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EISENSTEIN SERIES ON WEAKLY SPHERICAL HOMOGENEOUS SPACES OF $GL(n)$

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Abstract. A homogeneous space of a reductive group is called weakly spherical if the action of some proper parabolic subgroup is prehomogeneous. We associate Dirichlet series with weakly spherical homogeneous spaces defined over the rational number field and prove their functional equations in the case where the space under consideration is a homogeneous space of the general linear group.

Introduction.

0.1. Let G be a connected reductive algebraic group and P a proper parabolic subgroup. A homogeneous space $X=G/H$ of G is said to be *P-spherical* if there exists a Zariski-open P-orbit in X. In this case we also say that (G, H, P) is a *spherical triple*. We call X *spherical* (resp. *weakly spherical*) if X is B-spherical (resp. P-spherical) for a Borel subgroup B (resp. for some proper parabolic subgroup P). It is well-known that symmetric spaces are spherical (cf. [V]).

In [S3], [S5], [S6] and [HS], we introduced generalized Eisenstein series attached to (not necessarily Riemannian) symmetric spaces with $\boldsymbol{Q}$-structure and proved that, in a number of cases, the generalized Eisenstein series have nice analytic properties (analytic continuation, functional equations) similar to the properties of the Selberg-Langlands Eisenstein series. However, in the definition of the generalized Eisenstein series given in [HS], the assumption that a homogeneous space in question is a symmetric space is irrelevant and what is essential is that it is (weakly) spherical. Therefore one can naturally ask to what extent the results in the papers cited above can be generalized to general weakly spherical homogeneous spaces.

In [S7], we have shown that the theory of zeta functions in one variable associated with prehomogeneous vector spaces developed in [SS] gives an affirmative answer to the question above in the case where $G=GL(n)$ and P is its maximal parabolic subgroup.

In the present paper, we consider the case where G is a product of several general linear groups and P is its (not necessarily maximal) parabolic subgroup.

0.2. Set $G=GL(m_1)\times\cdots\times GL(m_l)$ and $\Gamma=SL(m_1, \boldsymbol{Z})\times\cdots\times SL(m_l, \boldsymbol{Z})$. Let P be a standard parabolic subgroup and H a reductive $\boldsymbol{Q}$-subgroup of G such that $X=G/H$ is P-spherical. Let Ω be the open P-orbit in X. We put $\mathfrak{a}^*_{P,\boldsymbol{C}}=\mathrm{Hom}_{\boldsymbol{Q}}(P, \boldsymbol{G}_m)\otimes\boldsymbol{C}$. Then

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an element $\lambda \in \mathfrak{a}^*_{P,C}$ defines a quasi-character $p \mapsto p^\lambda$ of the identity component P^+ of $P(\boldsymbol{R})$. We denote by δ_P the half sum of negative roots, which we regard as an element in $\mathfrak{a}^*_{P,C}$.

Let $\Omega_1, \ldots, \Omega_\nu$ be the connected components of $\Omega(\boldsymbol{R})$. Put $\Gamma_P = \Gamma \cap P^+$. Under a certain assumption (Assumption (1.2)) on the isotropy subgroup $P_x = \{p \in P \mid p \cdot x = x\}$, we can define the density $\mu(x)$ for each Γ_P-orbit $\Gamma_P \cdot x$ ($x \in \Omega(\boldsymbol{Q})$). Then, for $x \in X(\boldsymbol{Q})$ and $\lambda \in \mathfrak{a}^*_{P,C}$, the Eisenstein series are defined to be the infinite series

$$E_i(P; x, \lambda) = \sum_{y \in \Gamma_P \backslash (\Gamma \cdot x \cap \Omega_i)} \mu(y) |f(P; y)|^{-(\lambda + \delta_P)} \qquad (i = 1, \ldots, \nu),$$

where $|f(P; y)|^\lambda$ is the function on $\Omega(\boldsymbol{R})$ satisfying $|f(P; p \cdot y)|^\lambda = p^\lambda |f(P; y)|^\lambda$ (for the reason why we call the Dirichlet series *Eisenstein series*, see [S7, §2.1, Remark (1)]).

We assume the convergence of these series, (a sufficient condition for convergence is given in Theorem 3.1). Then we can obtain the following theorem (Theorem 3.2) on analytic continuation of the Eisenstein series:

THEOREM A. *Suppose that*

$$P_x \ (x \in \Omega) \text{ are reductive and } H \text{ is a reductive subgroup of } SL(m_1) \times \cdots \times SL(m_l).$$

Then the Eisenstein series $E_i(P; x, \lambda)$ have analytic continuations to meromorphic functions on $\mathfrak{a}^*_{P,C}$.

To formulate the functional equations, we need the notion of "X-associatedness" of parabolic subgroups (for the definition, see §3.2). Let $\mathscr{P}$ be an X-associated class of parabolic subgroups. For $P, P' \in \mathscr{P}$, we define a certain subset $W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{P'})$ of the Weyl group of G (§3.2). Then we have the following (Theorem 3.5):

THEOREM B. *Under the same assumptions as in Theorem* A, *the following functional equation holds for $x \in X(\boldsymbol{Q})$, $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{P'})$:*

$$\begin{pmatrix} E_1(P'; x, w\lambda) \\ \vdots \\ E_\nu(P'; x, w\lambda) \end{pmatrix} = C_{\mathrm{sph}}(w, \lambda) \begin{pmatrix} E_1(P; x, \lambda) \\ \vdots \\ E_\nu(P; x, \lambda) \end{pmatrix} C_{\mathrm{Eis}}(w, \lambda),$$

where $C_{\mathrm{sph}}(w, \lambda)$ is a ν by ν matrix whose entries have an elementary expression in terms of the gamma function and the exponential function, and $C_{\mathrm{Eis}}(w, \lambda)$ has an expression as a product of the Riemann zeta function and the gamma function.

Let K be a maximal compact subgroup of $G(\boldsymbol{R})$. It will turn out that $C_{\mathrm{sch}}(w, \lambda)$ has its origin in the functional equation satisfied by the K-invariant spherical functions on $X(\boldsymbol{R})$, hence it depends only on P and the real structure of the homogeneous space X (Theorem 3.7). On the other hand, $C_{\mathrm{Eis}}(w, \lambda)$ comes from the functional equation satisfied by the Selberg Eisenstein series of Γ and is independent of H.

0.3. There exists an intimate relation between prehomogeneous vector spaces and weakly spherical homogeneous spaces of $GL(n)$ and the proof of the theorems above is based on the theory of prehomogeneous vector spaces ([S1], [S2]). As is discussed in [S7], in some sense, the theory of zeta functions associated with prehomogeneous vector spaces is nothing but the theory of Eisenstein series on weakly spherical homogeneous spaces of A_n-type in disguise. To extend the theorems to other G than $GL(n)$, we can no longer appeal to the theory of prehomogeneous vector spaces. The functional equations in Theorem B suggest another possible way of proving the theorems, namely, the use of the Rankin-Selberg method applied to functions of non-rapid decay. It is quite probable that our generalized Eisenstein series can be obtained by regularizing the so-called Eisenstein periods. We shall discuss this topic in [S9].

0.4. The organization of the present paper is as follows. In §1, following [S7], we give a definition of the Eisenstein series on weakly spherical homogeneous spaces. In §2, we associate to a spherical triple (G, H, P) with $G=GL(m_1)\times\cdots\times GL(m_l)$ a prehomogeneous vector space, which plays a crucial role in the later part of the present paper. In §3, we present a precise formulation of our main results (Theorem 3.1, Theorem 3.2, Theorem 3.5, Theorem 3.6, Theorem 3.7) and discuss the following examples:

1. $(G, H, P)=(GL(n), O(n), P)$, $P=$ a parabolic subgroup,
2. $(G, H, P)=(GL(2)^3, SL(2), P)$, $P=$ the Borel subgroup,
3. $(G, H, P)=(GL(n), N, P)$, $P=$ the Borel subgroup and $N=$ its unipotent radical.

The first example is a generalization of the result in [S3] to the case of not necessarily minimal parabolic subgroups. Another example will be given in [S8], where we make a detailed investigation of the Eisenstein series on a weakly spherical homogeneous space related to the half-spin representation of $Spin_{10}$. In §4, we prove a convergence criterion of the Eisenstein series and the theorem on analytic continuation. The proof of the functional equations satisfied by our Eisenstein series is given in §5. In Appendix we show how to calculate the explicit formulas for the functional equations in the case of $(GL(n), O(n), P)$.

NOTATION. As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ stand for the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. For a linear algebraic group G defined over the rational number field $\boldsymbol{Q}$, we denote by G° the identity component of G and by $\mathfrak{X}(G)$ the group of rational characters of G defined over $\boldsymbol{Q}$. The unipotent radical of G is denoted by $R_u(G)$. For a real vector space V, $\mathscr{S}(V)$ stands for the space of rapidly decreasing functions on V. The symmetric group in n letters is denoted by $\mathfrak{S}_n$. We denote by 1_m the identity matrix of size m and by $0_{m,n}$ the $m\times n$ zero matrix. For a matrix A, we denote by ${}^tA$ the transposed matrix.

1. Eisenstein series on weakly spherical homogeneous spaces. Following §1 of [S7], we recall the definition of Eisenstein series on weakly spherical homogeneous spaces.

Let G be a connected reductive algebraic group defined over $\boldsymbol{Q}$ and H a $\boldsymbol{Q}$-subgroup of G. Put $X=G/H$. Let P be a proper $\boldsymbol{Q}$-parabolic subgroup of G and assume that

(1.1) *(G, H, P) is a spherical triple, namely, there exists a Zariski-open P-orbit Ω in X.*

For an $x\in\Omega$, we put

$$P_x=\{p\in P \mid p\cdot x=x\}.$$

We also assume that

(1.2) *for any $x\in\Omega(\boldsymbol{Q})$, P_x is unimodular and the identity component of P_x has no non-trivial $\boldsymbol{Q}$-rational characters.*

We denote by $\mathfrak{X}(P)$ the group of rational characters of P defined over $\boldsymbol{Q}$. Put

$$\mathfrak{X}_X(P)=\{\chi\in\mathfrak{X}(P) \mid \chi_{|P_x}\equiv 1\ (x\in\Omega)\}.$$

Then, for any $\chi\in\mathfrak{X}_X(P)$, there exists a non-zero rational function $f\in\boldsymbol{Q}(X)^\times$ satisfying

$$f(p\cdot x)=\chi(p)f(x) \qquad (p\in P,\ x\in X),$$

which is unique up to a constant factor. The group $\mathfrak{X}_X(P)$ is a free abelian group of finite rank. Let l be the rank of $\mathfrak{X}_X(P)$. Choose a system of generators $\{\chi_1, \dots, \chi_l\}$ of $\mathfrak{X}_X(P)$. For each $i=1, \dots, l$, fix a relative invariant $f_i\in\boldsymbol{Q}(X)^\times$ satisfying $f_i(p\cdot x)=\chi_i(p)f_i(x)$.

For $K=\boldsymbol{Q}$, $\boldsymbol{R}$ or $\boldsymbol{C}$, we put $\mathfrak{a}^*_{P,K}=\mathfrak{X}_X(P)\otimes_{\boldsymbol{Z}}K$. By Assumption (1.2), $\mathfrak{X}_X(P)$ is of finite index in $\mathfrak{X}(P)$; hence we may identify $\mathfrak{a}^*_{P,K}$ with $\mathfrak{X}(P)\otimes_{\boldsymbol{Z}}K$.

For $\lambda=\sum_{i=1}^{l}\lambda_i\chi_i\in\mathfrak{a}^*_{P,\boldsymbol{C}}$, we define a function $|f(P;x)|^\lambda$ on $\Omega(\boldsymbol{R})$ and $|\chi(p)|^\lambda$ on $P(\boldsymbol{R})$ by

$$(1.3)\qquad |f(P;x)|^\lambda=\prod_{i=1}^{l}|f_i(x)|^{\lambda_i}, \quad |\chi(p)|^\lambda=\prod_{i=1}^{l}|\chi_i(p)|^{\lambda_i}.$$

Then we have

$$|f(P;p\cdot x)|^\lambda=|\chi(p)|^\lambda|f(P;x)|^\lambda.$$

Let P^+ be an open subgroup of the real Lie group $P(\boldsymbol{R})$ and

$$\Omega(\boldsymbol{R})=\Omega_1\cup\cdots\cup\Omega_\nu$$

the P^+-orbit decomposition of $\Omega(\boldsymbol{R})$.

We fix a right invariant $\boldsymbol{Q}$-rational gauge form ω_P on P and let $d\omega_P(p)$ the right invariant measure on P^+ induced by ω_P. Let Δ_P be the module of P, namely, the character of P given by $\omega_P(gp)=\Delta_P(g)\omega_P(p)$ $(p, g\in P)$. By Assumption (1.2), there exists a relatively P-invariant $\boldsymbol{Q}$-rational gauge form ω_Ω on Ω such that

$$\omega_\Omega(p\cdot x)=\Delta_P(p)\omega_\Omega(x) \qquad (p\in P,\ x\in\Omega).$$

We denote by $d\omega_\Omega(x)$ the relatively P^+-invariant measure on $\Omega(\boldsymbol{R})$ induced by ω_Ω. For an $x\in\Omega(\boldsymbol{Q})$, we define an invariant $\boldsymbol{Q}$-rational gauge form ω_x on P_x by $\omega_x=\omega_P/\omega_\Omega$. Let

$\iota_P\colon P_x \to P_{p\cdot x}$ be the isomorphism defined by $\iota_p(g) = pgp^{-1}$. Then we have

$$\iota_p^*(\omega_{p\cdot x}) = \omega_x . \tag{1.4}$$

This identity characterizes the invariant gauge forms ω_x $(x \in \Omega)$ uniquely up to a constant factor independent of x. We denote by $d\mu_x$ the Haar measure on $P_x^+ = P^+ \cap P_x$ induced by ω_x. Thus we can specify the normalization of the Haar measure on P_x^+ for all $x \in \Omega(\boldsymbol{Q})$ uniquely (up to a constant factor depending only on the normalization of ω_P and ω_Ω).

We consider Δ_P as an element of $\mathfrak{a}_{P,\boldsymbol{R}}^*$ and put

$$\delta_P = -\frac{1}{2}\Delta_P .$$

Take an arithmetic subgroup Γ of $G(\boldsymbol{Q})$ and put $\Gamma_P = \Gamma \cap P^+$. For an $x \in X(\boldsymbol{Q})$, we define the Eisenstein series $E_i(P; x, \lambda)$ $(1 \le i \le \nu)$ by

$$E_i(P; x, \lambda) = \sum_{y \in \Gamma_P \diagdown \Omega_i \cap \Gamma \cdot x} \mu(y)/|f(P; y)|^{\lambda + \delta_P} , \tag{1.5}$$

where

$$\mu(y) = \int_{P_y^+/P_y^+ \cap \Gamma_P} d\mu_y .$$

Assumption (1.2) implies the finiteness of the volume $\mu(y)$.

REMARKS. (1) In [HS, §3], on the basis of a measure theoretic interpretation of $\mu(x)$, we gave an apparently different definition of Eisenstein series. The relation of these two definitions is given in [HS, Proposition 3.2]. Note that, in the argument in [HS, §3], the assumption that X is a symmetric space is not necessary.

(2) We have defined the Eisenstein series only for rational points x. We can not expect the convergence of the infinite series (1.5) for irrational points unless the group $G_x(\boldsymbol{R}) = \{g \in G(\boldsymbol{R}) \mid g \cdot x = x\}$ is compact. To see this, let us consider the case where $G = GL(n)$, $H = O(n)$ and X is the space of nondegenerate symmetric matrices of size n. Then, if $x \in X(\boldsymbol{R})$ is indefinite and is not a multiple of a rational symmetric matrix, the generalized Raghunathan conjucture proved by Ratner implies the divergence of the Eisenstein series (see [R, §5], and [Mar, §5.2, Remark]).

