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$L_{p,q}$-ESTIMATES FOR CERTAIN SEMI-LINEAR PARABOLIC EQUATIONS

TAKEYUKI NAGASAWA

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1. Introduction and results. In this paper we study the $L_{p,q}$-estimates for the solution of the equations

$$
(1.1)\qquad \begin{cases} u_t + Au = F_1(u, \partial u) + F_2(u), \quad t>0\,, \\ \quad u(0)=a\,, \\ \qquad u=(u^1, \cdots, u^N) \quad (N \in \boldsymbol{N}) \end{cases}
$$

in the $(L_p(\Omega))^N$-space. Here Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary, or $\boldsymbol{R}^n$ itself $(n \geq 2)$. We assume that A is of the form

$$
(1.2)\qquad A = -PL\,,
$$

where L is the realization in $(L_p(\Omega))^N$ of an elliptic operator of the second order (with certain boundary condition if $\partial\Omega \neq \varnothing$) and P is a bounded operator from $(L_p(\Omega))^N$ into the closed subspace $(PL_p(\Omega))^N$. The non-linear terms F_1 and F_2 of the types

$$
(1.3)\qquad \begin{cases} F_1(u, \partial u) = PN_1(u, \partial u)\,, \\ N_1(u, \partial u) = \sum_{i,j,k} a_{ijk} u^i \partial_j u^k\,, \\ \qquad (\partial u = (\partial_j u^k), j=1, \cdots, n;\, k=1, \cdots, N)\,, \\ \quad F_2(u) = PN_2(u)\,, \\ \quad N_2(u) = \sum_{i,j,k} b_{ijk} u^i u^j u^k\,, \end{cases}
$$

where a_{ijk} and b_{ijk} are bounded functions, and $\partial_j = \partial/\partial x_j$ $(j=1, 2, \cdots, n)$.

Our main purpose is to establish the $L_{p,q}$-estimates for solutions to this system. We know some examples of such a system in mathematical physics and differential geometry (see Section 2). For the Navier-Stokes system, which is one of the typical examples, the $L_{p,q}$-estimates play an important role in showing the regularity of weak solutions [2], [3]. Kato [4] and Giga [3] obtained such estimates for the system using certain special feature of non-linear terms. We shall show similar results using only the non-linearity (1.3). We shall also study an application of our results to the system of semi-linear heat equations having the non-linearity (1.3), because the gradient flow of the Yang-Mills functional is described by such a system.

Before stating our results, we describe our assumption more precisely. We denote $(L_p(\Omega))^N$ simply by $L_p(\Omega)$, and utilize other notation basically found in [7]. We assume that the restriction of P to $C_0(\Omega)$ is independent of p, $1<p<\infty$, and that the space $C_0(\Omega)\cap PL_p(\Omega)$ is dense in $PL_p(\Omega)$.

A defined by (1.2) is assumed to have the following property: $-A$ is an infinitesimal generator of a strongly continuous semigroup $\{e^{-tA}\}$ simultaneously on $PL_p(\Omega)$ for all $p\in(1, \infty)$ satisfying

$$\|e^{-tA}u\|_{p,\Omega}\leq C(p, q, n, \Omega)t^{-(n/q-n/p)/2}\|u\|_{q,\Omega}\,, \tag{1.4}$$

$$\|\partial e^{-tA}u\|_{p,\Omega}\leq C(p, q, n, \Omega)t^{-(1+n/q-n/p)/2}\|u\|_{q,\Omega}\,, \tag{1.5}$$

$(1<q\leq p<\infty,\ 0<t<T,\ T\in(0, \infty])$ for $u\in PL_q(\Omega)$. Since our system is parabolic type, the equation (1.1) can be converted into

$$\begin{cases} u(t)=e^{-tA}a+S_1(u)+S_2(u)\,, \\ S_1(u)=\displaystyle\int_0^t e^{-(t-\tau)A}F_1(u(\tau), \partial u(\tau))d\tau\,, \\ S_2(u)=\displaystyle\int_0^t e^{-(t-\tau)A}F_2(u(\tau))d\tau\,. \end{cases} \tag{1.6}$$

Our examples in §2 below satisfy the above assumptions for $T=\infty$.

