

$$L(n) = \{(t_1, t_2, \rho) \in \boldsymbol{R}_{\mathrm{I}}^3 \,|\, t_2 = T_{2,n}(t_1, \rho),\ \rho_n(t_1) \leqq \rho < 1/t_1^n,\ 0 < t_1 < 1\}$$

and

$$L(-n) = \{(t_1, t_2, \rho) \in \boldsymbol{R}_{\mathrm{I}}^3 \mid t_2 = T_{2,-n}(t_1, \rho),\ t_1^n \leqq \rho \leqq \rho_{-n}(t_1),\ 0 < t_1 < 1\}\ ,$$

where $\boldsymbol{R}_{\mathrm{I}}^3$ is as defined in §3.6. Then we easily see the following by Lemma 2.1 (1) and (3).

PROPOSITION 4.3. (1) *$N_1(L(n)) = L(-n)$ for $n = \pm 1, \pm 2, \cdots$.*
*(2) $N_3(L(n)) = L(n+1)$ for $n = 1, 2, \cdots$ or $n = -2, -3, \cdots$.*
*(3) $N_3(L(-1)) = L(1)$.*

REMARK. For a constant $c$, we set

$$D_c = \{(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2 \mid 1 < \rho < 1/t_1,\ 0 < t_1 < 1,\ t_2 = c\}\ .$$

Then the limit as $c \to 1$ of the images of $D_c$ under $N_3$ is the set $F^-(2)$ defined in §3.

4.2. Now we will determine fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{I}}\mathfrak{S}_2$. We set

$$S = N_1 N_3 N_1 \quad \text{and} \quad T = N_1 N_2\ .$$

Then $S$ and $T$ are elements in $\mathrm{Mod}(\mathfrak{S}_2)$. We easily see that $T^2 \sim 1$ and $(TS)^3 = 1$, where $T^2 \sim 1$ means that $T^2$ is equivalent to the identity mapping (see Definition 2.1). We introduce $\rho(t_1, t_2)$ for $0 < t_1 < 1$ and $0 < t_2 < 1$ as follows:

$$\rho(t_1, t_2) = (1 + t_1^{1/2} t_2)/(t_1^{1/2} + t_2)\ .$$

REMARK. $T_{2,1}(t_1, \rho(t_1, t_2)) = t_2$ and $\rho(t_1, T_{2,1}(t_1, \rho)) = \rho$.

THEOREM 2. *Set*

$$F_{\mathrm{I}}([\Phi_2]) = \{(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2 \mid 1 < \rho < \rho(t_1, t_2),\ 0 < t_2 < t_1,\ 0 < t_1 < 1\}$$

$$\begin{aligned} F_{\mathrm{I}}(\mathrm{Mod}(\mathfrak{S}_2)) &= F_{\mathrm{I}}([\Phi_2]) \cup N_1(F_{\mathrm{I}}([\Phi_2])) \\ &= \{(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2 \mid \rho(t_1, t_2)^{-1} < \rho < \rho(t_1, t_2),\ \rho \neq 1,\ 0 < t_2 < t_1,\ 0 < t_1 < 1\}\ . \end{aligned}$$

*Then $F_{\mathrm{I}}([\Phi_2])$ and $F_{\mathrm{I}}(\mathrm{Mod}(\mathfrak{S}_2))$ are fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{I}}\mathfrak{S}_2$, respectively.*

We obtain the proof by combining Theorem 3 and Proposition 4.4 which follow.

4.3. LEMMA 4.1. *$N_1^2 = 1$, $N_2^2 = 1$, $N_1 N_2 \sim N_2 N_1$, $N_1 N_3 N_1 N_3 \sim 1$ and $S \sim N_3^{-1}$.*

This lemma and the following lemma come from straightforward calculations.

LEMMA 4.2. *$N_1 N_3^n N_1 = S^n$, $N_1 N_3^n N_2 = S^n T$, $N_2 N_3^n N_1 \sim T S^n$ and $N_2 N_3^n N_2 \sim T S^n T$ for each integer $n$.*

Let $A, B, C, \cdots$ be distinct Möbius transformations. A word $W$ in $A, B, C, \cdots$ is a finite sequence

$$f_1 f_2 f_3 \cdots f_{m-1} f_m ,$$

where each of the $f_k (k = 1, 2, \cdots, m)$ is one of the transformations $A, B, C, \cdots, A^{-1}, B^{-1}, C^{-1}, \cdots$. The length $L(W)$ of $W$ is the integer $m$. For convenience we introduce the empty word 1 of length zero. If we wish to exhibit the transformations involved in $W$, then we write $W(A, B, C, \cdots)$.