We also define local zeta functions by

$$\Psi_i(P; \phi, \lambda) = \int_{\Omega_i} |f(P; x)|^{\lambda + \delta_P} \phi(x) d\omega_\Omega(x) \qquad (\phi \in C_0^\infty(X(\boldsymbol{R}))) . \tag{1.6}$$

Let $\mathfrak{X}_X(P)^+$ be the subset of $\mathfrak{X}_X(P)$ of characters corresponding to relative P-invariants that are regular everywhere on X. Let $\mathfrak{a}_{P,\boldsymbol{R}}^{*+}$ be the interior of the cone in $\mathfrak{a}_{P,\boldsymbol{R}}^*$ generated by $\mathfrak{X}_X(P)^+$. We put

$$\mathfrak{a}_{P,\boldsymbol{C}}^{*+} = \mathfrak{a}_{P,\boldsymbol{R}}^{*+} + \sqrt{-1}\,\mathfrak{a}_{P,\boldsymbol{R}}^* \subset \mathfrak{a}_{P,\boldsymbol{C}}^* .$$

The local zeta functions $\Psi_i(P; \phi, \lambda)$ $(1 \leq i \leq \nu)$ converge absolutely for $\lambda \in -\delta_P + \mathfrak{a}_{P,C}^{*+}$.

The following is the conjecture posed in [S7].

CONJECTURE 1.1. Under certain mild assumptions (including Assumption (1.2)), the Eisenstein series $E_i(P; x, \lambda)$ $(1 \leq i \leq \nu)$ have the following properties:

(1) $E_i(P; x, \lambda)$ are absolutely convergent on $\delta_P + \mathfrak{a}_{P,C}^{*+}$.

(2) $E_i(P; x, \lambda)$ have analytic continuations to meromorphic functions on $\mathfrak{a}_{P,C}^{*}$.

(3) Under the action of (some subquotient of) the Weyl group of G, the Eisenstein series $E_i(P; x, \lambda)$ satisfy functional equations of the form of the tensor product of the functional equations of the Langlands Eisenstein series and those of the local zeta functions $\Psi_i(P; \phi, \lambda)$ for functions ϕ that are invariant under the action of a maximal compact subgroup of $G(\boldsymbol{R})$.

At present we do not have any precise formulation of the conditions for general (G, H, P) that guarantee the validity of the conjecture. Therefore it might be better to understand the conjecture as a problem of finding a good condition under which the three properties above hold. In the rest of this paper we give an affirmative answer to this problem in the case where $G = GL(m_1) \times \cdots \times GL(m_l)$ and $\Gamma = SL(m_1, \boldsymbol{Z}) \times \cdots \times SL(m_l, \boldsymbol{Z})$.

2. Weakly spherical homogeneous spaces of $GL(m)$ and prehomogeneous vector spaces.

2.1. Since our whole argument is based on the relation between weakly spherical homogeneous spaces of $GL(m)$ and prehomogeneous vector spaces, we begin by recalling some basic definitions in the theory of prehomogeneous vector spaces.

Let k be a field of characteristic 0. Let G be a connected linear algebraic group defined over k and $\rho: G \to GL(V)$ a rational representation of G on a finite dimensional vector space. Assume that V has a k-structure for which ρ is defined over k. Denote by $\bar{k}$ the algebraic closure of k. The triple (G, ρ, V) is called a *prehomogeneous vector space* if there exists a proper algebraic subset S such that the complement $V(\bar{k}) - S(\bar{k})$ is a single $G(\bar{k})$-orbit.

A rational function f on V is called a *relative* (*G-*)*invariant* if there exists a rational character χ of G such that $f(\rho(g)v) = \chi(g)f(v)$ $(g \in G, v \in V)$. In this case we say that f (resp. χ) corresponds to χ (resp. f). If χ is defined over k, then one can choose the corresponding relative invariant f from $k(V)$.

Let $S_1, \ldots, S_r$ be the k-irreducible hypersurfaces contained in S. For $i = 1, \ldots, r$, let f_i be a k-irreducible polynomial on V defining S_i. The polynomial f_i is unique up to a non-zero constant factor in k. Then $f_1, \ldots, f_r$ are relative invariants and any relative invariant f in $k(V)$ is of the form $f = cf_1^{i_1} \cdots f_r^{i_r}$ $(c \in k, i_j \in \boldsymbol{Z})$. We call $f_1, \ldots, f_r$ the *fundamental* relative invariants over k.

A prehomogeneous vector space (G, ρ, V) is called *regular* if there exists a relative invariant polynomial $f(v)$ such that the Hessian $\det(\partial^2 f / \partial v_i \partial v_j)$ does not vanish identically.

For further details, we refer to [SK] and [S1].

2.2. For an ordered partition $e_1+\cdots+e_r=n$ of n, let $P_{e_1,\dots,e_r}$ be the standard parabolic subgroup of $GL(n)$ consisting of matrices in block form $g=(g_{ij})_{1\le i,j\le r}$ where g_{ij} is an e_i by e_j matrix and $g_{ij}=0$ for $i<j$. For a standard parabolic subgroup $P=P_{e_1,\dots,e_r}$, we denote by L_P the standard Levi subgroup of P consisting of matrices of diagonal form. In the following, parabolic subgroups are always standard in this sense.

Let $G=GL(m_1)\times\cdots\times GL(m_l)$ and $P=P_1\times\cdots\times P_l$ its standard parabolic subgroup. Let $e_1^{(k)}+\cdots+e_{r_k}^{(k)}=m_k$ be the ordered partition of m_k corresponding to the standard parabolic subgroup P_k of $GL(m_k)$. Put $n_i^{(k)}=e_1^{(k)}+\cdots+e_i^{(k)}$ $(1\le i\le r_k)$. We also put $n_{r_k+1}^{(k)}=n_{r_k}^{(k)}=m_k$. Let

$$G_P=G_{P_1}\times\cdots\times G_{P_l} \qquad \text{for} \quad G_{P_k}=\prod_{i=1}^{r_k-1} GL(n_i^{(k)})\,,$$

$$\tilde G_P=\tilde G_{P_1}\times\cdots\times \tilde G_{P_l} \qquad \text{for} \quad \tilde G_{P_k}=\prod_{i=1}^{r_k} GL(n_i^{(k)})$$

and

$$V_P=V_{P_1}\oplus\cdots\oplus V_{P_l} \qquad \text{for} \quad V_{P_k}=\bigoplus_{i=1}^{r_k-1} M(n_i^{(k)}, n_{i+1}^{(k)})\,,$$

$$\tilde V_P=\tilde V_{P_1}\oplus\cdots\oplus \tilde V_{P_l} \qquad \text{for} \quad \tilde V_{P_k}=\bigoplus_{i=1}^{r_k} M(n_i^{(k)}, n_{i+1}^{(k)})\,.$$

Then we have $\tilde G_P=G_P\times G$ and $\tilde V_P=V_P\oplus(M(m_1)\oplus\cdots\oplus M(m_l))$. For a

$$g=(g_i^{(k)})_{\substack{k=1,\dots,l\\ i=1,\dots,r_k+1}}\in\prod_{k=1}^{l}\prod_{i=1}^{r_k+1} GL(n_i^{(k)})=\tilde G_P\times G$$

and a

$$v=(v_i^{(k)})_{\substack{k=1,\dots,l\\ i=1,\dots,r_k}}\in\prod_{k=1}^{l}\prod_{i=1}^{r_k} M(n_i^{(k)}, n_{i+1}^{(k)})=\tilde V_P\,,$$

set

$$\tilde\rho_P(g)v=(g_i^{(k)}v_i^{(k)}g_{i+1}^{(k)^{-1}})_{\substack{k=1,\dots,l\\ i=1,\dots,r_k}}\,.$$

Then $\tilde\rho_P$ define a rational representation of $\tilde G_P\times G$ on $\tilde V_P$. The subspace V_P is an invariant subspace and $W_P=M(m_1)\oplus\cdots\oplus M(m_l)$ is the complementary subspace. We denote the subrepresentations of V_P and W_P by ρ_P and ρ_P', respectively.

Let H be a closed subgroup of G and consider the triples $(\tilde G_P\times H, \tilde\rho_P, \tilde V_P)$ and $(G_P\times H, \rho_P, V_P)$, where we regard $\tilde G_P\times H$ and $G_P\times H$ as subgroups of $\tilde G_P\times G$ and $\tilde G_P$, respectively.

PROPOSITION 2.1. *The following three assertions are equivalent*:

(1) (G, H, P) *is a spherical triple.*

(2) $(G_P \times H, \rho_P, V_P)$ *is a prehomogeneous vector space.*

(3) $(\tilde{G}_P \times H, \tilde{\rho}_P, \tilde{V}_P)$ *is a prehomogeneous vector space.*

PROOF. Both of the following two conditions are equivalent to the third assertion:

(2.1) (i) $(\tilde{G}_P, \rho_P, V_P)$ is a prehomogeneous vector space, and
(ii) for a point $\boldsymbol{v}$ in the open orbit of $(\tilde{G}_P, \rho_P, V_P)$, denote by $\tilde{G}_{P,\boldsymbol{v}}$ the isotropy subgroup at $\boldsymbol{v}$. Then $(\tilde{G}_{P,\boldsymbol{v}} \times H, \rho'_P, W_P)$ is also a prehomogeneous vector space.

(2.2) (i) $(\tilde{G}_P \times H, \rho'_P, W_P)$ is a prehomogeneous vector space, and
(ii) for a point w in the open orbit of $(\tilde{G}_P \times H, \rho'_P, W_P)$, denote by $(\tilde{G}_P \times H)_w$ the isotropy subgroup at w. Then $((\tilde{G}_P \times H)_w, \rho_P, V_P)$ is also a prehomogeneous vector space.

We prove that (2.1) (resp. (2.2)) is equivalent the first (resp. second) assertion. First consider (2.1). It is obvious that $(\tilde{G}_P, \rho_P, V_P)$ is a prehomogeneous vector space and

$$\boldsymbol{v} = (I_{n_i^{(k)}, n_{i+1}^{(k)}})_{\substack{k=1,\dots,l \\ i=1,\dots,r_k-1}}, \qquad I_{m,n} = (1_m, 0_{m,n-m}) \tag{2.3}$$

belongs to the open orbit. Then we have an isomorphism of P onto $\tilde{G}_{P,\boldsymbol{v}}$ given by

$$P \ni p = (p^{(1)}, \dots, p^{(l)}) \longmapsto ([p^{(k)}]_i)_{\substack{k=1,\dots,l \\ i=1,\dots,r_k}} \in \tilde{G}_{P,\boldsymbol{v}}, \tag{2.4}$$

where $[p^{(k)}]_i$ is the upper left $n_i^{(k)}$ by $n_i^{(k)}$ block of $p^{(k)}$. Hence $\tilde{G}_{P,\boldsymbol{v}} \times H$ acts on W_P through the left action of P and the right action of H. This implies that (2.1) is equivalent to the first assertion. Now we consider (2.2). In this case it is again obvious that $(\tilde{G}_P \times H, \rho'_P, W_P)$ is a prehomogeneous vector space and the isotropy subgroup $(\tilde{G}_P \times H)_w$ at a point w in the open orbit is isomorphic to $G_P \times wHw^{-1}$. Moreover $((\tilde{G}_P \times H)_w, \rho_P, V_P)$ is a prehomogeneous vector space if and only if $(G_P \times H, \rho_P, V_P)$ is a prehomogeneous vector space. This shows that (2.2) is equivalent to the second assertion. ∎

DEFINITION 2.2. (i) The prehomogeneous vector space $(G_P \times H, \rho_P, V_P)$ is called the prehomogeneous vector space *of flag type* attached to the spherical triple (G, H, P).

(ii) A spherical triple (G, H, P) is called *regular* if the prehomogeneous vector space $(G_P \times H, \rho_P, V_P)$ is regular.

A prehomogeneous vector spaces of flag type was introduced in [S3] for $G = GL(n)$ and $H = O(n)$ in order to understand descending chains of quadratic forms of Selberg [Se] from the viewpoint of the theory of prehomogeneous vector spaces. The relation between Eisenstein series on G/H and the zeta functions associated to the prehomogeneous vector space of flag type is the key of our whole argument.

From the proof of Proposition 2.1, it follows that generic isotropy subgroups of $(G_P \times H, \rho_P, V_P)$ and $(\tilde{G}_P \times H, \tilde{\rho}_P, \tilde{V}_P)$ and P_x $(x \in \Omega)$ are all isomorphic. Hence, by [SK, §4, Remark 26], we have the following lemma.

LEMMA 2.3. *Assume that H is reductive. Then, a spherical triple (G, H, P) is regular if and only if P_x is reductive for an $x \in \Omega$.*

3. Statement of the main results. In this section, we describe our main results (§3.1, Theorems 3.1, 3.2, §3.2, Theorems 3.5, 3.6), whose proofs will be given in §4 and §5. In §3.3, we present several examples. We keep the notation in §2.2.

3.1. Let (G, H, P) be a spherical triple satisfying Assumption (1.2).

For $p=(p^{(1)}, \dots, p^{(l)}) \in P = P_1 \times \cdots \times P_l$, we write

$$p^{(k)}=(p_{ij}^{(k)}), \quad p_{ij}^{(k)} \in M(e_i^{(k)}, e_j^{(k)}), \quad p_{ij}^{(k)}=0 \qquad \text{for} \quad i<j .$$

We put

$$\Lambda_i^{(k)}(p)=\det p_{11}^{(k)} \cdots \det p_{ii}^{(k)} \in \mathfrak{X}(P_k), \qquad i=1, \dots, r_k .$$

Then, $\{\Lambda_i^{(k)}\}_{\substack{k=1,\dots,l \\ i=1,\dots,r_k}}$ forms a basis of $\mathfrak{a}_{P,C}^* = = \mathfrak{a}_{P_1,C}^* \oplus \cdots \oplus \mathfrak{a}_{P_l,C}^*$. It is obvious that $\{\det p_{ii}^{(k)}\}_{\substack{k=1,\dots,l \\ i=1,\dots,r_k}}$ gives another basis of $\mathfrak{a}_{P,C}^*$. For $\lambda \in \mathfrak{a}_{P,C}^*$, we write

$$\lambda = \sum_{i=k}^{l} \lambda^{(k)}, \quad \lambda^{(k)} \in \mathfrak{a}_{P_k,C}^*$$

$$\lambda^{(k)} = \sum_{i=1}^{r_k} \lambda_i^{(k)} \Lambda_i^{(k)} = \sum_{i=1}^{r_k} z_i^{(k)}(\lambda) \det p_{ii}^{(k)} .$$

We write simply $z_i^{(k)}$ instead of $z_i^{(k)}(\lambda)$, if there is no fear of confusion.

Using the symbols above, we have

$$\delta_P = \sum_{k=1}^{l} \delta_{P_k}, \qquad \delta_{P_k} = \frac{1}{2} \sum_{i=1}^{r_k} (e_i^{(k)} + e_{i+1}^{(k)}) \Lambda_i^{(k)}, \tag{3.1}$$

where we understand that $e_{r_k+1}^{(k)} = -m_k$. Put

$$\mathfrak{a}_{P,C}^{*++} = \{\lambda \mid \mathrm{Re}(\lambda_i^{(k)})>0,\ k=1, \dots, l,\ i=1, \dots, r_k-1\} . \tag{3.2}$$

As is easily seen, we have

$$\mathfrak{a}_{P,C}^{*+} \subseteq \mathfrak{a}_{P,C}^{*++} .$$

Let P^+ be the identity component of the real Lie group $P(\boldsymbol{R})$ and put $\Gamma = SL(m_1, \boldsymbol{Z}) \times \cdots \times SL(m_l, \boldsymbol{Z})$. Put

$$P^{(1)} = \{p=(p^{(1)}, \dots, p^{(l)}) \in P \mid \det p_{ii}^{(k)}=1,\ 1 \le k \le l,\ 1 \le i \le r_k\} .$$

Then we have the following sufficient condition for the convergence and the existence of the analytic continuations of the Eisenstein series attached to (G, H, P, Γ).

THEOREM 3.1. *If the identity component H° of H has no non-trivial rational characters and $P_x^{(1)} = P^{(1)} \cap P_x$ is connected semisimple or trivial for some $x \in \Omega$, then*

$E_i(P; x, \lambda)$ are absolutely convergent on $\delta_P + \mathfrak{a}^{*+}_{P,\boldsymbol{C}}$.