We first establish the existence theorem.

THEOREM 1.1. *Let a be in $PL_n(\Omega)$. Then there exists a positive constant λ such that if $\|a\|_{n,\Omega}<\lambda$ then there exists a unique solution to* (1.1) *satisfying*

$$t^{(1-n/p)/2}u\in BC([0, T); PL_p(\Omega)) \qquad for \quad n\leq p<\infty\,,$$

$$t^{(1-n/(2q))}\partial u\in BC([0, T); PL_q(\Omega)) \qquad for \quad n\leq q<\infty$$

with values zero at $t=0$ except $u(0)=a$ in the case $p=n$.

PROOF. The solution is constructed by means of a standard successive approximation

$$\begin{cases} u_0=e^{-tA}a\,, \\ u_{m+1}=u_0+S_1(u_m)+S_2(u_m) \quad (m=0, 1, 2, \cdots)\,. \end{cases} \tag{1.7}$$

Our result then follows from an argument analogous to that in [6, Theorem 2 (i)]. ■

Let $Q_T:=\Omega\times(0, T)$. We would like to establish the $L_{p,q}$-estimates for u and its derivatives ∂u. To begin with, we have the following:

PROPOSITION 1.1. *Assume that $a\in PL_n(\Omega)$ and that its norm is sufficiently small.*

Let p_1, p_2, q_1 and q_2 be positive numbers satisfying the relations

$$\frac{1}{q_1}=\left(\frac{1}{n}-\frac{1}{p_1}\right)\frac{n}{2}, \qquad \frac{1}{q_2}=\left(\frac{2}{n}-\frac{1}{p_2}\right)\frac{n}{2},$$

with

$$\max\{3, n\}<p_1\leq 3p_2, \qquad \frac{n}{2}<p_2,$$

$$\max\{3, n\}<q_1, \qquad n<q_2, \qquad \frac{3}{p_1}-\frac{1}{p_2}<\frac{1}{n},$$

and

$$\frac{1}{p_1}+\frac{1}{p_2}<1.$$

Then the solution u to (1.1) *which is constructed in Theorem* 1.1 *satisfies*

$$u\in L_{q_1}(0, T; PL_{p_1}(\Omega))\subset L_{p_1,q_1}(Q_T),$$

$$\partial u\in L_{q_2}(0, T; PL_{p_2}(\Omega))\subset L_{p_2,q_2}(Q_T),$$

and $\|u\|_{p_1,q_1,Q_T}+\|\partial u\|_{p_2,q_2,Q_T}\to 0$ as $\|a\|_{n,\Omega}\to 0$.

To show this proposition we need the following lemma:

LEMMA 1.1. *Let p_1, p_2, q, r and s satisfy*

$$\frac{1}{q}=\left(\frac{1}{s}-\frac{1}{p_1}\right)\frac{n}{2}, \qquad \frac{1}{r}=\left(\frac{1}{n}+\frac{1}{s}-\frac{1}{p_2}\right)\frac{n}{2}$$

and

$$p_1, q, r>s>1, \qquad p_2>\left(\frac{1}{n}+\frac{1}{s}\right)^{-1}.$$

Then we have

$$\|e^{-tA}u\|_{p_1,q,Q_T}\leq C(p_1, q, s, n, \Omega)\|u\|_{s,\Omega}, \tag{1.8}$$

$$\|\partial e^{-tA}u\|_{p_2,r,Q_T}\leq C(p_2,r, s, n, \Omega)\|u\|_{s,\Omega} \tag{1.9}$$

for $u\in PL_s(\Omega)$.

PROOF. We find in [3] the proof of (1.8) by means of the Marcinkiewicz interpolation theorem and (1.4). (1.9) is proved in a similar manner. ■

PROOF OF PROPOSITION 1.1. We prove that $\{u_m\}$ defined by (1.7) satisfies

$$\|u_m\|_{p_1,q_1,Q_T}+\|\partial u_m\|_{p_2,q_2,Q_T}\leq K,$$

where K is a positive constant independent of m. Since u_m converges to u, we get the first part of our assertions from this estimate.