By Theorem A in §2, an element $\psi$ in $\mathrm{Mod}(\mathfrak{S}_2)$ is represented in one of the following seven ways: (1) $\psi = W(N_1) = N_1^{2n}$, $n \in \boldsymbol{Z}$; (2) $\psi = W(N_2) = N_2^{2n}$, $n \in \boldsymbol{Z}$; (3) $\psi = W(N_3) = N_3^n$, $n \in \boldsymbol{Z}$; (4) $\psi = W(N_1, N_2)$ where $L(W(N_1, N_2))$ is even and $\psi \neq W(N_1), W(N_2)$; (5) $\psi = W(N_1, N_3)$ where the number of $N_1^{\pm 1}$ in $W(N_1, N_3)$ is even and $\psi \neq W(N_1), W(N_3)$; (6) $\psi = (N_2, N_3)$ where the number of $N_2^{\pm 1}$ in $W(N_2, N_3)$ is even and $\psi \neq W(N_2), W(N_3)$; (7) $\psi = W(N_1, N_2, N_3)$, where the sum of the number of $N_1^{\pm 1}$ and $N_2^{\pm 1}$ in $W(N_1, N_2, N_3)$ is even and $W(N_1, N_2, N_3)$ is neither of (1) through (6) above.

THEOREM 3. *The Schottky modular group* $\mathrm{Mod}(\mathfrak{S}_2)$ *is generated by* $S$ *and* $T$, *that is,*

$$\mathrm{Mod}(\mathfrak{S}_2) = \langle T, S \mid T^2 \sim 1, (TS)^3 = 1 \rangle .$$

PROOF. We will show that $\psi \in \mathrm{Mod}(\mathfrak{S}_2)$ is represented by $T$ and $S$ in each of the above seven cases.

Case (1). Since $\psi = N_1^{2n}$ and $N_1^2 = 1$, we have $\psi = 1$.

Case (2). For the same reason as in Case (1), $\psi = 1$.

Case (3). Since $N_3 \sim S^{-1}$, we have $\psi = W(N_3) = N_3^n = S^{-n}$.

Case (4). From Lemma 4.1, we see that $\psi = N_1 N_2 = T$, $\psi = N_2 N_1 \sim T$, or $\psi = 1$.

Case (5). In this case we have four subcases: (i) $\psi = N_1 N_3^{n_1} N_1 \cdots N_1 N_3^{n_k} N_1$, (ii) $\psi = N_1 N_3^{n_1} N_1 \cdots N_1 N_3^{n_k}$, (iii) $\psi = N_3^{n_1} N_1 \cdots N_1 N_3^{n_k} N_1$, (iv) $\psi = N_3^{n_1} N_1 \cdots N_1 N_3^{n_k}$. Since the number of $N_1$ is even, we have $\psi = W(N_1, N_3) \sim S^m$ for some integer $m$ in each case by Lemmas 4.1 and 4.2.

Case (6). Similarly as in Case (5), we have four subcases:

( i ) $\psi = N_2 N_3^{n_1} N_2 \cdots N_2 N_3^{n_k} N_2 \sim (TS^{n_1} T) S^{-n_2} (TS^{n_3} T) \cdots S^{n_{k-1}} (TS^{n_k} T)$,

( ii ) $\psi = N_2 N_3^{n_1} N_2 \cdots N_3^{n_k} \sim (TS^{n_1} T) S^{-n_2} \cdots (TS^{n_{k-1}} T) S^{-n_k}$,

(iii) $\psi = N_3^{n_1} N_2 \cdots N_2 N_3^{n_k} N_2 \sim S^{-n_1} (TS^{n_2} T) S^{-n_2} \cdots (TS^{n_k} T)$, and

(iv) $\psi = N_3^{n_1} N_2 N_3^{n_2} \cdots N_2 N_3^{n_k} \sim S^{-n_1} (TS^{n_2} T) S^{-n_3} \cdots (TS^{n_{k-1}} T) S^{-n_k}$.

Case (7). We have the following four subcases:

$(\mathrm{i})_k$ $\psi = U_1 N_3^{n_1} U_2 N_3^{n_2} U_3 \cdots U_k N_3^{n_k} U_{k+1}$,

$(\mathrm{ii})_k$ $\psi = U_1 N_3^{n_1} U_2 N_3^{n_2} U_3 \cdots U_k N_3^{n_k}$,

$(\mathrm{iii})_k$ $\psi = N_3^{n_1} U_1 N_3^{n_2} U_2 \cdots U_{k-1} N_3^{n_k} U_k$, and

(iv)$_k$ $\psi = N_3^{n_1} U_1 N_3^{n_2} U_2 \cdots U_k N_3^{n_{k+1}}$,
where $U_i$ $(i = 1, 2, \cdots, k+1)$ are words $W_i(N_1, N_2)$ in $N_1$ and $N_2$ such that $\sum_{i=1}^{k} L(W_i(N_1, N_2))$ is even.

By induction we will show that $\psi = W(N_1, N_2, N_3)$ is represented by $S$ and $T$. First we note that $N_1N_2 \sim N_2N_1$. For $k=1$,

(i)$_1$ $\psi = U_1 N_3^{n_1} U_2$. There are three possibilities: $\psi = N_1 N_3^{n_1} N_2 = S^{n_1}T$; $\psi = N_2 N_3^{n_1} N_1 \sim TS^{n_1}$; $\psi \sim N_1 N_2 N_3^{n_1} N_1 N_2 \sim TS^{-n_1}T$.

(ii)$_1$ $\psi = U_1 N_3^{n_1} \sim N_1 N_2 N_3^{n_1} \sim TS^{-n_1}$.