Let $\zeta(s)$ be the Riemann zeta function and put

$$\zeta_P(\lambda) = \prod_{k=1}^{l} \prod_{1 \le i < j \le r_k} \prod_{\mu=0}^{e_i^{(k)}-1} \zeta\left(z_i^{(k)} - z_j^{(k)} + \frac{e_i^{(k)} + e_j^{(k)}}{2} - \mu \right). \tag{3.3}$$

For our later purpose, we also put

$$\Gamma_P(\lambda) = \prod_{k=1}^{l} \prod_{1 \le i < j \le r_k} \prod_{\mu=0}^{e_i^{(k)}-1} \Gamma_{\boldsymbol{R}}\left(z_i^{(k)} - z_j^{(k)} + \frac{e_i^{(k)} + e_j^{(k)}}{2} - \mu \right) \tag{3.4}$$

and

$$\hat{\zeta}_P(\lambda) = \Gamma_P(\lambda)\zeta_P(\lambda)\,, \tag{3.5}$$

where $\Gamma_{\boldsymbol{R}}(z) = \pi^{-z/2}\Gamma(z/2)$.

THEOREM 3.2. *If H is a reductive subgroup of $SL(m_1) \times \cdots \times SL(m_l)$, (G, H, P) is regular, and $E_i(P; x, \lambda)$ are absolutely convergent in $\delta + \mathfrak{a}^{*+}_{P,\boldsymbol{C}}$ for some $\delta \in \mathfrak{a}^*_{P,\boldsymbol{R}}$, then $E_i(P; x, \lambda)$ have analytic continuations to meromorphic functions of λ on $\mathfrak{a}^*_{P,\boldsymbol{C}}$. Moreover, there exist a finite number of linear forms $L_1, \dots, L_d$ on $\mathfrak{a}^*_{P,\boldsymbol{C}}$ with $\boldsymbol{Z}$-values on $\mathfrak{X}(P)$ and rational numbers $a_1, \dots, a_d$ such that the functions*

$$\prod_{j=1}^{d} (\langle L_j, \lambda \rangle + a_j) \cdot \zeta_P(\lambda) E_i(P; x, \lambda)$$

are entire functions on $\mathfrak{a}^*_{P,\boldsymbol{C}}$.

REMARK. There exists a relation between $\prod_{j=1}^{d}(\langle L_j, \lambda \rangle + a_j)$ and b-functions of relative invariants (see the proof of Theorem 3.2 in §4, and Proposition 5.13).

Combining two theorems above, we obtain the following:

COROLLARY 3.3. *If H is semisimple and $P_x^{(1)}$ is connected semisimple or trivial for some $x \in \Omega$, then $E_i(P; x, \lambda)$ are absolutely convergent on $\delta_P + \mathfrak{a}^{*+}_{P,\boldsymbol{C}}$ and have analytic continuations to meromorphic functions of λ on $\mathfrak{a}^*_{P,\boldsymbol{C}}$.*

3.2. To formulate the functional equation, we need some preliminaries. In the following (except in Example 3 in §3.3), we always assume that H is reductive and (G, H, P) is regular. Then, by Lemma 2.3, P_x $(x \in \Omega)$ is also reductive.

Take a parabolic subgroup Q containing P and let L_Q be the standard Levi subgroup of Q. Put $P_Q = P \cap L_Q$. The group P_Q is a standard parabolic subgroup of L_Q.

Let $w = w_Q$ be the permutation matrix that represents the longest element of the Weyl group of L_Q. We define a parabolic subgroup ${}^wP$ associated to P by

$${}^wP = w\,{}^tP_Q w^{-1} \cdot R_u(Q)\,.$$

Since the inner automorphism defined by w maps the standard Levi subgroup of P onto that of ${}^{w}P$, w also defines a canonical linear isomorphism $\mathfrak{a}^*_{P,\mathbf{C}} \to \mathfrak{a}^*_{{}^wP,\mathbf{C}}$, which we denote by the same symbol w.

Fix an $x \in \Omega$ and put $Q_x = \{h \in Q \mid hx = x\}$. Since P_x is reductive, the canonical surjection $Q \to L_Q \cong Q/R_u(Q)$ maps Q_x isomorphically into L_Q. Let H_Q be the image of Q_x in L_Q.

PROPOSITION 3.4. (i) *(L_Q, H_Q, P_Q) is a spherical triple.*

(ii) *Suppose that (L_Q, H_Q, P_Q) is a regular spherical triple. Then $(G, H, {}^wP)$ is also a regular spherical triple and P_x $(x \in \Omega)$ is isomorphic to $({}^wP)_{x'}$ $(x' \in {}^w\Omega)$, where ${}^w\Omega$ is the open ${}^wP$-orbit in X.*

We postpone the proof of the proposition above until §5.1.

REMARK. Let P and P' be parabolic subgroups of G associated to each other, namely, the partition corresponding to P is a permutation of the partition corresponding to P'. In general it may happen that (G, H, P) is a spherical triple and (G, H, P') is not. For example, let $G = GL(m)$ $(m = n(2n+1))$ and H the image of the second skew symmetric tensor representation of $SL(2n+1)$. Then $(G, H, P_{m-2,1,1})$ is a spherical triple, however $(G, H, P_{1,m-2,1})$ is not (cf. [KKO, Proposition 2.3]).

We say that two parabolic subgroups P and P' of G are *(X, Q)-associated* if Q is a parabolic subgroup containing P, (L_Q, H_Q, P_Q) is regular and $P' = {}^wP$ $(w = w_Q)$. Note that this condition does not depend on the choice of the point $x \in \Omega$. We say that P and P' are *X-associated* if there exists a sequence $Q_1, \dots, Q_s$ of parabolic subgroups satisfying

(i) $Q_1 \supset P_1 := P$,

(ii) for any $i = 1, \dots, s-1$, $Q_{i+1} \supset P_{i+1} := {}^{w_i}P_i$ $(w_i = w_{Q_i})$ and $P' = P_{s+1} := {}^{w_s}P_s$,

(iii) for any $i = 1, \dots, s$, P_i and P_{i+1} are (X, Q_i)-associated.

We denote by $W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{P'})$ the set of the mappings $w = w_s \circ w_{s-1} \circ \cdots \circ w_1 : \mathfrak{a}^*_{P,\mathbf{C}} \to \mathfrak{a}^*_{P',\mathbf{C}}$ obtained from sequences $Q_1, \dots, Q_s$ satisfying the conditions above. The set $W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{P'})$ is empty, if P and P' are not X-associated.

Let $\mathscr{P}$ be an X-associated class of standard parabolic subgroups of G.

We formulate our main results under the same assumption as in Theorem 3.2. Namely, we assume that

(3.6) *H is a reductive subgroup of $SL(m_1) \times \cdots \times SL(m_l)$. Moreover, for any $P \in \mathscr{P}$, (G, H, P) is a regular spherical triple and the Eisenstein series $E_i(P; x, \lambda)$ are absolutely convergent in $\delta + \mathfrak{a}^{*+}_{P,\mathbf{C}}$ for some $\delta \in \mathfrak{a}^*_{P,\mathbf{R}}$.*

The assumption implies the meromorphic continuation of the Eisenstein series.

It follows from Proposition 3.4 that, if (G, H, P) is regular for some $P \in \mathscr{P}$, then (G, H, P') is regular for any $P' \in \mathscr{P}$ and if (G, H, P) satisfies the assumption in Theorem 3.1 for some $P \in \mathscr{P}$, then so does (G, H, P') for any $P' \in \mathscr{P}$.

Let G^+ be the identity component of $G(\mathbf{R})$ and fix a G^+-orbit $X^{(0)}$ in $X(\mathbf{R})$. Let

$\Omega_1, \dots, \Omega_\nu$ be the connected components of $X^{(0)} \cap \Omega$, which are P^+-orbits. The number ν of the connected components depends only on the X-associated class $\mathscr{P}$ (cf. Lemma 5.4). We may restrict our attention to the Eisenstein series defined for these P^+-orbits, which we denote by $E_1(P; x, \lambda), \dots, E_\nu(P; x, \lambda)$, since the functional equations are relations between the Eisenstein series corresponding to the open P^+-orbits in a fixed G^+-orbit (for various $P \in \mathscr{P}$).

For $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$, we define $C_{\mathrm{Eis}}(w, \lambda)$ as follows. First assume that P and P' are (X, Q)-associated. Then, for $w = w_Q \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$, we put

$$C_{\mathrm{Eis}}(w_Q, \lambda) = \frac{\hat{\zeta}_{P_Q}(\lambda)}{\hat{\zeta}_{P_Q}(-\lambda)} \tag{3.7}$$

(for the definition of $\hat{\zeta}_{P_Q}$, see (3.5)). Here we can define $\hat{\zeta}_{P_Q}(\lambda)$ for $\lambda \in \mathfrak{a}_{P,C}^*$, since $\mathfrak{a}_{P_Q,C}^*$ can naturally be identified with $\mathfrak{a}_{P,C}^*$. For general $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$, we define $C_{\mathrm{Eis}}(w, \lambda)$ by the identity

$$C_{\mathrm{Eis}}(w'w, \lambda) = C_{\mathrm{Eis}}(w', w\lambda) C_{\mathrm{Eis}}(w, \lambda) \qquad (w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*),\ w' \in W_X(\mathfrak{a}_{P'}^*, \mathfrak{a}_{P''}^*)).$$

THEOREM 3.5. *For $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$, the following functional equation holds*:

$$\begin{pmatrix} E_1(P'; x, w\lambda) \\ \vdots \\ E_\nu(P'; x, w\lambda) \end{pmatrix} = C_{\mathrm{sph}}(w, \lambda) \begin{pmatrix} E_1(P; x, \lambda) \\ \vdots \\ E_\nu(P; x, \lambda) \end{pmatrix} C_{\mathrm{Eis}}(w, \lambda)\,,$$

where $C_{\mathrm{sph}}(w, \lambda)$ is a ν by ν matrix whose entries have an elementary expression in terms of the gamma function and the exponential function.

Put $K = SO(m_1) \times \cdots \times SO(m_l)$. The matrix $C_{\mathrm{sph}}(w, \lambda)$ appearing in the functional equation above has its origin in the functional equation satisfied by the K-invariant spherical functions on $X^{(0)}$ and depends only on the real structure of $X = G/H$. Namely we have the following theorem.

THEOREM 3.6. *For any K-invariant function $\phi \in C_0^\infty(X)$, the local zeta functions $\Psi_i(P; \phi, \lambda)$ have analytic continuations to meromorphic functions of λ in $\mathfrak{a}_{P,C}^*$ and the following functional equation holds for any $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$*:

$$\begin{pmatrix} \Psi_1(P; \phi, \lambda) \\ \vdots \\ \Psi_\nu(P; \phi, \lambda) \end{pmatrix} = {}^t C_{\mathrm{sph}}(w, \lambda) \begin{pmatrix} \Psi_1(P'; \phi, w\lambda) \\ \vdots \\ \Psi_\nu(P'; \phi, w\lambda) \end{pmatrix}.$$

For $i = 1, \dots, \nu$, we define the function $|f(P; x)|_i^\lambda$ by

$$|f(P; x)|_i^\lambda = \begin{cases} |f(P; x)|^\lambda & \text{for} \quad x \in \Omega_i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Put

$$\omega_i(P; x, \lambda) = \int_K |f(P; k \cdot x)|_i^{\lambda - \delta_P} dk \,.$$

The integrals $\omega_i(P; x, \lambda)$ $(i, \dots, \nu)$ are absolutely convergent for $\lambda \in \delta_P + \mathfrak{a}_{P,C}^{*+}$. Since the integrals are analogous to the Harish Chandra integral representation of the zonal spherical function of a semisimple Lie group, we call them the *K-invariant spherical functions* on $X^{(0)} \cong G^+/H^+$.

Now Theorem 3.6 has the following reformulation:

THEOREM 3.7. *The spherical functions $\omega_i(P; x, \lambda)$ have analytic continuations to meromorphic functions of λ in $\mathfrak{a}_{P,C}^*$ and the following functional equation holds for any $P, P' \in \mathscr{P}$ and $w \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$:*

$$\begin{pmatrix} \omega_1(P; x, \lambda) \\ \vdots \\ \omega_\nu(P; x, \lambda) \end{pmatrix} = {}^t C_{\mathrm{sph}}(w, \lambda) \begin{pmatrix} \omega_1(P'; x, w\lambda) \\ \vdots \\ \omega_\nu(P'; x, w\lambda) \end{pmatrix}.$$

If the G^+-orbit $X^{(0)}$ is a Riemannian symmetric space, then $\nu = 1$ and associated parabolic subgroups are X-associated. In this case, the integral $\omega_1(P; x, \lambda)$ gives the zonal spherical function of G^+ and the Eisenstein series $E_1(P; x; \lambda)$ is the series investigated in [L], [M], [T]. Since the zonal spherical function is invariant under the action of the Weyl group, we have $C_{\mathrm{sph}}(w, \lambda) = 1$ and the functional equation in Theorem 3.5 coincides with the one given in [M] and [T]. This shows that the origin of $C_{\mathrm{Eis}}(w, \lambda)$ is the functional equation of the Selberg-Langlands Eisenstein series.

Thus we see that the functional equations of $E_i(P; x, \lambda)$ are of the form of the tensor products of the functional equations of the K-invariant spherical functions and those of the Langlands Eisenstein series as alluded to in the third part of Conjecture 1.1.

We expect that there exists a generalization of the notion of regular spherical triple to weakly spherical homogeneous spaces of other reductive groups than $GL(m)$ and the two theorems above remain valid. In the case of reductive symmetric spaces, the regularity seems closely related to the notion of "σ-split" parabolic subgroups in the sense of [He] ("σ-anisotropic" in [V], "$\sigma\theta$-stable" in [Ba]). In this case Theorem 3.7 has been already known and is contained in the functional equations satisfied by the Eisenstein integrals given in [Ba, Proposition 16.4], while Theorem 3.5 has been proved only in some special cases other than the case of $G = GL(n)$ (cf. [S5], [S6]).

3.3. In this subsection we keep the notation in the preceding subsections except that we omit the superscript $^{(k)}$ when $l = 1$ and $G = GL(m)$.

EXAMPLE 1. $(G, H, P) = (GL(m), O(m), P_{e_1, \dots, e_r})$ $(e_1 + \cdots + e_r = m)$.

We identify the homogeneous space $X = GL(m)/O(m)$ with the set of symmetric matrices of size m with non-zero determinant. The action of G on X is then given by $x \mapsto g \cdot x = gx{}^tg$. For any $1 \leq i \leq r$, let $f_i(P; x)$ be the determinant of the upper left n_i by

n_i block of x. Then we have

$$f_i(P; p\cdot x)=\Lambda_i(p)^2 f_i(P; x) \qquad (p\in P,\, x\in X)\,.$$

The semigroup $\mathfrak{X}_X(P)^+$ is generated by $\Lambda_1^2, \ldots, \Lambda_{r-1}^2, \Lambda_r^2, \Lambda_r^{-2}$ and we have $\mathfrak{a}_{P,C}^{*+}=\mathfrak{a}_{P,C}^{*++}$.

It is easy to check that the open P-orbit Ω is given by

$$\Omega=\{x\in X \mid f_i(P; x)\neq 0 \quad (1\leq i\leq r-1)\}\,.$$

The point $x_0=1_m$, the identity matrix, is in Ω and $P_{x_0}\cong O(e_1)\times\cdots\times O(e_r)$. Hence, by Lemma 2.3, the spherical triple (G, H, P) is regular for any $e_1, \ldots, e_r$.

Since $P_{x_0}^{(1)}\cong SO(e_1)\times\cdots\times SO(e_r)$, the triple (G, H, P) satisfies the assumptions in Theorem 3.2 if $e_i\neq 2$ for any $i=1, \ldots, r$. In this case, the convergence of the Eisenstein series is an immediate consequence of Theorem 3.1 in the case $m\geq 3$. In the case $m=2$, the convergence of the Eisenstein series is well-known, since the series coincide with the zeta functions of binary quadratic forms up to the Riemann zeta function factor. If $e_i=2$ for some i, then Assumption (1.2) is not necessarily satisfied for indefinite $x\in\Omega(\boldsymbol{Q})$ and the situation becomes rather complicated (cf. [S4] and [S6]). In the following we assume that $e_i\neq 2$ for any i.

Let $X^{(n,m-n)}$ be the set of real nondegenerate symmetric matrices with exactly n positive and $m-n$ negative eigenvalues. Then

$$X(\boldsymbol{R})=\bigcup_{n=0}^{m} X^{(n,m-n)}$$

gives the G^+-orbit decomposition of $X(\boldsymbol{R})$.