First we have the estimate

$$\|u_0\|_{p_1,q_1,Q_T}+\|\partial u_0\|_{p_2,q_2,Q_T}\leq C\|a\|_{n,\Omega}$$

by Lemma 1.1 with $s=n$.

By (1.4) with $p=p_1$, $q=r$, where $1/r=1/p_1+1/p_2$, we have

$$\|S_1(u_m)\|_{p_1,\Omega}\leq C\int_0^t(t-\tau)^{-n/(2p_2)}\|u_m(\tau)\|_{p_1,\Omega}\|\partial u_m(\tau)\|_{p_2,\Omega}d\tau\ .$$

An application of the Hardy-Littlewood-Sobolev inequality [8, Corollary 1 of Lemma 7.1] gives us

$$\|S_1(u_m)\|_{p_1,q_1,Q_T}\leq C\|u_m\|_{p_1,q_1,Q_T}\|\partial u_m\|_{p_2,q_2,Q_T}\ .$$

Similarly we obtain

$$\|S_2(u_m)\|_{p_1,\Omega}\leq C\int_0^t(t-\tau)^{-n/p_1}\|u_m(\tau)\|^3_{p_1,\Omega}d\tau$$

and

$$\|S_2(u_m)\|_{p_1,q_1,Q_T}\leq C\|u_m\|^3_{p_1,p_1,Q_T}\ .$$

By suitable use of (1.5), we can get similar estimates for ∂S_i's, i.e.,

$$\|\partial S_1(u_m)\|_{p_2,\Omega}\leq C\int_0^t(t-\tau)^{-(1+n/p_1)/2}\|u_m(\tau)\|_{p_1,\Omega}\|\partial u_m(\tau)\|_{p_2,\Omega}d\tau\ ,$$

$$\|\partial S_2(u_m)\|_{p_2,\Omega}\leq C\int_0^t(t-\tau)^{-(1+(3n)/p_1-n/p_2)/2}\|u_m(\tau)\|^3_{p_1,\Omega}d\tau\ .$$

These yield

$$\|\partial S_1(u_m)\|_{p_2,q_2,Q_T}\leq C\|u_m\|_{p_1,q_1,Q_T}\|\partial u_m\|_{p_2,q_2,Q_T}\ ,$$

$$\|\partial S_2(u_m)\|_{p_2,q_2,Q_T}\leq C\|u_m\|^3_{p_1,q_1,Q_T}\ .$$

Summing up these estimates, we get

$$\begin{aligned}&\|u_{m+1}\|_{p_1,q_1,Q_T}+\|\partial u_{m+1}\|_{p_2,q_2,Q_T}\\&\qquad\leq C_1\|a\|_{n,\Omega}+C_2(\|u_m\|_{p_1,q_1,Q_T}\|\partial u_m\|_{p_2,q_2,Q_T}+\|u_m\|^3_{p_1,q_1,Q_T})\ .\end{aligned}$$

In the same manner as in [6, Lemma 3.3], we get the desired estimate if $\|a\|_{n,\Omega}$ is sufficiently small. It is easy to see $\|u\|_{p_1,q_1,Q_T}+\|\partial u\|_{p_2,q_2,Q_T}\to 0$ as $\|a\|_{n,\Omega}\to 0$ from the above argument. ■

$$p_1 = q_1 = n+2\,,\quad p_2 = \frac{n+1}{2}\,,\quad q_2 = n+1$$

satisfy the conditions in Proposition 1.1, in view of our basic assumption $n \geq 2$. Therefore we have:

COROLLARY 1.1. *The solution u to* (1.1) *has properties*

$$u \in L^{n+2}(Q_T)\,, \qquad \partial u \in L_{n+1}(0, T; PL_{(n+1)/2}(\Omega))\,,$$

$$\|u\|_{n+2,Q_T} + \|\partial u\|_{(n+1)/2,n+1,Q_T} \to 0 \qquad \text{as} \quad \|a\|_{n,\Omega} \to 0\,.$$

We are in a position to state one of our results.