(iii)$_1$ $\psi = N_3^{n_1} U_2 \sim N_3^{n_1} N_1 N_2 \sim S^{-n_1}T$.

(iv)$_1$ $\psi = N_3^{n_1} U_1 N_3^{n_2} \sim N_3^{n_1} N_1 N_2 N_3^{n_2} = S^{-n_1}TS^{-n_2}$.

We assume that in each of (i)$_j$ through (iv)$_j$ $(j = 1, 2, \cdots, k)$ $\psi$ is represented by $S$ and $T$, that is, $\psi = W_{j,l}(S, T)$ $(j = 1, 2, \cdots, k;\ l = 1, 2, 3, 4)$, where the latter numbers 1, 2, 3, 4 correspond to i, ii, iii, iv, respectively. Then we will show that in each of (i)$_{k+1}$ through (iv)$_{k+1}$, $\psi$ is represented by $S$ and $T$.

(i)$_{k+1}$

$$\psi = \begin{cases} (U_1 N_3^{n_1} U_2 \cdots U_k N_3^{n_k} U_{k+1})(N_3^{n_{k+1}} U_{k+2}) & \text{for } U_{k+2} = N_1N_2 \text{ or } N_2N_1 \\ (U_1 N_3^{n_1} U_2 \cdots U_k N_3^{n_k} U_{k+1,1})(U_{k+1,2} N_3^{n_{k+1}} U_{k+2}) & \text{otherwise}, \end{cases}$$

where $U_{k+1,1}U_{k+1,2} = U_{k+1}$ and $L(U_{k+1,2})$ is odd. In the first case $\psi = W_{k,1}(S, T)W_{1,3}(S, T)$ and in the second case $\psi = W_{k,1}(S, T)W_{1,1}(S, T)$.

(ii)$_{k+1}$

$$\psi = \begin{cases} (U_1 N_3^{n_1} U_2 \cdots U_k N_3^{n_k})(U_{k+1} N_3^{n_{k+1}}) & \text{for } U_{k+1} = N_1N_2 \text{ or } N_2N_1 . \\ (U_1 N_3^{n_1} U_2 \cdots U_k N_3^{n_k} U_{k+1,1})(U_{k+1,2} N_3^{n_{k+1}}) & \text{otherwise}, \end{cases}$$

where $U_{k+1,1}U_{k+1,2} = U_{k+1}$, and $L(U_{k+1,1}) = 1$ and $L(U_{k+1,2}) = 2$. In the first case $\psi = W_{k,2}(S, T)W_{1,2}(S, T)$ and in the second case $\psi = W_{k,1}(S, T)W_{1,2}(S, T)$.

(iii)$_{k+1}$

$$\psi = \begin{cases} (N_3^{n_1} U_1 \cdots U_{k-1} N_3^{n_k} U_k)(N_3^{n_{k+1}} U_{k+1}) & \text{for } U_{k+1} = N_1N_2 \text{ or } N_2N_1 \\ (N_3^{n_1} U_1 \cdots U_{k-1} N_3^{n_k} U_{k,1})(U_{k,2} N_3^{n_{k+1}} U_{k+1}) & \text{otherwise}, \end{cases}$$

where $U_{k,1}U_{k,2} = U_k$ and $L(U_{k,2})$ is odd. In the first case $\psi = W_{k,3}(S, T)W_{1,3}(S, T)$ and in the second case $\psi = W_{k,3}(S, T)W_{1,1}(S, T)$.

(iv)$_{k+1}$

$$\psi = \begin{cases} (N_3^{n_1} U_1 \cdots U_k N_3^{n_{k+1}})(U_{k+1} N_3^{n_{k+2}}) & \text{for } U_{k+1} = N_1N_2 \text{ or } N_2N_1 \\ (N_3^{n_1} U_1 \cdots U_k N_3^{n_{k+1}} U_{k+1}) N_3^{n_{k+2}} & \text{otherwise}. \end{cases}$$

In the first case $\psi = W_{k,4}(S, T)W_{1,2}(S, T)$ and in the second case $\psi = W_{k,3}(S, T)S^{-n_{k+2}}$. Our proof is now complete.

4.4. The boundary of $F_{\mathrm{I}}(\mathrm{Mod}(\mathfrak{S}_2))$ consists of the following seven surfaces $B_1$ through $B_7$: We recall the terminology $\rho(t_1, t_2)$ introduced in §4.2 and we set

$$B_1 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid \rho = \rho(t_1, t_2),\ 0 < t_2 \leqq t_1,\ 0 < t_1 < 1\}\ ,$$
$$B_2 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid \rho = \rho(t_1, t_2)^{-1},\ 0 < t_2 \leqq t_1,\ 0 < t_1 < 1\}\ ,$$
$$B_3 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid 1 < \rho \leqq \rho(t_1, t_2),\ t_2 = t_1,\ 0 < t_1 < 1\}\ ,$$
$$B_4 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid \rho(t_1, t_2)^{-1} \leqq \rho < 1,\ t_2 = t_1,\ 0 < t_1 < 1\}\ ,$$
$$B_5 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid 0 < t_2 \leqq t_1,\ 0 < t_1 < 1,\ \rho = 1\}\ ,$$
$$B_6 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid 1 < \rho \leqq t_1^{-1/2},\ 0 < t_1 < 1,\ t_2 = 0\}\ ,$$

and

$$B_7 = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_1^{1/2} \leqq \rho < 1,\ 0 < t_1 < 1,\ t_2 = 0\}\ .$$

REMARKS. (1) $B_1$, $B_2$, $B_3$ and $B_4$ are in the interior of $R_{\mathrm{I}}\mathfrak{S}_2$.
(2) $B_5$, $B_6$ and $B_7$ are on the boundary of $R_{\mathrm{I}}\mathfrak{S}_2$.