The P^+-orbit decomposition of $\Omega(\boldsymbol{R})$ is given by

$$\Omega(\boldsymbol{R})=\bigcup_{\substack{\varepsilon=(\varepsilon_1,\ldots,\varepsilon_r)\\ 0\leq\varepsilon_i\leq e_i}} \Omega_\varepsilon\,,$$

where $x=(x_{ij})_{1\leq i,j\leq r}\in\Omega(\boldsymbol{R})$ $(x_{ij}\in M(e_i, e_j; \boldsymbol{R}))$ is in Ω_ε if and only if $x_{ii}\in X^{(\varepsilon_i, e_i-\varepsilon_i)}$ $(1\leq i\leq r)$.

We write $\mathrm{sgn}(\varepsilon)=(n, m-n)$ if Ω_ε is contained in $X^{(n,m-n)}$, equivalently, $\varepsilon_1+\cdots+\varepsilon_r=n$.

For $x\in X(\boldsymbol{Q})\cap X^{(n,m-n)}$ and ε with $\mathrm{sgn}(\varepsilon)=(n, m-n)$, the Eisenstein series is defined by

$$E_\varepsilon(P; x, \lambda)=\sum_{y\in\Gamma_P\backslash\Gamma\cdot x\cap\Omega_\varepsilon}\frac{\mu(y)}{\prod_{i=1}^{r}|f_i(P; y)|^{(2z_i-2z_{i+1}+e_i+e_{i+1})/4}} \qquad \left(\lambda=\sum_{i=1}^{r} z_i \det p_{ii}\right),$$

where $z_{r+1}=0$ and $e_{r+1}=-m$. The coefficients $\mu(y)$ are quantities just like the measure of representation in the theory of quadratic forms. In fact, if $r=2$ and $e_1=1$, $e_2=m-1$, then $\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ coincides with the Epstein or Siegel zeta function of x according as x is definite or indefinite (cf. [Si], [SS]).

In another special case where $r=m$, $e_1=\cdots=e_m=1$, the Eisenstein series were investigated in [S3] (and in [S5, §6]).

Theorems 3.1 and 3.2 imply that $E_\varepsilon(P; x, \lambda)$ are absolutely convergent for

$$\mathrm{Re}(z_i)-\mathrm{Re}(z_{i+1})>(e_i+e_{i+1})/2 \qquad (1\leq i\leq r-1)$$

and have meromorphic continuations in $\mathfrak{a}^*_{P,\boldsymbol{C}}\cong \boldsymbol{C}^r$.

Put

$$b_{i,j}(s)=\prod_{\mu=1}^{j}\left(s+\frac{\mu-1}{2}\right)\left(s+\frac{i-\mu-1}{2}\right).$$

Note that the function $b_{i,j}(s)$ is the b-function of the prehomogeneous vector space $(SO(i)\times GL(j), M(i,j))$ (see [Ki]). Then we can obtain the following result:

THEOREM 3.8. *The functions* $\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ *multiplied by*

$$\prod_{1\leq i<j\leq r} b_{e_i+e_j,e_i}\left(\frac{2z_i-2z_j-e_i-e_j}{4}\right)$$

are entire functions of λ *in* $\mathfrak{a}^*_{P,\boldsymbol{C}}$.

For any parabolic subgroup Q containing P, the triple (L_Q, H_Q, P_Q) is a product of spherical triples of the same type as (G, H, P), hence regular. Therefore the X-associated class $\mathscr{P}$ containing $P=P_{e_1,\dots,e_r}$ is given by

$$\mathscr{P}=\{{}^\sigma P=P_{e_{\sigma(1)},\dots,e_{\sigma(r)}}\,|\,\sigma\in\mathfrak{S}_r\}.$$

Note that, if $Q=Q_i=P_{e_1,\dots,e_{i-1},e_i+e_{i+1},e_{i+2},\dots,e_r}$ and $w=w_{Q_i}$, then ${}^wP={}^\sigma P$ with $\sigma=(i,i+1)$. From Theorem 3.5 it follows that there exists a functional equation that connects $E_\varepsilon(P; x, \lambda)$ to $E_\eta({}^\sigma P; x, \sigma\lambda)$ for any $\sigma\in\mathfrak{S}_r$. The functional equation is of the form

$$E_\eta({}^\sigma P; x, \sigma\lambda)=C_{\mathrm{Eis}}(\sigma,\lambda)\sum_{\mathrm{sgn}(\varepsilon)=(n,m-n)} C_{\mathrm{sph}}(\sigma,\lambda)_{\eta,\varepsilon}E_\varepsilon(P; x, \lambda),$$

where $\sigma\in\mathfrak{S}_r$ is identified with $\sigma\in W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{{}^\sigma P})$ given by

$$\sigma\lambda=\sum_{i=1}^{r} z_{\sigma(i)}\det p'_{ii} \qquad (p'=(p'_{ij})\in{}^\sigma P,\ p'_{ij}\in M(e_{\sigma(i)}, e_{\sigma(j)})).$$

The calculation of $C_{\mathrm{sph}}(\sigma,\lambda)_{\eta,\varepsilon}$ and $C_{\mathrm{Eis}}(\sigma,\lambda)$ can be reduced to the case where σ is the transposition $(i,i+1)$ $(i=1,\dots,r-1)$. In this case, by (3.5) and (3.7), $C_{\mathrm{Eis}}(\sigma,\lambda)$ is given by

$$C_{\mathrm{Eis}}((i,i+1),\lambda)=\prod_{\mu=0}^{e_i-1}\frac{\hat{\zeta}\left(z_i-z_{i+1}+\dfrac{e_i+e_{i+1}}{2}-\mu\right)}{\hat{\zeta}\left(z_{i+1}-z_i+\dfrac{e_i+e_{i+1}}{2}-\mu\right)}.$$

To give an explicit formula for $C_{\mathrm{sph}}(\sigma,\lambda)_{\eta,\varepsilon}$ for $\sigma=(i,i+1)$, we prepare the following notation.

For an $i=1,\dots,r-1$, we put

$$\varepsilon_+^{(i)}=(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}-1, \varepsilon_i+1, \varepsilon_{i+2}, \dots, \varepsilon_r),$$

$$\varepsilon_-^{(i)}=(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}+1, \varepsilon_i-1, \varepsilon_{i+2}, \dots, \varepsilon_r).$$

We also put

$$u_i(s)=\prod_{\mu=1}^{e_i}\cos\pi\left(s+\frac{\mu}{2}\right),$$

$$u_{\varepsilon,i}^0(s)=\begin{cases}(-1)^{\varepsilon_i\varepsilon_i^*}\prod_{\mu=1}^{\varepsilon_i}\cos\pi\left(s+\dfrac{\varepsilon_i^*+\varepsilon_{i+1}^*+\mu}{2}\right)\prod_{\mu=\varepsilon_i+1}^{e_i}\cos\pi\left(s+\dfrac{\varepsilon_i+\varepsilon_{i+1}+\mu}{2}\right) & \text{if } e_{i+1}\equiv\varepsilon_{i+1} \pmod 2,\\ (-1)^{\varepsilon_i\varepsilon_i^*}\prod_{\mu=1}^{\varepsilon_i^*}\cos\pi\left(s+\dfrac{\varepsilon_i+\varepsilon_{i+1}+\mu}{2}\right)\prod_{\mu=\varepsilon_i^*+1}^{e_i}\cos\pi\left(s+\dfrac{\varepsilon_i^*+\varepsilon_{i+1}^*+\mu}{2}\right) & \text{if } e_{i+1}\not\equiv\varepsilon_{i+1} \pmod 2,\end{cases}$$

$$u_{\varepsilon,i}^+(s)=\begin{cases}\sin\pi\left(\dfrac{\varepsilon_{i+1}^*+1}{2}\right)\prod_{\mu=1}^{\varepsilon_i^*-1}\cos\pi\left(s+\dfrac{\varepsilon_i+\varepsilon_{i+1}+\mu}{2}\right)\prod_{\mu=\varepsilon_i^*+1}^{e_i}\cos\pi\left(s+\dfrac{\varepsilon_i^*+\varepsilon_{i+1}^*+\mu}{2}\right) & \text{if } \varepsilon_i\equiv 0 \pmod 2,\\ 0 & \text{if } \varepsilon_i\equiv 1 \pmod 2,\end{cases}$$

$$u_{\varepsilon,i}^-(s)=\begin{cases}\sin\pi\left(\dfrac{\varepsilon_{i+1}+1}{2}\right)\prod_{\mu=1}^{\varepsilon_i-1}\cos\pi\left(s+\dfrac{\varepsilon_i^*+\varepsilon_{i+1}^*+\mu}{2}\right)\prod_{\mu=\varepsilon_i+1}^{e_i}\cos\pi\left(s+\dfrac{\varepsilon_i+\varepsilon_{i+1}+\mu}{2}\right) & \text{if } \varepsilon_i^*\equiv 0 \pmod 2,\\ 0 & \text{if } \varepsilon_i^*\equiv 1 \pmod 2,\end{cases}$$

where $\varepsilon_i^*=e_i-\varepsilon_i$.

Now we can give an explicit formula for $C_{\mathrm{sph}}((i, i+1), \lambda)$.

THEOREM 3.9. *For $\sigma=(i, i+1)$ $(1\leq i\leq r-1)$, we have*

$$C_{\mathrm{sph}}(\sigma, \lambda)_{\eta,\varepsilon}=\begin{cases}\dfrac{u_{\varepsilon,i}^0((2z_i-2z_{i+1}-e_i-e_{i+1})/4)}{u_i((2z_i-2z_{i+1}-e_i-e_{i+1})/4)} & \text{if } \eta=\varepsilon,\\ \dfrac{u_{\varepsilon,i}^+((2z_i-2z_{i+1}-e_i-e_{i+1})/4)}{u_i((2z_i-2z_{i+1}-e_i-e_{i+1})/4)} & \text{if } \eta=\varepsilon_+^{(i)},\\ \dfrac{u_{\varepsilon,i}^-((2z_i-2z_{i+1}-e_i-e_{i+1})/4)}{u_i((2z_i-2z_{i+1}-e_i-e_{i+1})/4)} & \text{if } \eta=\varepsilon_-^{(i)},\\ 0 & \text{otherwise}.\end{cases}$$

We give a proof of Theorems 3.8 and 3.9 in Appendix.

The space $X=GL(m)/O(m)$ we have just investigated is a typical example of symmetric spaces. Now let us consider a non-symmetric spherical homogeneous space.

EXAMPLE 2. $(G, H, P)=(GL(2)\times GL(2)\times GL(2), SL(2), B(2)\times B(2)\times B(2))$.

Here $B(2)=P_{1,1}$, the group of 2×2 nondegenerate lower triangular matrices and we identify $SL(2)$ with the subgroup $H=\{(h, h, h)\in G \mid h\in SL(2)\}$ of G.

For any $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL(2)$, we put $\check{g}=\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$. For $x=(x_1, x_2, x_3)H\in X=G/H$, we define

$$\begin{aligned} &f_1(P; x)=(x_2\check{x}_3)_{12}\,, \quad f_2(P; x)=(x_3\check{x}_1)_{12}\,, \quad f_3(P; x)=(x_1\check{x}_2)_{12}\,, \\ &f_4(P; x)=\det x_1\,, \qquad f_5(P; x)=\det x_2\,, \qquad f_6(P; x)=\det x_3\,, \end{aligned}$$

where A_{12} denotes the $(1, 2)$-entry of a matrix A. Then these $f_i(P; x)$ $(1\le i\le 6)$ are relative P-invariants and the corresponding characters are given by

$$\begin{aligned} &\chi_1(p_1, p_2, p_3)=\Lambda_1^{(2)}\Lambda_1^{(3)}\,, \quad \chi_2(p_1, p_2, p_3)=\Lambda_1^{(3)}\Lambda_1^{(1)}\,, \quad \chi_3(p_1, p_2, p_3)=\Lambda_1^{(1)}\Lambda_1^{(2)}\,, \\ &\chi_4(p_1, p_2, p_3)=\Lambda_2^{(1)}\,, \qquad \chi_5(p_1, p_2, p_3)=\Lambda_2^{(2)}\,, \qquad \chi_6(p_1, p_2, p_3)=\Lambda_2^{(3)}\,. \end{aligned}$$

The semigroup $\mathfrak{X}_X(P)^+$ is generated by $\chi_1, \chi_2, \chi_3, \chi_4, \chi_4^{-1}, \chi_5, \chi_5^{-1}, \chi_6, \chi_6^{-1}$ and we have

$$\mathfrak{a}_{P,C}^{*++}=\left\{\lambda=\sum_{k=1}^{3}\sum_{i=1}^{2}\lambda_i^{(k)}\Lambda_i^{(k)} \;\middle|\; \operatorname{Re}(\lambda_1^{(k)})>0 \quad (k=1, 2, 3)\right\},$$

$$\mathfrak{a}_{P,C}^{*+}=\left\{\lambda=\sum_{k=1}^{3}\sum_{i=1}^{2}\lambda_i^{(k)}\Lambda_i^{(k)} \;\middle|\; \begin{array}{l} \operatorname{Re}(\lambda_1^{(1)})+\operatorname{Re}(\lambda_1^{(2)})>\operatorname{Re}(\lambda_1^{(3)})\,, \\ \operatorname{Re}(\lambda_1^{(2)})+\operatorname{Re}(\lambda_1^{(3)})>\operatorname{Re}(\lambda_1^{(1)})\,, \\ \operatorname{Re}(\lambda_1^{(3)})+\operatorname{Re}(\lambda_1^{(1)})>\operatorname{Re}(\lambda_1^{(2)}) \end{array}\right\}.$$

The open P-orbit Ω is given by

$$\Omega=\{x\in X \mid f_i(P; x)\neq 0 \quad (i=1, 2, 3)\}\,.$$

For an $x\in\Omega$, we have $P_x=\{(1_2, 1_2, 1_2), (-1_2, -1_2, -1_2)\}$ and $P_x^{(1)}=\{(1_2, 1_2, 1_2)\}$. Hence, by Lemma 2.3, the spherical triple (G, H, P) is regular and satisfies the assumptions in Theorems 3.1 and 3.2.

The set of real points $X(\boldsymbol{R})$ is a single $G(\boldsymbol{R})$-orbit and decomposes into eight G^+-orbits as follows:

$$X(\boldsymbol{R})=\bigcup_{\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\{\pm 1\}^3} X^{(\varepsilon)}\,, \quad X^{(\varepsilon)}=\left\{x\in X(\boldsymbol{R}) \;\middle|\; \frac{f_{i+3}(P; x)}{|f_{i+3}(P; x)|}=\varepsilon_i \quad (i=1, 2, 3)\right\}.$$

Since the structure of these eight G^+-orbits are quite the same, we consider only $X^{(1,1,1)}$ and denote it by $X^{(0)}$. The P^+-orbit decomposition of $X^{(0)}$ is given by

$$X^{(0)}=\bigcup_{\eta=(\eta_1,\eta_2,\eta_3)\in\{\pm 1\}^3} X_\eta^{(0)}\,, \quad X_\eta^{(0)}=\left\{x\in X^{(0)} \;\middle|\; \frac{f_i(P; x)}{|f_i(P; x)|}=\eta_i \quad (i=1, 2, 3)\right\}.$$

Let $(z_i^{(k)})_{\substack{k=1,2,3\\ i=1,2}}$ be the coordinate system on $\mathfrak{a}_{P,C}^*$ introduced in §3.1. Then, for $x\in X^{(0)}\cap X(\boldsymbol{Q})$ and $\eta\in\{\pm 1\}^3$, the Eisenstein series is defined by

$$E_\eta(P; x, \lambda) = \prod_{k=1}^{3} |\det x_k|^{-z_2^{(k)}+1/2} \sum_{y \in \Gamma_P \backslash \Gamma \cdot x \cap X_\eta^{(0)}} \prod_{i=1}^{3} |f_i(P; y)|^{-(1+\sum_{j=1}^{3}(-1)^{\delta_{ij}}(z_1^{(j)}-z_2^{(j)}))/2},$$

where δ_{ij} is the Kronecker delta. The series does not depend on the signature η; hence we simply write $E(P; x, \lambda)$. By Theorems 3.1 and 3.2, we have the following result:

THEOREM 3.10. *The series $E(P; x, \lambda)$ is absolutely convergent for*

$$\sum_{j=1}^{3} (-1)^{\delta_{ij}}(\mathrm{Re}(z_1^{(j)}) - \mathrm{Re}(z_2^{(j)})) > 1 \qquad (i=1, 2, 3)$$

and the function $\zeta_P(\lambda) \cdot E(P; x, \lambda)$, $\zeta_P(\lambda) = \prod_{i=1}^{3} \zeta(z_1^{(i)} - z_2^{(i)} + 1)$ multiplied by

$$\left\{ \sum_{j=1}^{3} (z_1^{(j)} - z_2^{(j)}) - 1 \right\} \prod_{i=1}^{3} \left\{ \sum_{j=1}^{3} (-1)^{\delta_{ij}}(z_1^{(j)} - z_2^{(j)}) - 1 \right\}$$

is an entire function in $\mathfrak{a}_{P,C}^*$.