THEOREM 1.2. *Assume that $a \in PL_n(\Omega) \cap PL_s(\Omega)$ $(s > 1)$ and that $\|a\|_{n,\Omega}$ is sufficiently small. Suppose p and q satisfy the relations*

$$\frac{1}{q} = \left(\frac{1}{s} - \frac{1}{p}\right)\frac{n}{2}\,, \qquad p > \max\left\{\frac{n+1}{n-1}, s\right\},$$

and

$$q > \max\left\{\frac{n+1}{n}, s\right\}.$$

Then the solution u to (1.1) *which is constructed in Theorem* 1.1 *satisfies*

$$u \in L_q(0, T; PL_p(\Omega)) \subset L_{p,q}(Q_T)\,.$$

PROOF. By Lemma 1.1, we get

$$\|u_0\|_{p,q,Q_T} \leq C\|a\|_{s,\Omega}\,.$$

Making use of (1.4), we obtain

$$\|S_1(u)\|_{p,\Omega} \leq C\int_0^t (t-\tau)^{-n/(n+1)}\|u(\tau)\|_{p,\Omega}\|\partial u(\tau)\|_{(n+1)/2,\Omega}d\tau$$

and

$$\|S_1(u)\|_{p,q,Q_T} \leq C\|u\|_{p,q,Q_T}\|\partial u\|_{(n+1)/2,n+1,Q_T}\,.$$

Similarly we get

$$\|S_2(u)\|_{p,\Omega} \leq C\int_0^t (t-\tau)^{-n/(n+2)}\|u(\tau)\|_{p,\Omega}\|u(\tau)\|_{n+2,\Omega}^2 d\tau$$

and

$$\|S_2(u)\|_{p,q,Q_T} \leq C\|u\|_{p,q,Q_T}\|u\|_{n+2,n+2,Q_T}^2\,.$$

If $\|a\|_{n,\Omega}$ is sufficiently small, we then obtain the boundedness of the $L_{p,q}$-norm for u by virtue of Corollary 1.1. ■

For $s>(n^2+n)/(n^2+n-2)$,

$$p=q=\frac{s(n+2)}{2}$$

fulfill the conditions in Theorem 1.2. Thus we have:

COROLLARY 1.2. *If the hypotheses of Theorem* 1.2 *for* $s>(n^2+n)/(n^2+n-2)$ *are satisfied, then*

$$u\in L_{(s(n+2))/n}(0, T; PL_{(s(n+2))/n}(\Omega))\subset L_{(s(n+2))/n}(Q_T),$$

$$\|u\|_{(s(n+2))/n,Q_T}\to 0 \qquad as \quad \|a\|_{n,\Omega}+\|a\|_{s,\Omega}\to 0$$

hold for the solution u to (1.1).

Making use of Corollaries 1.1 and 1.2, we get an esyimate for ∂u.

THEOREM 1.3. *Assume that* $a\in PL_n(\Omega)\cap PL_s(\Omega)$ $(s>(n^2+n)/(n^2+n-2))$ *and that* $\|a\|_{n,\Omega}$ *is sufficiently small. Let p and r be positive numbers satisfying*

$$\frac{1}{r}=\left(\frac{1}{n}+\frac{1}{s}-\frac{1}{p}\right)\frac{n}{2}, \qquad p>\max\left\{\frac{n+2}{n}, \left(\frac{1}{n}+\frac{1}{s}\right)^{-1}\right\}, \qquad r>\max\left\{\frac{n+2}{n+1}, s\right\},$$

and

$$\frac{1}{p}-\frac{1}{n}<\frac{1}{n+2}\left(2+\frac{n}{s}\right)<\frac{1}{p}+\frac{1}{n}.$$