We easily see the following by Lemma 2.1.

PROPOSITION 4.4. (1) *$S(B_1) = B_2$ and $T(B_3) = B_4$.*
(2) *$S(B_6) = B_7$.*
(3) *$T(B_5) = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_1 \leqq t_2 < 1,\ 0 < t_1 < 1,\ \rho = 1\}$.*

## 5. The first type—Riemann surfaces.

5.1. The boundary of the Schottky space consists of the following two kinds of points (cf. Rodriguez [11]):

(1) marked Möbius transformation groups which are limits of marked Schottky groups;

(2) points which represent Riemann surfaces with nodes.

For (1), every boundary group $G$ is discrete (Chuckrow [2]) and either $G$ is a cusp or $\Omega(G) = \varnothing$ (Rodriguez [11]), where $\Omega(G)$ is the region of discontinuity of $G$. For (2), Bers [1] studied compact Riemann surfaces with non-dividing nodes, and Sato [12], [13] studied Riemann surfaces with non-dividing nodes and dividing nodes corresponding to points on the augmented Schottky space. Furthermore Rodriguez [10], [11] studied relationship between Schottky-type groups and Riemann surfaces with nodes and cusps. In this section we will consider the following problem: Which Riemann surface does a point on the closure of $R_{\mathrm{I}}\mathfrak{S}_2$ represent?

DEFINITION 5.1. A Riemann surface with nodes is a complex space each point $P$ of which has a neighborhood isomorphic either to a disk $|z| < 1$ in $\boldsymbol{C}$ (with $P$ corresponding to $z = 0$) or to the set $|z| < 1$,

$|w| < 1$, $zw = 0$ in $C^2$ (with $P$ corresponding to $z = w = 0$). In the latter case, $P$ is called a *node*.

PROPOSITION 5.1. *A point in $\mathfrak{S}_2(\Sigma_0)$ represents a compact Riemann surface of genus two without nodes.*

This proposition is well-known.

PROPOSITION 5.2 (cf. Sato [12]). *A point on the surface $F_0 = \{(t_1, t_2, 1) \mid 0 < t_1 < 1, 0 < t_2 < 1\}$ represents a compact Riemann surface of genus two with one dividing node.*

PROPOSITION 5.3 (cf. [12]). *A point on the surfaces $H(n)$, $H(-n)$, $H(1, n)$ and $H(-1, -n)$ defined in §3.2 represents a compact Riemann surface of genus two with one non-dividing node.*

PROPOSITION 5.4 (cf. [12]). *A point on the set $\{(t_1, 0, 1) \mid 0 < t_1 < 1\}$ and $\{(0, t_2, 1) \mid 0 < t_2 < 1\}$ represents a compact Riemann surface of genus two with one dividing node and one non-dividing node.*

PROPOSITION 5.5 (cf. [12]). *A point on the set $\{(0, 0, \rho) \mid 0 < \rho, \rho \neq 1\}$ represents a compact Riemann surface of genus two with two non-dividing nodes.*

PROPOSITION 5.6 (cf. [12]). *A point $(0, 0, 1)$ represents a compact Riemann surface of genus two with one dividing node and two non-dividing nodes.*

5.2. Throughout this section, let $p > 1$ and $c > 0$. We set

$$\hat{A}(z) = \{(pc + 1)z - p^2c\}/\{cz - (pc - 1)\} .$$

Then we have the following Lemmas 5.1, 5.2 and Corollary by straightforward computations.

LEMMA 5.1. *$\hat{A}$ is a parabolic transformation whose fixed point is $z = p$.*

LEMMA 5.2. *Let $x > 0$. (1) The minimum value of $A(x)/x$ is $((pc + 1)/(pc - 1))^2$, which is attained at the point $x = p(pc - 1)/(pc + 1)$.*
*(2) $\hat{A}(p(pc - 1)/(pc + 1)) = p(pc + 1)/(pc - 1)$.*

COROLLARY. *Set*

$$C_2: |z - p^2c/(pc + 1)| = p/(pc + 1)$$

*and*

$$C_4: |z - p^2c/(pc - 1)| = p/(pc - 1) .$$

*Then the circle $C_2$ touches the circle $C_4$ at the point $z = p$, and $\hat{A}(C_2) = C_4$.*

Let $A_1$ and $A_2$ be the Möbius transformations as in §3.1. Then

$$A_1^{-1}A_2(z) = \{t_1(\rho - t_2)z + t_1\rho(t_2 - 1)\}/\{(1 - t_2)z + \rho t_2 - 1\}.$$

LEMMA 5.3. *Let* $t_2^{1/2} = (\rho^{1/2}t_1^{1/2} - 1)/(\rho^{1/2} - t_1^{1/2})$ $(1 < \rho \leqq \rho_2(t_1))$, *where* $\rho_2(t_2)$ *is as defined in* §3.1. *Then* (1) $A_1^{-1}A_2$ *is a parabolic transformation whose fixed point is* $t_1^{1/2}\rho^{1/2}$.