For any parabolic subgroup Q containing P, the triple (L_Q, H_Q, P_Q) is regular and ${}^wP$ coincides with P. Hence we have $\mathscr{P} = \{P\}$. Moreover $W_X(\mathfrak{a}_P^*, \mathfrak{a}_P^*)$ can be identified with the Weyl group of G and is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$. In the case $Q = G$, the prehomogeneous vector space of flag type attached to (G, H, P) is given by

$$(SL(2) \times GL(1)^3, \rho, M(2, 3)), \quad \rho(h, t_1, t_2, t_3)v = hv \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}^{-1}.$$

This is the space studied in [S1, §7.1]. The results obtained there contain essentially a proof of Theorem 3.10 and an explicit formula for $C_{\mathrm{sph}}(w; \lambda)$. The functional equation satisfied by $E(P; x, \lambda)$ can be formulated as follows:

THEOREM 3.11. *The function $\hat{\zeta}_P(\lambda)E(P; x, \lambda)$ multiplied by*

$$\Gamma\left(\frac{\sum_{j=1}^{3}(z_1^{(j)} - z_2^{(j)}) + 1}{4}\right) \prod_{i=1}^{3} \left\{ \Gamma\left(\frac{z_2^{(i)} - z_1^{(i)} + 1}{2}\right) \Gamma\left(\frac{\sum_{j=1}^{3}(-1)^{\delta_{ij}}(z_1^{(j)} - z_2^{(j)}) + 1}{4}\right) \right\}$$

is invariant under the action of $\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$. Here the action of $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$ on $\mathfrak{a}_{P,C}^*$ is given by

$$\sigma \cdot (z_j^{(i)})_{\substack{i=1,2,3 \\ j=1,2}} = (z_{\sigma_i(j)}^{(i)})_{\substack{i=1,2,3 \\ j=1,2}}.$$

We now discuss an example of spherical homogeneous space with non-reductive H.

EXAMPLE 3. $(G, H, P) = (GL(m), N(m), B(m))$.

Here $B(m) = P_{1,\dots,1}$, the group of m by m nondegenerate lower triangular matrices, and $N(m) = R_u(B(m))$, the group of m by m lower triangular unipotent matrices.

We put

$$w = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in GL(m).$$

Then $x_0 = w \cdot H$ is in the open P-orbit and $P_{x_0} = \{1\}$. Hence Theorem 3.1 holds for (G, H, P) and the Eisenstein series converges absolutely in $\delta_P + \mathfrak{a}^{*+}_{P,C}$. However (G, H, P) is not regular. Even (L_Q, H_Q, P_Q) is not regular for any parabolic subgroup Q containing P properly. Moreover H is not reductive. Therefore our results on analytic continuation and functional equations do not apply to the Eisenstein series attached to (G, H, P, Γ).

Let us examine the situation more closely.

For $i = 1, 2, \ldots, m$, denote by $f_i(P; x)$ $(x = gH \in X = G/H)$ the determinant of the upper right i by i block of g. Then $f_i(P; x)$ is the relative invariant corresponding to the character Λ_i. The semigroup $\mathfrak{X}_X(P)^+$ is generated by $\Lambda_1, \ldots, \Lambda_m$. Hence $\mathfrak{a}^{*+}_{P,C} = \mathfrak{a}^{*++}_{P,C}$.

The open P-orbit Ω is given by

$$\Omega = \{x \in X \mid f_i(P; x) \neq 0 \ (i = 1, \ldots, m-1)\} .$$

The G^+-orbit decomposition of $X(\boldsymbol{R})$ is given by

$$X(\boldsymbol{R}) = \bigcup_{\varepsilon = \pm 1} X^{(\varepsilon)}, \quad X^{(\varepsilon)} = \left\{ x \in X(\boldsymbol{R}) \,\middle|\, \frac{f_m(P; x)}{|f_m(P; x)|} = \varepsilon \right\}$$

and the P^+-decomposition of $\Omega(\boldsymbol{R})$ is given by

$$\Omega(\boldsymbol{R}) = \bigcup_{\eta = (\eta_1, \ldots, \eta_m) \in \{\pm 1\}^m} \Omega_\eta , \quad \Omega_\eta = \left\{ x \in \Omega(\boldsymbol{R}) \,\middle|\, \frac{f_i(P; x)}{|f_i(P; x)|} = \eta_i \ (1 \leq i \leq m) \right\} .$$

For $x \in X(\boldsymbol{Q})$ and $\eta \in \{\pm 1\}^m$, the Eisenstein series is defined by

$$E_\eta(P; x, \lambda) = |f_m(P; x)|^{-z_m + (m-1)/2} \sum_{y \in \Gamma_P \backslash \Gamma \cdot x \cap \Omega_\eta} \prod_{i=1}^{m-1} |f_i(P; y)|^{-(z_i - z_{i+1} + 1)} .$$

It is easy to see that $E_\eta(P; x, \lambda) = 0$ unless $\eta_m = f_m(P; x)/|f_m(P; x)|$, and non-vanishing Eisenstein series do not depend on η. Hence it is better to consider the series

$$E(P; x, \lambda) = |f_m(P; x)|^{-z_m + (m-1)/2} \sum_{y \in \Gamma_P \backslash \Gamma \cdot x \cap \Omega} \prod_{i=1}^{m-1} |f_i(P; y)|^{-(z_i - z_{i+1} + 1)} .$$

Since any Γ-orbit in $X(\boldsymbol{Q})$ contains a unique element of the form

$$x = \begin{pmatrix} & & t_1 \\ & \cdot^{\cdot^{\cdot}} & \\ t_m & & \end{pmatrix} \cdot H, \qquad t_1, \ldots, t_m \in \boldsymbol{Q}^\times ,$$

we have

$$E(P; x, \lambda) = \prod_{i=1}^{m} |t_i|^{-z_i - (m - 2i + 1)/2} \cdot E(P; 1_m, \lambda) .$$

The series $E(P; 1_m, \lambda)$ coincides with the series $D(1_m; \lambda)$ studied in [S5, §3] and, by [S5, Proposition 3.3], we have

$$E(P;1_m,\lambda)=2^m\prod_{1\le i<j\le m}\frac{\zeta(z_i-z_j)}{\zeta(z_i-z_j+1)}\,.$$

Thus we obtain the following explicit formula for the Eisenstein series:

$$E(P;x,\lambda)=2^m\prod_{i=1}^{m}|t_i|^{-z_i-(m-2i+1)/2}\prod_{1\le i<j\le m}\frac{\zeta(z_i-z_j)}{\zeta(z_i-z_j+1)}\,.$$

This proves the first and the second parts of Conjecture 1.1.

As for functional equation, it is obvious from the explicit formula that there exist no functional equations that are valid for all $x\in X(\mathbf{Q})$. This reflects the fact that there exist no parabolic subgroups Q other than P for which (L_Q, H_Q, P_Q) is regular. Since the little Weyl group of the horospherical homogeneous space $GL(m)/N(m)$ is considered to be trivial (cf. [Kn, Satz 9.1]), we may say that even the third part of Conjecture 1.1 is true for the present example.

4. Proof of Theorems 3.1 and 3.2. In this section, to avoid the complicated notation, we give a proof of Theorems 3.1, 3.2, 3.5 and 3.6 under the assumption that $l=1$, hence $G=GL(m)$. Therefore we omit the superscript $^{(k)}$ used in §2, and §3. For example, we write e_i for $e_i^{(k)}$ and n_i for $n_i^{(k)}$. The proof of the theorems for general l is quite the same.

For the proof of Theorems 3.1 and 3.2, we need precise information on the structure of the prehomogeneous vector space of flag type $(G_P\times H, \rho_P, V_P)$ attached to (G, H, P). For the moment, we assume that

(4.1) *H is a reductive subgroup of $SL(m)$,*

as well as (1.1) and (1.2).

For a $g\in G=GL(n)$, we put

$$v(g)=((1_{n_1},0),(1_{n_2},0),\dots,(1_{n_{r-1}},0)g)\in V_P\,. \tag{4.2}$$

For a $p\in P_{e_1,\dots,e_i}$ $(1\le i\le r)$, we denote by $[p]_j$ $(1\le j\le i)$ the upper left n_j by n_j block of p. We define an embedding of $P_{e_1,\dots,e_i}$ into G_P by

$$P_{e_1,\dots,e_i}\ni p\longmapsto\tilde{p}=([p]_1,\dots,[p]_i,1_{n_{i+1}},\dots,1_{n_{r-1}})\in G_P\,. \tag{4.3}$$

Fix a rational point $x\in X(\mathbf{Q})$ and put $H_x=\{g\in G\,|\,g\cdot x=x\}$. Then H_x is conjugate to H in G and we can consider the prehomogeneous vector space $(G_P\times H_x^\circ, \rho_P, V_P)$. Let Ω_V be the open $G_P\times H_x^\circ$-orbit in V_P. Note that Ω_V is $G_P\times H_x$-stable.

Take a $g_0\in G_{\mathbf{Q}}$ such that Pg_0H_x is open in G and put $P_{g_0}=P\cap g_0H_xg_0^{-1}$, which is the isotropy subgroup of P at $g_0\cdot x\in X$. We also put $P'_{g_0}=P\cap g_0H_x^\circ g_0^{-1}$. Then $v(g_0)$ is in Ω_V and the isotropy subgroup $(G_P\times H_x^\circ)_{v(g_0)}$ is isomorphic to P'_{g_0}. The isomorphism is given by

$$P'_{g_0} \ni p \longmapsto (\tilde{p}, g_0^{-1} p g_0) \in (G_P \times H_x^\circ)_{v(g_0)} .$$

Let α_i be the smallest positive integer such that the character $\det g_i^{\alpha_i}$ is trivial on $(G_P \times H_x^\circ)_{v(g_0)}$. Since P'_{g_0} is isomorphic to $(G_P \times H_x^\circ)_{v(g_0)}$, Assumption (1.2) implies the existence of such an α_i. Then, by [SK, §4, Proposition 19], the character $\chi_i(g, h) = \det g_i^{\alpha_i}$ corresponds to a relative invariant $F_i(v) \in \boldsymbol{Q}(V_P)^\times$. The relative invariant F_i is unique up to a rational constant factor.

For $\lambda = \sum_{i=1}^r \lambda_i \Lambda_i \in \mathfrak{a}_{P,\boldsymbol{C}}^*$, we define a function $|\chi|^\lambda$ on $G_P(\boldsymbol{R}) \times H_x(\boldsymbol{R})$ and $|F|^\lambda$ on $\Omega_V(\boldsymbol{R})$ by setting

$$|\chi(g, h)|^\lambda = \prod_{i=1}^{r-1} |\chi_i(g, h)|^{\lambda_i/\alpha_i} = \prod_{i=1}^{r-1} |\det g_i|^{\lambda_i} ,$$

$$|F(v)|^\lambda = \prod_{i=1}^{r-1} |F_i(v)|^{\lambda_i/\alpha_i} .$$

Note that F_i differs only by a factor equal to ± 1 under the action of $H_x(\boldsymbol{R})$, hence we have

$$|F(\rho_P(g, h)v)|^\lambda = |\chi(g, h)|^\lambda |F(v)|^\lambda \qquad ((g, h) \in G_P(\boldsymbol{R}) \times H_x(\boldsymbol{R}) , \quad v \in \Omega_V(\boldsymbol{R})) .$$

For $g \in G$, we have

$$F_i(v(pgh))^d = \Lambda_i(p)^{\alpha_i d} F_i(v(g))^d \qquad (p \in P, h \in H_x) ,$$

where $d = [H_x : H_x^\circ]$. Hence we can choose the relative invariants $F_1, \ldots, F_{r-1}$ so that

$$(4.4) \qquad |\det g|^{\lambda_r} |F(v(g))|^\lambda = |f(P; g \cdot x)|^\lambda \qquad (g \in GL(m; \boldsymbol{R}) , \quad g \cdot x \in \Omega(\boldsymbol{R})) ,$$

where the right hand side of the identity is the function on $\Omega(\boldsymbol{R})$ defined by (1.3).

Let G_P^+ be the identity component of $G_P(\boldsymbol{R})$ and $H_x^+ = H_x(\boldsymbol{R}) \cap GL(m; \boldsymbol{R})^+$.

LEMMA 4.1. *Let x be a rational point of X and $X^{(0)}$ the $GL(m; \boldsymbol{R})^+$-orbit containing x. Let*

$$\Omega(\boldsymbol{R}) \cap X^{(0)} = P^+ g_1 \cdot x \cup \cdots \cup P^+ g_\nu \cdot x$$

be the P^+-orbit decomposition. Then the $\rho_P(G_P^+ \times H_x^+)$-orbit decomposition of $\Omega_V(\boldsymbol{R})$ is given by

$$\Omega_V(\boldsymbol{R}) = \rho_P(G_P^+ \times H_x^+) v(g_1) \cup \cdots \cup \rho_P(G_P^+ \times H_x^+) v(g_\nu) .$$

PROOF. Put

$$V'_P = \{(v_1, \ldots, v_{r-1}) \in V_P(\boldsymbol{R}) \mid \operatorname{rank} v_i = n_i \ (i = 1, \ldots, r-1)\} ,$$

$$\tilde{G}_P^+ = GL(n_1; \boldsymbol{R})^+ \times \cdots \times GL(n_{r-1}; \boldsymbol{R})^+ \times GL(n_r; \boldsymbol{R})^+ .$$

Then, any $v \in V'_P$ can be written as $v = \rho_P(\tilde{g}) v_0$ ($\tilde{g} \in \tilde{G}_P^+$). Hence, under the action of G_P^+, any $v \in V'_P$ is moved to a point of the form $v(g)$ ($g \in G^+$). Two points $v(g)$, $v(g')$ ($g, g' \in G^+$) belong to the same $\rho_P(G_P^+ \times H_x^+)$-orbit if and only if $P^+ g H_x^+ = P^+ g' H_x^+$. Moreover the

$\rho_P(G_P^+ \times H_x^+)$-orbits in $\Omega_V(\boldsymbol{R})$ correspond to the open double cosets $P^+gH_x^+$. It is easy to see that the open double cosets $P^+gH_x^+$ correspond to the P^+-orbits in $\Omega(\boldsymbol{R}) \cap X^{(0)}$. ∎

By Assumption (4.1), $|\det g|$ is identically equal to 1 on $H(\boldsymbol{R})$. Hence we can define the function $|\det x|$ on $X^{(0)}$ by

$$|\det x| = |\det g| \quad \text{for} \quad x = gH \qquad (g \in G(\boldsymbol{R}) = GL(m; \boldsymbol{R}))\,.$$

Put $\Gamma_{G_P} = SL(n_1; \boldsymbol{Z}) \times \cdots \times SL(n_{r-1}; \boldsymbol{Z})$ and $\Gamma_x = H_x^+ \cap \Gamma$.

Using this notation, we have the following proposition giving a relation between the Eisenstein series $E_i(P; x, \lambda)$ attached to (G, H, P, Γ) and the zeta functions associated with $(G_P \times H_x^\circ, \rho_P, V_P)$.