Then the solution u to (1.1) *which is constructed in Theorem* 1.1 *satisfies*

$$\partial u\in L_r(0, T; PL_p(\Omega))\subset L_{p,r}(Q_T).$$

PROOF. By (1.5), we have

$$\|\partial u_0\|_{p,r,Q_T}\leq C\|a\|_{s,\Omega},$$

and

$$\|\partial S_1(u)\|_{p,\Omega}\leq C\int_0^t(t-\tau)^{-(n+1)/(n+2)}\|u(\tau)\|_{n+2,\Omega}\|\partial u(\tau)\|_{p,\Omega}d\tau.$$

By virtue of Corollary 1.1,

$$\|\partial S_1(u)\|_{p,r,Q_T}\leq C\|u\|_{n+2,Q_T}\|\partial u\|_{p,r,Q_T}$$

holds.

Let s' be a positive number satisfying

$$\frac{3}{s'}=\frac{2}{n}+\frac{1}{s}.$$

Since s' is between n and s, $a\in PL_{s'}(\Omega)$ and $s'\geq\min\{n, s\}>(n^2+n)/(n^2+n-2)$. It follows from Corollary 1.2 that $u\in L_{(s'(n+2))/n}(0, T; PL_{(s'(n+2))/n}(\Omega))$. By (1.5), we have

$$\|\partial S_2(u)\|_{p,\Omega}\leq C\int_0^t(t-\tau)^{-(1+(3n^2)/(s'(n+2))-n/p)/2}\|u(\tau)\|^3_{(s'(n+2))/n,\Omega}d\tau .$$

In view of our assumption

$$0<\frac{1}{2}\left(1+\frac{3n^2}{s'(n+2)}-\frac{n}{p}\right)<1 ,$$

we get

$$\|\partial S_2(u)\|_{p,r,Q_T}\leq C\|u\|^3_{(s'(n+2))/n,Q_T}$$

by [8, Corollary 1 to Lemma 7.1]. Hence our assertion follows if $\|a\|_{n,\Omega}$ is sufficiently small. ■

2. Applications. In this section we study some applications of our theorems.

2.1 The Navier-Stokes system. The motion of incompressible viscous fluid in Ω (with fixed boundary condition) is described by the following system of equations, called the Navier-Stokes system:

$$\text{(2.1.1)}\qquad \begin{cases} u_t=\Delta u-(u\cdot\operatorname{grad})u-\operatorname{grad} p ,\\ \operatorname{div} u=0 ,\\ u(0)=a ,\\ u|_{\partial\Omega}=0 \quad \text{if} \quad \partial\Omega\neq\varnothing . \end{cases}$$

Here, $u=(u^1, \cdots, u^n)$ and p represent the velocity and the pressure of the fluid, respectively. Let X_p by the closure in $L_p(\Omega)$ of all C^∞-solenoidal functions with compact support in Ω. We define G_p by

$$G_p=\{f=\operatorname{grad}\phi\,|\,\phi\in W^1_p(\Omega)\} .$$

It is well-known that the Helmholtz decomposition

$$L_p(\Omega)=X_p\oplus G_p$$

holds and that the projection P from $L_p(\Omega)$ to X_p is a bounded operator (cf. e.g., [1]).

Applying P to both sides of the first equation of (2.1.1), we have

$$\text{(2.1.2)}\qquad u_t+Au=-P(u\cdot\operatorname{grad})u ,$$

where $A=-P\Delta$ is the Stokes operator with domain

$$\mathscr{D}(A)=PL_p(\Omega)\cap\{u\in W_p^2(\Omega)\,|\,u|_{\partial\Omega}=0 \text{ if } \partial\Omega\neq\emptyset\}\,.$$

We can check our assumptions with $T=\infty$ on A and on the non-linear terms described in the previous section (see [4], [3]). Therefore we get the existence of a unique solution u to (2.1.2) with initial value a by Theorem 1.1 and the $L_{p,q}$-estimates for u and ∂u by Theorems 1.2 and 1.3. Looking at the proof of Theorem 1.3 more carefully, we find that the assumptions in the theorem

$$s>\frac{n^2+n}{n^2+n-2}\,,\qquad \frac{1}{p}-\frac{1}{n}<\frac{1}{n+2}\left(2+\frac{n}{s}\right)<\frac{1}{p}+\frac{1}{n}$$

are needed only for the estimate for $\|\partial S_2(u)\|_{p,r,Q_T}$. Since the term $F_2(u)$ does not appear in the Navier-Stokes system, we can replace the above conditions by $s>1$. Thus we have:

THEOREM 2.1. (i) *Assume that a is in $PL_n(\Omega)$ and that its norm is sufficiently small. Then there exists a unique solution u to* (2.1.2) *with initial value a satisfying*

$$t^{(1-n/p)/2}u\in BC([0,\infty);\,PL_p(\Omega))\qquad \textit{for}\quad n\le p<\infty\,,$$

$$t^{(1-n/(2q))}\partial u\in BC([0,\infty);\,PL_q(\Omega))\qquad \textit{for}\quad n\le q<\infty$$

with values zero at $t=0$ except $u(0)=a$ in the case $p=n$.

(ii) *Assume that $a\in PL_n(\Omega)\cap PL_s(\Omega)$ $(s>1)$ and that $\|a\|_{n,\Omega}$ is sufficiently small. Let p_1, p_2, q and r be positive numbers satisfying*

$$\frac{1}{q}=\left(\frac{1}{s}-\frac{1}{p_1}\right)\frac{n}{2}\,,\qquad \frac{1}{r}=\left(\frac{1}{n}+\frac{1}{s}-\frac{1}{p_2}\right)\frac{n}{2}\,,$$

$$p_1>\max\left\{\frac{n+1}{n-1},\,s\right\},\qquad q>\max\left\{\frac{n+1}{n},\,s\right\},\qquad p_2>\max\left\{\frac{n+2}{n},\,\left(\frac{1}{n}+\frac{1}{s}\right)^{-1}\right\},$$

and

$$r>\max\left\{\frac{n+2}{n+1},\,s\right\}.$$

Then the solution u constructed in (i) *has the properties*

$$u\in L_q(0,\infty;\,PL_{p_1}(\Omega))\subset L_{p_1,q}(Q_\infty)\,,$$

$$\partial u\in L_r(0,\infty;\,PL_{p_2}(\Omega))\subset L_{p_2,r}(Q_\infty)\,.$$

Kato [4] and Giga [3] already obtained similar results, using the special feature of the non-linear term

$$(u\cdot\mathrm{grad})u^i=\mathrm{div}(u^iu)\,,$$

whereas we do not need such a feature.

The $L_{p,q}$-estimates for the Navier-Stokes system give the criteria for the regularity of weak solutions. For various regularity theorems on this system, the reader is referred to [2], [3] and references cited therein.

2.2 Semi-linear heat equations. The second example is the simplest case $A=-\Delta$ (P=identity, Δ=Laplacian (with the Dirichlet condition if $\partial\Omega\neq\varnothing$)), i.e.,

$$\tag{2.2.1} \begin{cases} u_t=\Delta u+N_1(u,\partial u)+N_2(u)\,, \\ u(0)=a\,, \\ u|_{\partial\Omega}=0 \text{ if } \partial\Omega\neq\varnothing\,. \end{cases}$$

As is shown in [6], our assumptions on A are fulfiled for $T=\infty$. Theorems 1.2 and 1.3 yield the regularity of the solutions which are constructed in Theorem 1.1. For simplicity, we assume $a\in\bigcap_{s\geq n}L_s(\Omega)$. For $\varepsilon\in(0,1)$, k, $l\in(0,2)$, set

$$p_1=\frac{n^2}{(1-\varepsilon)(n-k)}\,,\qquad q_1=\frac{2n}{k(1-\varepsilon)}\,,$$

$$p_2=\frac{n^2}{(2-\varepsilon)n-l(1-\varepsilon)}\,,\qquad q_2=\frac{2n}{l(1-\varepsilon)}\,,$$