(2) *The minimum value of* $A_1^{-1}A_2(x)/x$ *for* $x > 0$ *is*

$$(t_1^{1/2}(\rho - 2t_1^{1/2}\rho^{1/2} + 1)/(2\rho^{1/2} - \rho t_1^{1/2} - t_1^{1/2}))^2$$

*which is attained at* $x = \rho^{1/2}(2\rho^{1/2} - \rho t_1^{1/2} - t_1^{1/2})/(\rho - 2t_1^{1/2}\rho^{1/2} + 1)$.

PROOF. This follows from Lemmas 5.1 and 5.2 by substituting $t_1^{1/2}\rho^{1/2}$ and $(1 - t_1)/t_1^{1/2}(\rho^{1/2}t_1^{1/2} - 1)(\rho^{1/2} - t_1^{1/2})$ for $p$ and $c$, respectively. q.e.d.

LEMMA 5.4. *Let* $p$ *and* $c$ *be as in the proof of Lemma* 5.3. *Set*

$$\varepsilon = (p/(1 + t_1))\{(pc - 1)/(pc + 1) - t_1(pc + 1)/(pc - 1)\}.$$

*Then* (1) $A_1(p(pc - 1)/(pc + 1) - \varepsilon) = p(pc + 1)/(pc - 1) + \varepsilon$.

(2) $\varepsilon = 0$ *if and only if*

$$\rho^{1/2} = \{t_1^{3/2} + 1 + (1 - t_1)(1 + t_1)^{1/2}\}/t_1^{1/2}(t_1^{1/2} + 1).$$

This lemma comes from straightforward computations.

COROLLARY. *Set*

$$C_1\colon |z| = p/(pc + 1) - \varepsilon$$

*and*

$$C_3\colon |z| = p/(pc - 1) + \varepsilon,$$

*where* $\varepsilon$ *is as in Lemma* 5.4. *Then* $A_1(C_1) = C_3$.

LEMMA 5.5. *Let* $G = \langle A_1, A_2\rangle$ *be a marked discontinuous group. Let* $\phi(G)$ *be the image of* $G$ *under a mapping* $\phi$ *in* $\mathrm{Mod}(\mathfrak{S}_2)$. *Then* $\phi(G)$ *is a marked discontinuous group and represents the same Riemann surface as* $G$ *does, that is,* $\Omega(\phi(G))/\phi(G)$ *is conformally equivalent to* $\Omega(G)/G$.

This lemma is well-known.

PROPOSITION 5.7. *A point on the surfaces* $F^+(1) - P(1)$, $F^+(-1) - P(-1)$ *and* $F^-(2) - P(2)$ *represents a Riemann surface of genus one with two punctures, where* $P(1)$, $P(-1)$ *and* $P(2)$ *are the sets defined in* §3.1.

PROOF. This proposition for $F^+(1) - P(1)$ and $F^+(-1) - P(-1)$ follows from Proposition 3.1. Therefore we will only prove it for $F^-(2) - P(2)$.

First we consider a marked group $\langle A_1, A_1^{-1}A_2\rangle$. Let $C_1$, $C_2$, $C_3$ and $C_4$ be the circles as in corollaries to Lemmas 5.2 and 5.4, where $p$ and

$c$ are as in the proof of Lemma 5.3, and $\varepsilon$ is as in Lemma 5.4. Then by corollaries to Lemmas 5.2 and 5.4 we see that the region bounded by the four circles is a fundamental region for $\langle A_1, A_1^{-1}A_2\rangle$, and that the group $\langle A_1, A_1^{-1}A_2\rangle$ represents a Riemann surface of genus one with two punctures.

Next we consider a marked group $\langle A_1, A_2\rangle$. We easily see that $\langle A_1, A_2\rangle = N_3(\langle A_1, A_1^{-1}A_2\rangle)$. Hence by $N_3 \in \mathrm{Mod}(\mathfrak{S}_2)$, Lemma 5.5 and the first part of this proof, $\langle A_1, A_2\rangle$ represents a Riemann surface of genus one with two punctures. q.e.d.