PROPOSITION 4.2. *Let x be a point in $X(\boldsymbol{Q})$ and $X^{(0)}$ as in Lemma* 4.1. *For $\phi \in \mathscr{S}(V_P(\boldsymbol{R}))$, put*

$$Z_P(x, \phi, \lambda) = |\det x|^{-(\lambda+\delta_P)_r} \int_{G_P^+/\Gamma_{G_P} \times H_x^+/\Gamma_x} |\chi(g, h)|^{\lambda+\delta_P} \sum_{v \in L \cap \Omega_V(\boldsymbol{R})} \phi(\rho_P(g, h)v)dgdh\,,$$

$$\Psi_{P,i}(\phi; \lambda) = \int_{\Omega_{V,i}} |F(v)|^{\lambda-\delta_P} \phi(v)dv\,,$$

where $L = \bigoplus_{i=1}^{r-1} M(n_i, n_{r+1}; \boldsymbol{Z})$ and dv is the standard Euclidean measure on $V_P(\boldsymbol{R})$. Then, under a suitable normalization of the Haar measure $dgdh$ on $G_P^+ \times H_x^+$, we have

$$Z_P(x, \phi, \lambda) = \zeta_P(\lambda) \sum_{i=1}^{\nu} E_i(P; x, \lambda) \Psi_{P,i}(\phi; \lambda)$$

(*for the definition of $\zeta_P(\lambda)$, see* (3.3)). *The absolute convergence of one side of the identity implies the absolute convergence of the other side.*

Once Proposition 4.2 is established, then Theorems 3.1 and 3.2 are immediate consequences of the general theory of zeta functions associated with prehomogeneous vector spaces.

PROOF OF THEOREMS 3.1 AND 3.2. By Proposition 4.2, $\zeta_P(\lambda)E_i(P; x, \lambda)$ can be considered as the zeta functions associated to the prehomogeneous vector space $(G_P \times H_x^\circ, \rho_P, V_P)$ (cf. [S1, §4]). Since $P_{g\cdot x}^{(1)}$ is isomorphic to $(G_P \times H_x)_{v(g)} \cap (SL(n_1) \times \cdots \times SL(n_{r-1}) \times H_x)$, the assumption of Theorem 3.1 implies that $(G_P \times H_x^\circ, \rho_P, V_P)$ is $\boldsymbol{Q}$-split and $(G_P \times H_x^\circ)_{v(g)} \cap (SL(n_1) \times \cdots \times SL(n_{r-1}) \times H_x^\circ)$ is connected semisimple. Hence [S2, Theorem 1] can be applied to $(G_P \times H_x^\circ, \rho_P, V_P)$. It is easy to see that

$$\det \rho_P(g_1, \ldots, g_{r-1}, g_r) = \prod_{i=1}^{r} \det g_i^{e_i+e_{i+1}} \quad (g = (g_1, \ldots, g_r) \in \tilde{G}_P)\,,$$

where $e_{r+1} = -m$. From this identity and (3.1) it follows that the domain of absolute convergence given in [S2, Theorem 1] coincides with $\delta_P+\mathfrak{a}_{P,C}^{*+}$. The assumption of Theorem 3.2 implies that $G_P\times H_x^\circ$ is reductive and $(G_P\times H_x^\circ, \rho_P, V_P)$ is regular. Hence Theorem 3.2 follows immediately from [S1, §6, Corollary 1 to Theorem 2]. The product $\prod_{i=1}^d(\langle L_i, \lambda\rangle+a_i)$ of linear forms describing the singularities of the Eisenstein series is given by the b-function of the prehomogeneous vector space $(G_P\times H_x^\circ, \rho_P, V_P)$. The rationality of L_j and a_j is due to Sabbah [Sab] and Gyoja [G]. ∎

REMARK. In general, the b-function of the prehomogeneous vector space $(G_P\times H_x^\circ, \rho_P, V_P)$ does not give the best possible result on the singularities of the Eisenstein series. A much better result will often be obtained by looking at the functional equations for various Q (cf. §5.5, Proposition 5.13).

Put $\Gamma_P=\Gamma\cap P^+$ as in §1. For the proof of the proposition, we need the following lemma (cf. [M, §17, pp. 280–282]).

LEMMA 4.3. *Put*

$$L'=\{(v_1,\dots,v_{r-1})\in V_P(\boldsymbol{Z})\mid \operatorname{rank} v_k=n_k\ (k=1,\dots,r-1)\}$$

$$\Delta^{(k)}=\left\{\begin{pmatrix} a_1 & & 0\\ & \ddots & \\ a_{ij} & & a_{n_k}\end{pmatrix}\in M(n_k;\boldsymbol{Z})\ \middle|\ \begin{matrix} a_i>0\ (i=1,\dots,r-1)\\ a_{ij}=0,1,\dots,a_j-1\ (i>j)\end{matrix}\right\}.$$

Fix a complete set of representatives of $\Gamma_P\backslash\Gamma$. Then the set

$$\left\{((D_1,0),\dots,(D_{r-2},0),(D_{r-1},0)U)\ \middle|\ \begin{matrix} D_k\in\Delta^{(k)}\ (k=1,\dots,r-1)\\ U\in\Gamma_P\backslash\Gamma\end{matrix}\right\}$$

gives a complete set of representatives of Γ_{G_P}-equivalence classes in L'.

PROOF OF PROPOSITION 4.2. Let $\Omega_1,\dots,\Omega_\nu$ be the P^+-orbits in $\Omega(\boldsymbol{R})\cap X^{(0)}$ and $\Omega_{V,1},\dots,\Omega_{V,\nu}$ be the corresponding $\rho_P(G_P^+\times H_x^+)$-orbits in $\Omega_V(\boldsymbol{R})$ (cf. Lemma 4.1). Since

$$((D_1,0),\dots,(D_{r-2},0),(D_{r-1},0)U)=\rho_P(\tilde D_1)\cdots\rho_P(\tilde D_{r-1})v(U)$$

(for the definitions of $\tilde D_i$ and $v(U)$, see (4.3) and (4.2), respectively), the point

$$((D_1,0),\dots,(D_{r-2},0),(D_{r-1},0)U)$$

belongs to $\Omega_{V,i}$ if and only if $U\cdot x\in\Omega_i$. We further note that $L\cap\Omega_{V,i}=L'\cap\Omega_{V,i}$ and Γ_{G_P} acts on L' freely. Therefore, by Lemma 4.3, we have

$$Z_P(x,\phi,\lambda)=|\det x|^{-(\lambda+\delta_P)_r}\left(\sum_{D_1,\dots,D_{r-1}}|\chi(\tilde D_1\cdots\tilde D_{r-1},1)|^{-(\lambda+\delta_P)}\right)$$

$$\times\sum_{i=1}^{\nu}\int_{G_P^+\times H_x^+/\Gamma_x}|\chi(g,h)|^{\lambda+\delta_P}\sum_{\substack{U\in\Gamma_P\backslash\Gamma\\ U\cdot x\in\Omega_i}}\phi(\rho_P(g,h)v(U))dgdh\,.$$

Since $\chi_j(\tilde{D}_i, 1) = d_{n_j}(D_i)$ or 1 according as $j \leq i$ or $j > i$, we have

$$\sum_{D_1,\dots,D_{r-1}} |\chi(\tilde{D}_1 \cdots \tilde{D}_{r-1}, 1)|^{-(\lambda+\delta_P)} = \prod_{i=1}^{r-1} \sum_{D_i \in \Delta^{(i)}} \prod_{j=1}^{i} |d_{n_j}(D_i)|^{-(z_{j+1}-z_j+(e_j+e_{j+1})/2)}$$

$$= \prod_{i=1}^{r-1} \sum_{a_1,\dots,a_{n_i}=1}^{\infty} a_1^{n_i-1} a_2^{n_i-2} \cdots a_{n_i-1} \prod_{j=1}^{i} (a_1 \cdots a_{n_j})^{-(z_{j+1}-z_j+(e_j+e_{j+1})/2)}$$

$$= \prod_{i=1}^{r-1} \prod_{j=1}^{i} \prod_{k=0}^{e_j-1} \prod_{a=1}^{\infty} a^{-(z_j-z_{i+1}+(e_j+e_{i+1})/2-k)}$$

$$= \zeta_P(\lambda)\,.$$

Hence

$$Z_P(x, \phi, \lambda) = |\det x|^{-(\lambda+\delta_P)_r} \zeta_P(\lambda)$$

$$\times \sum_{i=1}^{\nu} \sum_{\substack{U \in \Gamma_P \backslash \Gamma/\Gamma_x \\ U \cdot x \in \Omega_i}} \int_{G_P^+ \times (H_x^+/U^{-1}\Gamma_{P,U\cdot x}U)} |\chi(g, h)|^{\lambda+\delta_P} \phi(\rho_P(g, h)v(U))dgdh$$

$$= |\det x|^{-(\lambda+\delta_P)_r} \zeta_P(\lambda)$$

$$\times \sum_{i=1}^{\nu} \sum_{\substack{U \in \Gamma_P \backslash \Gamma/\Gamma_x \\ U \cdot x \in \Omega_i}} \int_{G_P^+ \times H_x^+/(\Gamma_{G_P} \times \Gamma_x)_{v(U)}} |\chi(g, h)|^{\lambda+\delta_P} \phi(\rho_P(g, h)v(U))dgdh$$

$$= |\det x|^{-(\lambda+\delta_P)_r} \zeta_P(\lambda) \sum_{i=1}^{\nu} \left(\sum_{\substack{U \in \Gamma_P \backslash \Gamma/\Gamma_x \\ U \cdot x \in \Omega_i}} \frac{\mu(v(U))}{|F(v(U))|^{\lambda+\delta_P}} \right) \Psi_{P,i}(\phi; \lambda)\,.$$

Here $\mu(v(U))$ is the density defined to be the volume of the fundamental domain of $(G_P^+ \times H_x^+)_{v(U)}$ with respect to $(\Gamma_{G_P} \times \Gamma_x)_{v(U)}$ (cf. [S1, §4], or [S7, §1]). Since the normalization of the Haar measures on $(G_P^+ \times H_x^+)_{v(U)}$ satisfies the invariance similar to (1.4) and $(G_P^+ \times H_x^+)_{v(U)}$ is isomorphic to $P_{U\cdot x}^+$, the density $\mu(v(U))$ differs from $\mu(U \cdot x)$, the coefficients of the Eisenstein series, only by a constant factor independent of U. We may normalize $dgdh$ so that $\mu(v(U)) = \mu(U \cdot x)$. Moreover, by (4.4), we have

$$|\det x|^{-(\lambda+\delta_P)_r} |F(v(U))|^{-(\lambda+\delta_P)} = |f(P; U \cdot x)|^{-(\lambda+\delta_P)}\,.$$

Hence we have

$$|\det x|^{-(\lambda+\delta_P)_r} \sum_{\substack{U \in \Gamma_P \backslash \Gamma/\Gamma_x \\ U \cdot x \in \Omega_i}} \frac{\mu(v(U))}{|F(v(U))|^{\lambda+\delta_P}} = E_i(P; x, \lambda)\,.$$

∎

5. Proof of Theorems 3.5 and 3.6. In this subsection, we always assume (3.6) and the notation is the same as in §3.

5.1. First we prove Proposition 3.4. For this purpose we need the following two lemmas.

LEMMA 5.1. *Let the notation be as in Proposition* 2.1. *Let V_P^* be the vector space dual to V_P and ρ_P^* the representation of $G_P \times H$ on V_P^* contragredient to ρ_P. Let w be the permutation matrix that represents the longest element of the Weyl group of G. Then the following three conditions are equivalent*:

(1) $(G, H, {}^wP)$ *is a spherical triple.*

(2) $(G, H, {}^tP)$ *is a spherical triple.*

(3) $(G_P \times H, \rho_P^*, V_P^*)$ *is a prehomogeneous vector space.*

PROOF. Since ${}^wP = w{}^tPw^{-1}$, the first assertion is equivalent to the second. Let us prove that the second assertion is equivalent to the third. We identify the vector space V_P with its dual vector space V_P^* via the symmetric bilinear form $(v, v^*) = \sum_{k,i} \operatorname{tr} {}^tv_i^{(k)}v_i^{*(k)}$. Let $\iota: G_P \times H \to G_P \times {}^tH$ be the isomorphism defined by $\iota((g_i^{(k)}), h) = (({}^tg_i^{(k)^{-1}}), {}^th^{-1})$. Then the contragredient representation ρ_P^* is given by $\rho_P^*(g, h) = \rho_P(\iota(g, h))$. Therefore $(G_P \times H, \rho_P^*, V_P^*)$ is a prehomogeneous vector space if and only if $(G_P \times {}^tH, \rho_P, V_P)$ is a prehomogeneous vector space. By Proposition 2.1, this is equivalent to that $(G, {}^tH, P)$ is a spherical triple. It is obvious that the latter condition is equivalent to the second assertion in the lemma. ∎

LEMMA 5.2. *Let P and Q be parabolic subgroups. Assume that Q contains P and (G, H, Q) is a spherical triple. We further assume that $R_u(Q) \cap Q_x = \{1\}$ for a point x in the open Q-orbit in $X = G/H$. Let H_Q be the image of Q_x under the canonical surjection $Q \to L_Q$. Then (G, H, P) is a spherical triple if and only if (L_Q, H_Q, P_Q) is a spherical triple.*

PROOF. We may assume that H is the isotropy subgroup at x. Then QH is Zariski-open in G and $Q_x = Q \cap H$. Since $R_u(Q) \cap Q_x = \{1\}$, Q_x is isomorphic to H_Q. Hence $\dim G = \dim Q + \dim H - \dim H_Q$. From this identity, we see that $P \cdot x$ is open in X if and only if $\dim L_Q = \dim H_Q + \dim P_Q - \dim P \cap H$. Denote by H_P the isomorphic image of $P \cap H$ in H_Q. Then it is easy to see that $H_P = H_Q \cap P_Q$. Hence $P \cdot x$ is open in X if and only if (L_Q, H_Q, P_Q) is a spherical triple and the base point of L_Q/H_Q is in the open P_Q-orbit. This proves the lemma. ∎

PROOF OF PROPOSITION 3.4. (i) Since P_x is assumed to be reductive, $R_u(Q) \cap Q_x$ is trivial. Hence Proposition 3.4 (i) follows immediately from the "only if" part of the lemma above.

(ii) Since ${}^wP$ is contained in Q, $(G, H, {}^wP)$ is a spherical triple if and only if $(L_Q, H_Q, w{}^tP_Qw^{-1})$ is a spherical triple. The latter condition is equivalent to that $(L_Q, H_Q, {}^tP_Q)$ is a spherical triple. By Lemma 5.1, this is again equivalent to the condition that the triple $(G_{P_Q} \times H_Q, \rho_{P_Q}^*, V_{P_Q}^*)$ dual to the prehomogeneous vector space of flag type attached to (L_Q, H_Q, P_Q) is a prehomogeneous vector space. Since the regularity of (L_Q, H_Q, P_Q) implies the last condition (cf. [SK, §4, Remark 11]), we see that $(G, H, {}^wP)$ is a spherical triple. It is easy to see that P_x (resp. $({}^wP)_{x'}$) is isomorphic to the generic isotropy subgroup of $(G_{P_Q} \times H_Q, \rho_{P_Q}, V_{P_Q})$ (resp. $(G_{P_Q} \times H_Q, \rho_{P_Q}^*, V_{P_Q}^*)$). Hence, by [S1, Lemma 2.4 (ii)], we see that P_x is isomorphic to $({}^wP)_{x'}$. The regularity of $(G, H, {}^wP)$ is now an immediate consequence of Lemma 2.3. ∎

5.2. It is sufficient to prove Theorems 3.5 and 3.6 in the case where P and P' are (X, Q)-associated. In the following, we always assume that $w = w_Q \in W_X(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$ and $P' = {}^wP$. For the proof of the theorems, we need a new integral representation of the Eisenstein series, which is a generalization of the integral representation used in [S5].

Since L_Q is a product of general linear groups and P_Q is its standard parabolic subgroup, we can define G_{P_Q} and V_{P_Q} as in §2.2. Put

$$G_{Q|P} = Q \times G_{P_Q} \quad \text{and} \quad X_{Q|P} = X \times V_{P_Q}.$$

Let $\pi_0 : Q \to L_Q \cong Q/R_u(Q)$ be the canonical surjection and define a homomorphism $\pi : G_{Q|P} = Q \times G_{P_Q} \to \tilde{G}_{P_Q} = G_{P_Q} \times L_Q$ by $\pi(q, g) = (g, \pi_0(q))$. Then $G_{Q|P}$ acts on $X_{Q|P}$ by

$$(q, g) \cdot (x, v) = (q \cdot x, \rho_{P_Q}(\pi(q, g))v).$$

It is easy to see that there exists an open $G_{Q|P}$-orbit $\Omega_{Q|P}$ in $X_{Q|P}$.