$$s_1=s_2=\frac{n}{1-\varepsilon}\,.$$

If $1-\varepsilon$ is sufficiently small, the conditions in Theorem 1.2 hold for $(p,q,s)=(p_1,q_1,s_1)$ and those in Theorem 1.3 do for $(p,r,s)=(p_1,q_2,s_2)$. Hence we have $N_1(u,\partial u)\in L_{p,q}(Q_\infty)$, where $1/p=l/p_1+1/p_2$, $1/q=1/q_1+1/q_2$. If $k+l$ is sufficiently small, then $p<q$ holds. Therefore $N_1(u,\partial u)\in L_p(Q_T)$ is valid for any $T\in(0,\infty)$. For any $\delta>0$, we take $1-\varepsilon$ and $k+l$ so small that $p\geq(n+2)/(2+\delta)$ holds. We choose δ sufficiently small.

On the other hand, Corollary 1.2 gives us $N_2(u)\in\bigcap_{r\geq(n+2)/3}L_r(Q_\infty)$. Therefore the nonlinear terms belong to $L_p(Q_T)$. A priori estimate of $W_p^{2,1}(Q_T)$-type [7, IV, Theorem 9.1 or VII, Theorem 10.4] gives $u\in W_p^{2,1}(Q_T)$ provided $a\in W_p^{2-2/p}(\Omega)$. Using [7, II, Lemma 3.3], we have $N_1(u,\partial u)+N_2(u)\in L_{p'}(Q_T)$ for some $p'>(n+2)/2$, and therefore $u\in W_{p'}^{2,1}(Q_T)$ provided $a\in W_{p'}^{2-2/p'}(\Omega)$. By the same procedure we obtain $u\in W_r^{2,1}(Q_T)$ provided $a\in W_r^{2-2/r}(\Omega)$ for some $r>n+2$. By virtue of [7, II, Lemma 3.3] again, the Hölder continuity of non-linear terms follows from the Hölder continuity of the coefficients a_{ijk}, b_{ijk}. Finally the Schauder estimate [7, IV, Theorems 5.1/5.2 or VII, Theorems 10.1/10.2] gives the fact $u\in H^{\alpha+2,\alpha/2+1}(\overline{Q_T})$ for some $\alpha\in(0,1)$ provided that a is Hölder continuous up to its second orderderivatives. Hence we have:

THEOREM 2.2. *We assume that a_{ijk} and b_{ijk} are Hölder continuous in $\overline{Q_\infty}$. If a belongs to $\bigcap_{s\geq n}W_s^{2-2/s}(\Omega)$, if $\|a\|_{n,\Omega}$ is small, and if it is hölder continuous up to its second order derivatives, then there exists a unique global classical solution to* (2.2.1).

REMARK. Looking at the above argument more carefully, we find that we can weaken the assumption on a.

Using a standard bootstrap argument, we get:

THEOREM 2.3. *Suppose that the hypotheses of Theorem 2.2 are satisfied and that a_{ijk}, b_{ijk}, a and $\partial\Omega$ (if it exists) are C^∞-smooth. Moreover when $\partial\Omega \neq \emptyset$, we assume that the compatibility conditions of any order hold between the initial and boundary data. Then the solution is also C^∞.*

2.3 The equations of Yang-Mills' gradient flow. Let (Ω, dx^2) be a smooth $(=C^\infty)$ n-dimensional Riemannian manifold. $E=(\Omega\times \boldsymbol{R}^m, \langle\, ,\rangle)$ is a Riemannian vector bundle over Ω of rank m. $\mathscr{C}_E$ is the space consisting of all smooth metric connections on E. For $\nabla \in \mathscr{C}_E$, we define the Hom(E, E)-valued 2-form R^∇, called the curvature, by

$$R^\nabla_{V,W} = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]}$$

for any smooth vector fields V, W on Ω. The Yang-Mills functional $\mathscr{YM} : \mathscr{C}_E \to [0, \infty]$ is defined by the square integral of R^∇:

$$\mathscr{YM}(\nabla) = \frac{1}{2}\int_\Omega \langle R^\nabla, R^\nabla\rangle_x .$$