From Proposition 5.7 and Lemma 5.5, we have the following:

COROLLARY. *Let $P(n)$ $(n = \pm 1, \pm 2, \cdots)$ be the sets defined in* §3.1. *Then a point on the surfaces $F^+(n_k, n_{k-1}, \cdots, n_0) - P(n_k)$ and $F^-(n_k, n_{k-1}, \cdots, n_0) - P(n_k)$ represents a Riemann surface of genus one with two punctures.*

PROPOSITION 5.8. *A point on the set $P(n)$ $(n = \pm 2, \pm 3, \cdots)$ defined in* §3.1 *represents a Riemann surface of genus zero with four punctures.*

PROOF. We only consider this proposition for $n = 2$, since the other cases follow from the case for $n = 2$ and Lemma 5.5. By Proposition 3.2,

$$\rho_2(t_1) = (t_1^{3/2} + 1 + (1 - t_1)(1 + t_1)^{1/2})/t_1^{1/2}(t_1^{1/2} + 1)$$

for $n = 2$. We easily see that both $A_1^{-1}A_2$ and $A_1^{-2}A_2$ are parabolic. Let $C_1$, $C_2$, $C_3$ and $C_4$ be the circles in the proof of Proposition 5.7 and let $\varepsilon$ be the number in Lemma 5.4. Then by Lemma 5.4 we see that $\varepsilon = 0$. Therefore the circle $C_2$ touches the circles $C_1$ and $C_4$, and the circle $C_3$ touches the circle $C_4$. Hence $\langle A_1, A_1^{-1}A_2\rangle$ represents a Riemann surface of genus zero with four punctures. By Lemma 5.5, $\langle A_1, A_2\rangle$ represents the same Riemann surface as $\langle A_1, A_1^{-1}A_2\rangle$ does. Our proof is now complete.

5.3. We will consider which Riemann surfaces the other boundary points of $R_{\mathrm{I}}\mathfrak{S}_2$ represent. In our coordinates $(t_1, t_2, \rho)$, it is natural to consider for the problem as follows:

(1) A point $(t_1, 1, 1)$ $(0 < t_1 < 1)$ or $(1, t_2, 1)$ $(0 < t_2 < 1)$ represents a Riemann surface of genus one with two punctures and one dividing node.

(2) The points $(0, 1, 1)$ and $(1, 0, 1)$ represent Riemann surfaces of genus one with two punctures, one dividing node and one non-dividing node.

(3) The point $(1, 1, 1)$ represents a Riemann surface of genus zero with four punctures and one dividing node.

(4) A point $(t_1, 0, 1/t_1)$ $(0 < t_1 < 1)$ or $(0, t_2, 1/t_2)$ $(0 < t_2 < 1)$ represents a Riemann surface of genus one with one puncture.

## 6. The real classical Schottky space of the fourth type.

6.1. In this section we will determine the shape of the real classical Schottky space of the fourth type $R_{\mathrm{IV}}\mathfrak{S}_2^0$ and fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{IV}}\mathfrak{S}_2^0$, and consider the relationship between points on the closure of $R_{\mathrm{IV}}\mathfrak{S}_2^0$ and Riemann surfaces. We denote by $\boldsymbol{R}_{\mathrm{IV}}^3$ the set $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid 0 < t_1 < 1, 0 < t_2 < 1, \rho < 0\}$.

PROPOSITION 6.1. *Let $(t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2 \rangle$. If*

$$(*) \qquad 2t_1^{1/2}t_2^{1/2}(1 - \rho) = (-\rho)^{1/2}(1 - t_1)(1 - t_2)\ ,$$

*then $A_1A_2A_1^{-1}A_2^{-1}$ is a parabolic transformation. Furthermore $G$ is a discontinuous group and the union of two open sets bounded by the four circles $C_j$: $|z - a_j| = r_j$ $(j = 1, 2, 3, 4)$ is a fundamental region for $G$, where*

$$\begin{aligned}
a_1 &= -\rho(1 + \rho)(1 - t_2)^2(1 + t_1)/4(1 - \rho t_2)(t_2 - \rho)\ ,\\
r_1 &= -\rho(1 + t_1)(1 - t_2^2)(1 - \rho)/4(1 - \rho t_2)(t_2 - \rho)\ ,\\
a_2 &= \rho(1 + t_1)^2(t_2 - 1)/4t_1(t_2 - \rho)\ ,\\
r_2 &= \rho(1 - t_1^2)(t_2 - 1)/4t_1(t_2 - \rho)\ ,\\
a_3 &= a_1/t_1\ , \quad r_3 = r_1/t_1\ , \quad a_4 = \rho(1 + t_1)^2(1 - t_2)/4t_1(1 - \rho t_2)\ ,\\
r_4 &= \rho(1 - t_1^2)(t_2 - 1)/4t_1(1 - \rho t_2)\ .
\end{aligned}$$

For given $0 < t_1 < 1$ and $\rho < 0$, we write $t_2^*(t_1, \rho)$ for $t_2$ $(0 < t_2 < 1)$ satisfying the above equation $(*)$. As in the proofs of Lemmas 3.5 and 3.6 and Proposition 3.4, we have the following.