Let Q^+, L_Q^+ and $G_{Q|P}^+$ be the identity components of $Q(\boldsymbol{R})$, $L_Q(\boldsymbol{R})$ and $G_{Q|P}(\boldsymbol{R})$, respectively. Since Q contains P, there exists an open Q-orbit Ω_Q in X. Let

$$\Omega_Q(\boldsymbol{R}) \cap X^{(0)} = Q^+ g_1 x_0 \cup \cdots \cup Q^+ g_\alpha x_0$$

be the Q^+-orbit decomposition. Each open Q^+-orbit is decomposed into a union of P^+-orbits. Let

$$P^+ h_{i1} g_i x_0 \cup \cdots \cup P^+ h_{i\nu_i} g_i x_0$$

be the open P^+-orbits in $Q^+ g_i x_0$. Then we have $\nu_1 + \cdots + \nu_\alpha = \nu$, the number of open P^+-orbits in $X^{(0)}$, and the P^+-orbit decomposition of $\Omega(\boldsymbol{R}) \cap X^{(0)}$ is given by

$$\Omega(\boldsymbol{R}) \cap X^{(0)} = \bigcup_{\substack{1 \le i \le \alpha \\ 1 \le j \le \nu_i}} \Omega_{ij}, \quad \Omega_{ij} = P^+ h_{ij} g_i x_0.$$

We can take h_{ij} from L_Q^+.

Let $\boldsymbol{v}$ be the point of V_{P_Q} given by (2.3). Considering L_Q as a subgroup of $\tilde{G}_{P_Q} = G_{P_Q} \times L_Q$, we put

$$v(h) = \rho_{P_Q}(h^{-1})\boldsymbol{v} \qquad \text{for} \quad h \in L_Q.$$

Then the proof of the following lemma is similar to that of Lemma 4.1.

LEMMA 5.3. *The $G_{Q|P}^+$-orbit decomposition of $\Omega_{Q|P} \cap (X^{(0)} \times V_{P_Q}(\boldsymbol{R}))$ is given by*

$$\Omega_{Q|P} \cap (X^{(0)} \times V_{P_Q}(\boldsymbol{R})) = \bigcup_{\substack{1 \le i \le \alpha \\ 1 \le j \le \nu_i}} \Omega_{Q|P,ij}, \quad \Omega_{Q|P,ij} = G_{Q|P}^+ \cdot (g_i x_0, v(h_{ij})).$$

We put $X_{Q|P}^* = X \times V_{P_Q}^*$. Let us define the dual action $*$ of $G_{Q|P}$ on $X_{Q|P}^*$ by

$$(q, g) * (x, v) = (q \cdot x, \rho_{P_Q}^*(\pi(q, g))v).$$

Lemma 5.4. *Assume that* (L_Q, H_Q, P_Q) *is a regular spherical triple.*

(1) *The dual action* $*$ *of* $G_{Q|P}$ *on* $X_{Q|P}$ *has an open orbit.*

(2) *Let* $w=w_Q$ *be the permutation matrix that represents the longest element of the Weyl group of* L_Q. *Then there exists a natural one to one correspondence between the open* ${}^wP^+$-*orbits and the open* P^+-*orbits in a* Q^+-*orbit* $Q^+g_ix_0$.

Proof. By Lemma 5.3, the open P^+-orbits in $Q^+g_ix_0$ correspond bijectively to the $G^+_{Q|P}$-orbits in $\Omega_{Q|P}\cap(Q^+g_ix_0\times V_{P_Q}(\boldsymbol{R}))$. Note that (g_ix_0, v) and (g_ix_0, v') $(v, v'\in V_{P_Q}(\boldsymbol{R}))$ belong to the same $G^+_{Q|P}$-orbit if and only if v and v' belong to the same $\rho_{P_Q}(G^+_{P_Q}\times\pi_0(Q^+\cap g_i^{-1}Hg_i))$-orbit. Hence there exists a one to one correspondence between the open P^+-orbits in $Q^+g_ix_0$ and the open $\rho_{P_Q}(G^+_{P_Q}\times\pi_0(Q^+\cap g_i^{-1}Hg_i))$-orbits in $V_{P_Q}(\boldsymbol{R})$. Using the realization of $\rho^*_{P_Q}$ given in the proof of Lemma 5.1, we can see that there exists a one to one correspondence between the open ${}^wP^+$-orbits in $Q^+g_ix_0$ and the open $\rho^*_{P_Q}(G^+_{P_Q}\times\pi_0(Q^+\cap g_i^{-1}Hg_i))$-orbits in $V^*_{P_Q}(\boldsymbol{R})$. Since (L_Q, H_Q, P_Q) is assumed to be regular, the prehomogeneous vector space $(G_{P_Q}\times\pi_0(Q\cap g_i^{-1}Hg_i), \rho_{P_Q}, V_{P_Q})$ is regular. Hence, by [S1, Lemma 5.1], the open $\rho_{P_Q}(G^+_{P_Q}\times\pi_0(Q^+\cap g_i^{-1}Hg_i))$-orbits in $V_{P_Q}(\boldsymbol{R})$ correspond bijectively to the open $\rho^*_{P_Q}(G^+_{P_Q}\times\pi_0(Q^+\cap g_i^{-1}Hg_i))$-orbits in $V^*_{P_Q}(\boldsymbol{R})$. ∎

By the lemma above, there exist h^*_{ij} $(1\le i\le\alpha,\ 1\le j\le\nu_i)$ in L^+_Q with which the ${}^wP^+$-orbit decomposition of ${}^w\Omega(\boldsymbol{R})\cap X_0$ is given by

$${}^w\Omega(\boldsymbol{R})\cap X_0=\bigcup_{\substack{1\le i\le\alpha\\ 1\le j\le\nu_i}}{}^w\Omega_{ij}\,,\quad {}^w\Omega_{ij}=P^+h^*_{ij}g_ix_0\,.$$

Let $\Omega^*_{Q|P}$ be the open $G_{Q|P}$-orbit in $X^*_{Q|P}$. Identifying $V^*_{P_Q}$ with V_{P_Q} as in the proof of Lemma 5.1, we put

$$\boldsymbol{v}^*=\rho_{P_Q}(\tilde{w})\boldsymbol{v}\,,$$

where $\tilde{w}$ is the permutation matrix that represents the longest element of the Weyl group of $\tilde{G}_{P_Q}$. We also put

$$v^*(h)=\rho^*_{P_Q}(h^{-1})\boldsymbol{v}^*\qquad\text{for}\quad h\in L_Q\,.$$

Lemma 5.5. *The* $G^+_{Q|P}$-*orbit decomposition of* $\Omega^*_{Q|P}\cap(X^{(0)}\times V^*_{P_Q}(\boldsymbol{R}))$ *is given by*

$$\Omega^*_{Q|P}\cap(X^{(0)}\times V^*_{P_Q}(\boldsymbol{R}))=\bigcup_{\substack{1\le i\le\alpha\\ 1\le j\le\nu_i}}\Omega^*_{Q|P,ij}\,,\quad \Omega^*_{Q|P,ij}=G^+_{Q|P}*(g_ix_0, v^*(h^*_{ij}))\,.$$

Let $\varepsilon: P_Q\to G_{P_Q}$ be the injective homomorphism defined to be the composition of the mapping given by (2.4) and the projection of $\tilde{G}_{P_Q}$ onto G_{P_Q}. Denote by $\pi_0: Q\to L_Q$ the canonical surjection as before. We define an embedding of P into $G_{Q|P}$ by

$$P\ni p\longmapsto\bar{p}=(p, \varepsilon(\pi_0(p)))\in G_{Q|P}\,.$$

We also define an embedding of ${}^wP$ into $G_{Q|P}$ by

$$ {}^{w}P \ni p \longmapsto \bar{p}^* = (p, {}^t\varepsilon(w {}^t\pi_0(p)^{-1} w^{-1})^{-1}) \in G_{Q|P}\,. $$

By pulling back characters of $G_{Q|P}$ to P and ${}^wP$, we obtain isomorphisms between $\mathfrak{X}(G_{Q|P})$ and $\mathfrak{X}(P)$ and between $\mathfrak{X}(G_{Q|P})$ and $\mathfrak{X}({}^wP)$. Then it is easy to see that the linear isomorphism

$$ \mathfrak{a}^*_{P,\boldsymbol{C}} \longrightarrow \mathfrak{X}(G_{Q|P}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{C} \longrightarrow \mathfrak{a}^*_{{}^wP,\boldsymbol{C}} $$

given by the identification of $\mathfrak{X}(P)$, $\mathfrak{X}({}^wP)$ and $\mathfrak{X}(G_{Q|P})$ coincides with the isomorphism $w\colon \mathfrak{a}^*_{P,\boldsymbol{C}} \to \mathfrak{a}^*_{{}^wP,\boldsymbol{C}}$ introduced in §3.2.

By abuse of notation, we denote by $\det \rho_{P_Q}$ the character of P given by $P \ni p \mapsto \det \rho_{P_Q}(\varepsilon(\pi_0(p)), \pi_0(p))$. Then the following lemma can easily be proved by direct computation.

LEMMA 5.6. *We have*

$$ w(\delta_P - \det \rho_{P_Q}) = \delta_{{}^wP}\,. $$

By the same discussion as the one leading to the construction of $|F(v)|^\lambda$ in §4, we can define functions $|F_{Q|P}(x, v)|^\lambda$ ($\lambda \in \mathfrak{a}^*_{P,\boldsymbol{C}}$) on $\Omega_{Q|P}(\boldsymbol{R})$ and $|F^*_{Q|P}(x, v^*)|^{\lambda^*}$ ($\lambda^* \in \mathfrak{a}^*_{{}^wP,\boldsymbol{C}}$) on $\Omega^*_{Q|P}(\boldsymbol{R})$ satisfying

$$ |F_{Q|P}((q, g)\cdot(x, v))|^\lambda = |\chi_{Q|P}(q, g)|^\lambda |F_{Q|P}(x, v)|^\lambda\,, $$
$$ |F^*_{Q|P}((q, g)*(x, v^*))|^{\lambda^*} = |\chi_{Q|P}(q, g)|^{\lambda^*} |F^*_{Q|P}(x, v^*)|^{\lambda^*} $$

and

$$ |F_{Q|P}((x, \boldsymbol{v}))|^\lambda = |f(P; x)|^\lambda\,, \qquad |F^*_{Q|P}((x, \boldsymbol{v}^*))|^\lambda = |f({}^wP; x)|^{\lambda^*}\,. $$

Let d_rq be the right invariant measure on Q^+ and dg the Haar measure on $G^+_{P_Q}$. Put

$$ \Gamma_{Q|P} = \Gamma_Q \times \Gamma_{G_{P_Q}} \subset Q^+ \times G^+_{P_Q} = G^+_{Q|P}\,. $$

Let $\mathscr{F}_0(X_{Q|P})$ (resp. $\mathscr{F}_0(X^*_{Q|P})$) be the space of C^∞-functions $\phi(x, v)$ (resp. $\phi^*(x, v^*)$) on $X^{(0)} \times V_{P_Q}(\boldsymbol{R})$ (resp. $X^{(0)} \times V^*_{P_Q}(\boldsymbol{R})$) satisfying that

(i) as a function of x, the support of $\phi(x, v)$ (resp. $\phi^*(x, v^*)$) is contained in a compact subset of $X^{(0)}$ independent of v (resp. v^*), and

(ii) as a function of v (resp. v^*), $\phi(x, v)$ (resp. $\phi^*(x, v^*)$) is rapidly decreasing.

For $\phi \in \mathscr{F}_0(X_{Q|P})$, $x \in X(\boldsymbol{Q}) \cap X^{(0)}$ and $\lambda \in \mathfrak{a}^*_{P,\boldsymbol{C}}$, we put

$$ Z_{Q|P}(\phi; x, \lambda) = \int_{G^+_{Q|P}/\Gamma_{Q|P}} |\chi_{Q|P}(q, g)|^{\lambda+\delta_P} \sum_{(y, v)} \phi((q, g)\cdot(y, v)) d_rq dg\,, $$

where the summation with respect to (y, v) is taken over $(\Gamma \cdot x \times V_{P_Q}(\boldsymbol{Z})) \cap \Omega_{Q|P}$, and

$$ \Psi_{ij}(Q\,|\,P; \phi, \lambda) = \int_{\Omega_{Q|P,ij}} |F_{Q|P}(y, v)|^{\lambda-\delta_P} \phi(y, v) d\omega_X(y) dv\,, $$

where $d\omega_X$ is the G^+-invariant measure on $X^{(0)}$ and dv is the standard Euclidean measure on $V_{P_Q}(\boldsymbol{R})$. For $\phi^* \in \mathscr{F}_0(X^*_{Q|P})$, $x \in X(\boldsymbol{Q}) \cap X^{(0)}$ and $\lambda^* \in \mathfrak{a}^*_{{}^wP,\boldsymbol{C}}$, we put

$$Z^*_{Q|P}(\phi^*; x, \lambda^*) = \int_{G^+_{Q|P}/\Gamma_{Q|P}} |\chi_{Q|P}(q, g)|^{\lambda^* + \delta_{{}^wP}} \sum_{(y, v^*)} \phi^*((q, g) * (y, v^*)) d_r q dg ,$$

where the summation with respect to (y, v^*) is taken over $(\Gamma \cdot x \times V^*_{P_Q}(\boldsymbol{Z})) \cap \Omega^*_{Q|P}$, and

$$\Psi^*_{ij}(Q|P; \phi^*, \lambda^*) = \int_{\Omega^*_{Q|P,ij}} |F^*_{Q|P}(y, v^*)|^{\lambda^* - \delta_{{}^wP}} \phi^*(y, v^*) d\omega_X(y) dv^* ,$$

where dv^* is the standard Euclidean measure on $V^*_{P_Q}(\boldsymbol{R})$. The integrals $\Psi_{ij}(Q|P; \phi, \lambda)$ (resp. $\Psi^*_{ij}(Q|P; \phi^*, \lambda^*)$) are absolutely convergent in $\delta_P + \mathfrak{a}^{*+}_{P,\boldsymbol{C}}$ (resp. $\delta_{{}^wP} + \mathfrak{a}^{*+}_{{}^wP,\boldsymbol{C}}$).

Now we have the following new integral representations of the Eisenstein series. The proof is similar to that of Proposition 4.2 and is omitted.

PROPOSITION 5.7. *The integral* $Z_{Q|P}(\phi; x, \lambda)$ *(resp.* $Z^*_{Q|P}(\phi^*; x, \lambda^*)$*) is absolutely convergent in* $\delta + \mathfrak{a}^{*+}_{P,\boldsymbol{C}}$ *(resp.* $\delta^* + \mathfrak{a}^{*+}_{{}^wP,\boldsymbol{C}}$*) for some* $\delta \in \mathfrak{a}^*_{P,\boldsymbol{C}}$ *(resp.* $\delta^* \in \mathfrak{a}^*_{{}^wP,\boldsymbol{C}}$*). Moreover, under a suitable normalization of the measures* $d_r q$, dg, $d\omega_X$, *we have*

$$Z_{Q|P}(\phi; x, \lambda) = \zeta_{P_Q}(\lambda) \sum_{\substack{1 \le i \le \alpha \\ 1 \le j \le \nu_i}} E_{ij}(P; x, \lambda) \Psi_{ij}(Q|P; \phi, \lambda)$$

and

$$Z^*_{Q|P}(\phi^*; x, \lambda^*) = \zeta_{P_Q}(-w^{-1}\lambda^*) \sum_{\substack{1 \le i \le \alpha \\ 1 \le j \le \nu_i}} E_{ij}({}^wP; x, \lambda^*) \Psi^*_{ij}(Q|P; \phi^*, \lambda^*) .$$

5.3. In this and the next subsections, we introduce two prehomogeneous vector spaces which play an important role in the proof of the functional equations of the Eisenstein series. The first one is the prehomogeneous vector space of flag type corresponding to the Eisenstein series of the Riemannian symmetric space of $L_Q(\boldsymbol{R})$.