We call ∇ the Yang-Mills connection, if it is a critical point of the functional. To find such a connection and to study its stability, we consider the flow

$$\frac{d\nabla(t)}{dt} = -\operatorname{grad}\mathscr{YM}(\nabla(t)) . \tag{2.3.1}$$

In [6], we studied the asymptotical stability of the flat connection ∇_0 by reducing (2.3.1) to certain system of heat equations. Taking the gauge invariance of the functional into consideration, we put

$$\nabla(t) = g(t)(\nabla_0 + A(t))g^{-1}(t) ,$$

where $A(t) \in \Omega^1_0(\mathfrak{g}_E)$, $g(t) \in \mathscr{G}$ (see [6, §1] for the definition of $\Omega^1_0(\mathfrak{g}_E)$ and $\mathscr{G}$). Then the principal part of the right-hand side of (2.3.1) is

$$-\delta^{\nabla_0} d^{\nabla_0} A(t) + [\nabla_0 + A(t), Y(t)] ,$$

where $Y(t) = g^{-1}(t)dg(t)/dt$, and δ^{∇_0} is a formal adjoint operator of the covariant derivative d^{∇_0}. This does not satisfy our assumption because of the lack of ellipticity. We impose Yokotani's idea [9] on $g(t)$, i.e., it satisfies

$$\frac{dg(t)}{dt} = -g(t)\delta^{\nabla_0}A(t) , \qquad g(0) = \text{identity} . \tag{2.3.2}$$

This condition makes $-d^{\nabla_0}\delta^{\nabla_0}A(t)$ of the term $[\nabla_0, Y(t)]$. Since $-(\delta^{\nabla_0}d^{\nabla_0} + d^{\nabla_0}\delta^{\nabla_0}) = \Delta$, the principal part recovers the ellipticity, and (2.3.1) is reduced to the following system of heat equations:

$$
(2.3.3)\quad \begin{cases} \dfrac{dA(t)}{dt}=\Delta A(t)+[A(t),\ -\delta^{\nabla_0}A(t)]-\delta^{\nabla_0}[A(t),A(t)] \\ \qquad +\sum_{\alpha=1}^{n}[A_{e_\alpha}(t), d^{\nabla_0}A_{e_\alpha}(t)]+\sum_{\alpha=1}^{n}[A_{e_\alpha}(t),[A_{e_\alpha}(t),A(t)]]\,, \\ A\,|_{\partial\Omega}=0 \quad \text{if } \partial\Omega\neq\varnothing\,, \end{cases}
$$

where $\{e_\alpha\}_{\alpha=1,\cdots,n}$ is an orthonormal basis on $T_x\Omega$. For the detailed derivation of above system, the reader is referred to [9] and [5].

Our results are applicable to the equations (2.3.3). The global solvability and the stability of (2.3.2) and (2.3.3) with the given initial data $A(0)$ are established in [6, Theorem 1], provided that the components of $A(0)$ belong to $\mathring{W}_n^1(\Omega)$ and $\|A(0)\|_{n,\Omega}$ is sufficiently small. (It is enough to assume $A(0)\in L_n(\Omega)$ to solve (2.3.3), but it is not sufficient to solve (2.3.2)).

The Sobolev imbedding theorem gives the fact $\mathring{W}_n^1(\Omega)\subset\bigcap_{s\geq n}L_s(\Omega)$. Therefore we can apply the argument of the previous subsection.

THEOREM 2.4. *If $A(0)$ belongs to $\mathring{W}_n^1(\Omega)\cap\bigcap_{s\geq n}W_s^{2-2/s}(\Omega)$, if $\|A(0)\|_{n,\Omega}$ is small and if it is Hölder continuous up to its second order derivatives, then there exists a unique global classical solution to* (2.3.2)–(2.3.3). *Moreover if $A(0)$ is C^∞ and the compatibility conditions of any order hold between the initial and boundary data in the case $\partial\Omega\neq\varnothing$, then the solution is also C^∞.*

PROOF. It is enough to see the regularity of $g(t)$. This follows from the theorems of regularity and continuous dependence on parameters of ordinary differential equations. ■

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DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
SENDAI 980
JAPAN