PROPOSITION 6.2.

$$R_{\mathrm{IV}}\mathfrak{S}_2^{00} = \{(t_1, t_2, \rho) \in \boldsymbol{R}_{\mathrm{IV}}^3 \mid 0 < t_2 < t_2^*(t_1, \rho), 0 < t_1 < 1, \rho < 0\}\ .$$

THEOREM B (Nielsen [8]). *Let $G = \langle A_1, A_2 \rangle$ and let $\tilde{A}_1$ and $\tilde{A}_2$ be the images of $A_1$ and $A_2$ under any element of $\Phi_2$. Then $\tilde{A}_2^{-1}\tilde{A}_1^{-1}\tilde{A}_2\tilde{A}_1 = U(A_2^{-1}A_1^{-1}A_2A_1)^{\pm 1}U^{-1}$, where $U$ is a word in $A_1$ and $A_2$.*

From Theorem B, we have the following.

PROPOSITION 6.3. $R_{\mathrm{IV}}\mathfrak{S}_2^0 = R_{\mathrm{IV}}\mathfrak{S}_2^{00}$.

The following theorem follows from Proposition 6.3 and a result of Purzitsky [9, Theorem 2].

THEOREM 4. $R_{\mathrm{IV}}\mathfrak{S}_2 = R_{\mathrm{IV}}\mathfrak{S}_2^0 = R_{\mathrm{IV}}\mathfrak{S}_2^{00}$.

6.2. Throughout this section let $N_j$ $(j = 1, 2, 3)$ be the automorphisms defined in §2.1. The following Propositions 6.4 through 6.7 follow from Lemma 2.1.

PROPOSITION 6.4. *The set of invariant points in* $\boldsymbol{R}^3_{\mathrm{IV}}$ *under the mapping* $N_1$ *is* $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3_{\mathrm{IV}} \mid 0 < t_1 < 1,\ 0 < t_2 < 1,\ \rho = -1\}$.

PROPOSITION 6.5. *The set of invariant points in* $R_{\mathrm{IV}}\mathfrak{S}_2$ *under the mapping* $N_2$ *is* $\{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho < 0,\ t_2 = t_1,\ 0 < t_1 < 1\}$.

We set $\rho^*(t_1, t_2) = (1 - t_1^{1/2}t_2)/(t_2 - t_1^{1/2})$ for $0 < t_1 < 1$ and $0 < t_2 < 1$.

PROPOSITION 6.6. *The set of invariant points in* $R_{\mathrm{IV}}\mathfrak{S}_2$ *under the mapping* $N_3$ *is* $B_8 = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho = \rho^*(t_1, t_2),\ \rho < 0,\ 0 < t_1 < 1,\ 0 < t_2 < 1\}$.

PROPOSITION 6.7. *The set of invariant points in* $R_{\mathrm{IV}}\mathfrak{S}_2$ *under the mapping* $(A_1, A_2) \to (A_1, A_1^{-1}A_2)$ *is* $B_9 = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho = \rho^*(t_1, t_2)^{-1},\ \rho < 0,\ 0 < t_1 < 1,\ 0 < t_2 < 1\}$.

Now we will determine fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{IV}}\mathfrak{S}_2$. As in §4, we set $S = N_1N_3N_1$ and $T = N_1N_2$. We set

$$B_{10} = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid -1 < \rho < \rho^*(t_1, t_2)^{-1},\ t_2 = t_1,\ 0 < t_1 < 1\}\ .$$

and

$$B_{11} = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho^*(t_1, t_2) < \rho < -1,\ t_2 = t_1,\ 0 < t_1 < 1\}\ .$$

PROPOSITION 6.8. $S(B_9) = B_8$ *and* $T(B_{10}) = B_{11}$.

From Proposition 6.8, we have the following.

THEOREM 5. *Set*

$$F_{\mathrm{IV}}(\mathrm{Mod}(\mathfrak{S}_2)) = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho^*(t_1, t_2) < \rho < \rho^*(t_1, t_2)^{-1},\ t_2 < t_1,\ 0 < t_2 < t_2^*(t_1, \rho),\ 0 < t_1 < 1\}$$

*and*

$$F_{\mathrm{IV}}([\Phi_2]) = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2 \mid \rho^*(t_1, t_2) < \rho < -1,\ t_2 < t_1\ ,\ 0 < t_2 < t_2^*(t_1, \rho),\ 0 < t_1 < 1\}\ ,$$

*respectively. Then* $F_{\mathrm{IV}}(\mathrm{Mod}(\mathfrak{S}_2))$ *and* $F_{\mathrm{IV}}([\Phi_2])$ *are fundamental regions for* $\mathrm{Mod}(\mathfrak{S}_2)$ *and* $[\Phi_2]$, *respectively, acting on* $R_{\mathrm{IV}}\mathfrak{S}_2$.

6.3. Finally, we will consider the following problem: Which Riemann surface does a point on the closure of $R_{\mathrm{IV}}\mathfrak{S}_2$ represent?

PROPOSITION 6.9. *A point in* $R_{\mathrm{IV}}\mathfrak{S}_2$ *represents a compact Riemann surface of genus two without nodes.*

This proposition is well-known.

PROPOSITION 6.10. *A point in the set* $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3_{\mathrm{IV}} \mid t_2 = t_2^*(t_1, \rho), 0 < t_1 < 1, \rho < 0\}$ *represents two Riemann surfaces each of which is of genus one with one puncture.*

This proposition comes from Proposition 6.1.

PROPOSITION 6.11 (cf. [12]). *A point on the sets* $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_1 = 0, 0 < t_2 < 1, \rho < 0\}$ *or* $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = 0, 0 < t_1 < 1, \rho < 0\}$ *represents a compact Riemann surface of genus two with one non-dividing node.*

PROPOSITION 6.12 (cf. [12]). *A point on the set* $\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_1 = t_2 = 0, \rho < 0\}$ *represents a compact Riemann surface of genus two with two non-dividing nodes.*

**7. Trace moduli.** In this section we will consider the relationship between Keen's results in [3] and our results in §6. In [3] keen sets $x = \text{trace}\, A_1$, $y = \text{trace}\, A_2$, $z = \text{trace}\, A_1A_2$, and $k = \text{trace}\, A_2^{-1}A_1^{-1}A_2A_1$. Then the equation $k = x^2 + y^2 + z^2 - xyz - 2$ is always satisfied. Keen considered Fuchsian groups representing Riemann surfaces of genus one with one hole or one puncture. In our terminology, she considered the real Schottky groups of the fourth type.

THEOREM C (Keen [3]). (1) *The moduli space for tori with a hole is the three-dimensional manifold M described by the equations* $x^2 + y^2 + z^2 - xyz - 2 - k = 0$, $x > 2$, $y > 2$, $z > 2$, $k < -2$.

(2) *The intersection of M with the half spaces* $x \leqq y$, $y \leqq z$, $z \leqq xy/2$ *is a fundamental region with respect to the action of* $\Phi_2$ *on M. The intersection of M with the half spaces* $x \leqq y$, $y \leqq z$, $xy \geqq 2z$, $xz \geqq 2y$ *is a fundamental region with respect to the action of* $\mathrm{Mod}(\mathfrak{S}_2)$ *on M.*

Let us represent the equation $x^2 + y^2 + z^2 - xyz = 0$, and the inequalities $x > 2$, $y > 2$ and $z > 2$ in our coordinates $(t_1, t_2, \rho)$. We easily see that $x^2 = t_1 + 1/t_1 + 2$, $y^2 = t_2 + 1/t_2 + 2$ and $z^2 = (t_1t_2\rho - t_2 + \rho - t_1)^2/(t_1t_2(\rho - 1)^2)$.

It is now straightforward to verify the following three propositions. We recall the notation $\rho^*(t_1, t_2) = (1 - t_1^{1/2}t_2)/(t_2 - t_1^{1/2})$ and $t_2 = t_2^*(t_1, \rho)$ introduced in §6. We set $\rho^{**}(t_1, t_2) = (1 - t_1t_2^{1/2})/(t_1 - t_2^{1/2})$.

PROPOSITION 7.1. (1) $2z \leqq xy$ *if and only if* $\rho \geqq -1$.

(2) $2y \leqq xz$ *if and only if* $\rho \leqq (t_2 - t_1)/(1 - t_1t_2)$.

(3) $x \leqq y$ *if and only if* $t_1 \geqq t_2$.

(4) $x \leqq z$ *if and only if* $\rho \leqq \rho^{**}(t_1, t_2)^{-1}$.

(5) $y \leqq z$ *if and only if* $\rho \leqq \rho^*(t_1, t_2)^{-1}$.

(6) $z > y(x - 1)$ *if and only if* $\rho > \rho^*(t_1, t_2)$.

(7) $x^2 + y^2 + z^2 - xyz = 0$ *if and only if* $t_2 = t_2^*(t_1, \rho)$.

Let $N_j$ ($j = 1, 2, 3$) be the automorphisms defined in §2. We set $S = N_1 N_3 N_1$ and $T = N_1 N_2$.

PROPOSITION 7.2. *The images of the sets* $\{(x, y, z) \in \boldsymbol{R}^3 \mid xz = 2y\}$ *and* $\{(x, y, z) \in \boldsymbol{R}^3 \mid y = z\}$ *under the mapping* $S$ *are* $\{(x, y, z) \in \boldsymbol{R}^3 \mid xy = 2z\}$ *and* $\{(x, y, z) \in \boldsymbol{R}^3 \mid z = y(x - 1)\}$, *respectively.*

PROPOSITION 7.3. *The image of the set* $\{(x, y, z) \in \boldsymbol{R}^3 \mid x^2 + y^2 + z^2 - xyz = 0, 2z < xy, y < z\}$ *under the mapping* $T$ *is* $\{(x, y, z) \in \boldsymbol{R}^3 \mid x^2 + y^2 + z^2 - xyz = 0, 2z > xy, z > y(x - 1)\}$.

From Theorem C and Propositions 7.2 and 7.3, we have the following.

PROPOSITION 7.4. *The three-dimensional manifold*

$$\{(x, y, z) \in \boldsymbol{R}^3 \mid y < z, z > y(x - 1), x < y\} \cap M$$

*is a fundamental region for* $\mathrm{Mod}(\mathfrak{S}_2)$ *acting on* $M$.

REMARK. The above fundamental region coincides with the set $F_{\mathrm{IV}}(\mathrm{Mod}(\mathfrak{S}_2))$ in §6.

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