Since L_Q is a direct product of general linear groups, we can write $L_Q = GL(m_1) \times \cdots \times GL(m_l)$. Put $K_Q = SO(m_1) \times \cdots \times SO(m_l)$ and let $(G_{P_Q} \times K_Q, \rho_{P_Q}, V_{P_Q})$ be the prehomogeneous vector space of flag type attached to the spherical triple (L_Q, K_Q, P_Q). We consider the standard real structure of this prehomogeneous vector space, for which $K_Q(\boldsymbol{R})$ is compact. Then there exists a unique real open $G^+_{P_Q} \times K_Q(\boldsymbol{R})$-orbit V'_{P_Q} in $V_{P_Q}(\boldsymbol{R})$, which is characterized by the same rank condition as in the definition of V'_P in the proof of Lemma 4.1. We denote by $|d_{P_Q}(v)|^\lambda$ ($\lambda \in \mathfrak{a}^*_{P_Q,\boldsymbol{C}} = \mathfrak{a}^*_{P,\boldsymbol{C}}$) the function on V'_{P_Q} satisfying

$$|d_{P_Q}(\rho_{P_Q}(g, k)v)|^\lambda = |\chi_{Q|P}(k, g)|^\lambda |d_{P_Q}(v)|^\lambda .$$

For $\phi \in \mathscr{S}(V_{P_Q}(\boldsymbol{R}))$, we put

$$\Psi_0(\phi, \lambda) = \int_{V'_{P_Q}} |d_{P_Q}(v)|^{\lambda - \delta_P} \phi(v) dv .$$

The integral is the local zeta function attached to $(G_{P_Q}\times K_Q, \rho_{P_Q}, V_{P_Q})$ defined in Proposition 4.2.

We define the local zeta function also for $(G_{P_Q}\times K_Q, \rho^*_{P_Q}, V^*_{P_Q})$, the prehomogeneous vector space contragredient to $(G_{P_Q}\times K_Q, \rho_{P_Q}, V_{P_Q})$. Let $V^{*\prime}_{P_Q}$ be the unique real open $G^+_{P_Q}\times K_Q(\boldsymbol{R})$-orbit in $V^*_{P_Q}(\boldsymbol{R})$. We denote by $|d^*_{P_Q}(v)|^{\lambda^*}$ $(\lambda^*\in\mathfrak{a}^*_{wP_Q,\boldsymbol{C}}=\mathfrak{a}^*_{wP,\boldsymbol{C}})$ the function on $V^{*\prime}_{P_Q}$ satisfying

$$|d^*_{P_Q}(\rho^*_{P_Q}(g, k)v^*)|^{\lambda^*}=|\chi_{Q|P}(k, g)|^{\lambda^*}|d^*_{P_Q}(v^*)|^{\lambda^*}.$$

For $\phi^*\in\mathscr{S}(V^*_{P_Q}(\boldsymbol{R}))$, we put

$$\Psi^*_0(\phi^*, \lambda^*)=\int_{V^{*\prime}_{P_Q}}|d^*_{P_Q}(v^*)|^{\lambda^*-\delta_{wP}}\phi^*(v^*)dv^*.$$

If we identify $V^*_{P_Q}$ with V_{P_Q} as in the proof of Lemma 5.1, then $V^{*\prime}_{P_Q}=V'_{P_Q}$ and

$$|d^*_{P_Q}(v)|^{-w\lambda}=|d_{P_Q}(v)|^{\lambda}. \tag{5.1}$$

PROPOSITION 5.8. *The integral $\Psi_0(\phi, \lambda)$ (resp. $\Psi^*_0(\phi^*, \lambda^*)$) is absolutely convergent for $\lambda\in\delta_{P_Q}+\mathfrak{a}^{*++}_{P_Q,\boldsymbol{C}}$ (resp. $\lambda^*\in\delta_{wP_Q}+\mathfrak{a}^{*++}_{wP_Q,\boldsymbol{C}}$) and has an analytic continuation to a meromorphic function of λ (resp. λ^*) in $\mathfrak{a}^*_{P,\boldsymbol{C}}$ (resp. $\mathfrak{a}^*_{wP,\boldsymbol{C}}$). Moreover they satisfy the functional equation*

$$\Psi_0(\hat{\phi}^*, \lambda)=\frac{\Gamma_{P_Q}(\lambda)}{\Gamma_{P_Q}(-\lambda)}\Psi^*_0(\phi^*, w\lambda),$$

where

$$\hat{\phi}^*(v)=\int_{V^*_{P_Q}(\boldsymbol{R})}\phi^*(v^*)\exp(2\pi i\langle v, v^*\rangle)dv^*.$$

REMARK. Note that, if we identify $\mathfrak{a}^*_{P_Q,\boldsymbol{C}}$ with $\mathfrak{a}^*_{P,\boldsymbol{C}}$, then we have $\delta_{P_Q}+\mathfrak{a}^{*+}_{P_Q,\boldsymbol{C}}\supset\delta_P+\mathfrak{a}^{*+}_{P,\boldsymbol{C}}$ and $\delta_{wP_Q}+\mathfrak{a}^{*+}_{wP_Q,\boldsymbol{C}}\supset\delta_{wP}+\mathfrak{a}^{*+}_{wP,\boldsymbol{C}}$.

PROOF OF PROPOSITION 5.8. The proof can be easily reduced to the case where $Q=G$ and $P_Q=P=P_{e_1,\dots,e_r}$. In this case, we have

$$V'_P=\{(v_1,\dots,v_{r-1})\mid V_i\in M(n_i, n_{i+1}; \boldsymbol{R}),\ \operatorname{rank} v_i=n_i\ (1\le i\le r-1)\}.$$

For $i=1, 2, \dots, r$, we put

$$\Lambda_i(p)=\det(p_1)\cdots\det(p_i)\qquad\text{for}\quad\begin{pmatrix} p_1 & & 0\\ & \ddots & \\ * & & p_r\end{pmatrix}$$

and write

$$\lambda=\sum_{i=1}^{r}\lambda_i\Lambda_i.$$

Then we have

$$\Psi_0(\phi, \lambda)=\int_{V'_P}\prod_{i=1}^{r-1}|\det((v_i\cdots v_{r-1})\cdot{}^t(v_i\cdots v_{r-1}))|^{(2\lambda_i-e_i-e_{i+1})/4}\phi(v)dv\,.$$

It is clear that the integral is absolutely convergent for $\mathrm{Re}(\lambda_i)>(e_i+e_{i+1})/2$ $(i=1,\ldots, r-1)$, namely, for $\lambda\in\delta_P+\mathfrak{a}^{*++}_{P,C}$. We identify $V^*_{P_Q}$ with V_{P_Q} via the inner product $\langle v, v^*\rangle=\sum_{i=1}^{r-1}\mathrm{tr}\, v_i{}^tv_i^*$. Since $w\delta_P=-\delta_{wP}$, the identity (5.1) implies that

$$\Psi^*_0(\phi, \lambda^*)=\Psi_0(\phi, -w^{-1}\lambda^*)\quad(\lambda^*\in\mathfrak{a}^*_{wP,C})\,.$$

It is easy to check that $\mathfrak{a}^{*++}_{wP,C}=-w\mathfrak{a}^{*++}_{P,C}$. Hence $\Psi^*_0(\phi^*, \lambda^*)$ is absolutely convergent for $\lambda^*\in\delta_{wP}+\mathfrak{a}^{*++}_{wP,C}$. From the general theory of prehomogeneous vector spaces ([S1, Theorem 1]), there exists a meromorphic function $\gamma(\lambda)$ such that the functional equation

$$\Psi_0(\hat{\phi}^*, \lambda)=\gamma(\lambda)\Psi^*_0(\phi^*, w\lambda)$$

holds for any $\phi^*\in\mathscr{S}(V^*_{P_Q}(\boldsymbol{R}))$. To obtain the explicit formula for $\gamma(\lambda)$, it is enough to calculate the integrals in the both sides of the functional equation for the function $\phi^*(v)=\exp(-\pi\langle v, v\rangle)$. This can be done by using the well known formulas

$$\hat{\phi}^*(v)=\phi^*(v)$$

and

$$\int_{M(n;\boldsymbol{R})}|\det v|^s\exp(-\pi\,\mathrm{tr}\, v{}^tv)dv=\frac{\prod_{i=1}^{n}\Gamma_{\boldsymbol{R}}(s+i)}{\prod_{i=1}^{n}\Gamma_{\boldsymbol{R}}(i)}\,.$$

In fact we can obtain the identities

$$\Psi_0(\hat{\phi}^*; \lambda)=\frac{\Gamma_P(\lambda)}{\Gamma_P(\delta_P)}\,,\quad \Psi^*_0(\phi^*; w\lambda)=\frac{\Gamma_P(-\lambda)}{\Gamma_P(\delta_P)}\,.$$

Hence we have $\gamma(\lambda)=\Gamma_P(\lambda)/\Gamma_P(-\lambda)$. ∎

The following proposition is the key to the proof of Theorem 3.6.

PROPOSITION 5.9. *Let* $\phi_1\in C^\infty_0(X^{(0)})$, $\phi_2\in\mathscr{S}(V_{P_Q}(\boldsymbol{R}))$ *and* $\phi^*_2\in\mathscr{S}(V^*_{P_Q}(\boldsymbol{R}))$. *Let* $K=SO_m(\boldsymbol{R})$, *the maximal compact subgroup of* G^+. *If* ϕ_1 *is K-invariant, then we have*

$$\Psi_{ij}(Q\,|\,P; \phi_1\otimes\phi_2, \lambda)=\Psi_{ij}(P; \phi_1, \lambda)\Psi_0(\phi_2, \lambda) \tag{5.2}$$

and

$$\Psi^*_{ij}(Q\,|\,P; \phi_1\otimes\phi^*_2, \lambda^*)=\Psi_{ij}({}^wP; \phi_1, \lambda^*)\Psi^*_0(\phi^*_2, \lambda^*)\,, \tag{5.3}$$

where $\Psi_{ij}(P; \phi_1, \lambda)$ (*resp.* $\Psi_{ij}({}^wP; \phi_1, \lambda^*)$) *are the integrals defined for the* P^+-*orbits* Ω_{ij}

(resp. the ${}^wP^+$*-orbits* ${}^w\Omega_{ij}$*) by* (1.6).

PROOF. We prove only (5.2), since the proof of (5.3) is quite the same. For any $v\in V'_{P_Q}$, there exists $(g, k)\in G^+_{P_Q}\times K_Q(\boldsymbol{R})$ such that $\rho_{P_Q}(g, k)\boldsymbol{v}=v$. Then we have

$$\begin{aligned}|F_{Q|P}(y, v)|^{\lambda-\delta_P}&=|F_{Q|P}((k, g)\cdot(k^{-1}\cdot y, \boldsymbol{v}))|^{\lambda-\delta_P}\\&=|F_{Q|P}(k^{-1}\cdot y, \boldsymbol{v})|^{\lambda-\delta_P}|\chi_{Q|P}(1, g)|^{\lambda-\delta_P}\\&=|f(P; k^{-1}\cdot y)|^{\lambda-\delta_P}|d_{P_Q}(v)|^{\lambda-\delta_P}.\end{aligned}$$

Hence

$$\begin{aligned}\Psi_{ij}(Q\,|\,P; \phi, \lambda)&=\int_{\Omega_{Q|P,ij}}|f(P; k^{-1}\cdot y)|^{\lambda-\delta_P}|d_{P_Q}(v)|^{\lambda-\delta_P}\phi_1(y)\phi_2(v)\omega_{Q|P}(y, v)\\&=\int_{V'_{P_Q}}|d_{P_Q}(v)|^{\lambda-\delta_P}\phi_2(v)dv\int_{\Omega_{ij}}|f(P; y)|^{\lambda-\delta_P}\phi_1(y)d\omega_X(y).\end{aligned}$$

Since $d\omega_\Omega(y)=|f(P; y)|^{-2\delta_P}d\omega_X(y)$, this implies the required identity

$$\Psi_{ij}(Q\,|\,P; \phi, \lambda)=\Psi_{ij}(P; \phi_1, \lambda)\Psi_0(\phi_2, \lambda).$$

∎

5.4. For the proof of Theorem 3.6, we need another prehomogeneous vector space.

We consider the prehomogeneous vector space $(G_{Q|P}\times H, \rho_{Q|P}, M(m)\oplus V_{P_Q})$ defined by

$$\rho_{Q|P}(q, g, h)(x, v)=(qxh^{-1}, \rho_{P_Q}(\pi(q, g))v)\quad(q\in Q, g\in G_{P_Q}, h\in H, x\in M(m), v\in V_{P_Q}).$$

Then the $\rho_{Q|P}(G^+_{Q|P}\times H^+)$-open orbits contained in $G^+\times V_{P_Q}(\boldsymbol{R})$ correspond bijectively to the $G^+_{Q|P}$-orbits in $\Omega_{Q|P}\cap(X^{(0)}\times V_{P_Q}(\boldsymbol{R}))$. In fact, with the same notation as in Lemma 5.3, we see that

$$\tilde{\Omega}_{Q|P,ij}=\rho_{Q|P}(G^+_{Q|P}\times H^+)(g_i, v(h_{ij}))\qquad(1\le i\le\alpha, 1\le j\le\nu_i)$$

are the $\rho_{Q|P}(G^+_{Q|P}\times H^+)$-open orbits contained in $G^+\times V_{P_Q}(\boldsymbol{R})$.

We note that the direct summand V_{P_Q} is a regular subspace (in the sense of [S1, §2]) of $(G_{Q|P}\times H, \rho_{Q|P}, M(m)\oplus V_{P_Q})$. Hence, by [S1, Lemma 2.4], the partial dual $(G_{Q|P}\times H, \rho^*_{Q|P}, M(m)\oplus V^*_{P_Q})$ with respect to V_{P_Q} is also a prehomogeneous vector space. With the notation as in Lemma 5.5, the $\rho^*_{Q|P}(G^+_{Q|P}\times H^+)$-open orbits contained in $G^+\times V^*_{P_Q}(\boldsymbol{R})$ are given by

$$\tilde{\Omega}^*_{Q|P,ij}=\rho^*_{Q|P}(G^+_{Q|P}\times H^+)(g_i, v^*(h^*_{ij}))\qquad(1\le i\le\alpha, 1\le j\le\nu_i)$$

and correspond bijectively to the $G^+_{Q|P}$-orbits in $\Omega^*_{Q|P}\cap(X^{(0)}\times V^*_{P_Q}(\boldsymbol{R}))$.

It is easy to see that generic isotropy subgroups of these prehomogeneous vector spaces are isomorphic to P_x $(x\in\Omega)$, which is reductive by Assumption (3.6) and Lemma 2.3. Hence the singular set of $(G_{Q|P}\times H, \rho_{Q|P}, M(m)\oplus V_{P_Q})$ is a hypersurface and we may apply the theory of functional equations of local zeta functions in [S1, §5].

In the present case, the local zeta functions are defined as follows:

$$\tilde{\Psi}_{ij}(Q\,|\,P;\phi,\lambda)=\int_{\tilde{\Omega}_{Q|P,ij}}|\tilde{F}_{Q|P}(x,v)|^{\lambda-\delta_P}\phi(x,v)d^\times x dv\,,$$

$$\tilde{\Psi}^*_{ij}(Q\,|\,P;\phi^*,\lambda^*)=\int_{\tilde{\Omega}^*_{Q|P,ij}}|\tilde{F}^*_{Q|P}(x,v^*)|^{\lambda^*-\delta_{wP}}\phi^*(x,v^*)d^\times x dv^*\,,$$

where $\phi\in\mathscr{S}(M(m;\boldsymbol{R})\oplus V_{P_Q}(\boldsymbol{R}))$, $\phi^*\in\mathscr{S}(M(m;\boldsymbol{R})\oplus V^*_{P_Q}(\boldsymbol{R}))$,

$$|\tilde{F}_{Q|P}(x,v)|^\lambda=|F_{Q|P}(xH,v)|^\lambda\quad(\lambda\in\mathfrak{a}^*_{P,\boldsymbol{C}})\,,$$

$$|\tilde{F}^*_{Q|P}(x,v^*)|^{\lambda^*}=|F^*_{Q|P}(xH,v^*)|^{\lambda^*}\quad(\lambda^*\in\mathfrak{a}^*_{wP,\boldsymbol{C}})\,,$$

$d^\times x=|\det x|^{-m}\prod_{i,j=1}^m dx_{ij}$ and the other notation is the same as in §5.2.

For $\phi^*\in\mathscr{S}(M(m;\boldsymbol{R})\oplus V^*_{P_Q}(\boldsymbol{R}))$, we define its partial Fourier transform with respect to V_{P_Q} by setting

$$\hat{\phi}^*(x,v)=\int_{V^*_{P_Q}(\boldsymbol{R})}\phi^*(x,v^*)\exp(2\pi i\langle v,v^*\rangle)dv^*\,.$$

Then Theorem 1 of [S1] gives the following functional equation.

PROPOSITION 5.10. *The integrals* $\tilde{\Psi}_{ij}(Q\,|\,P;\phi,\lambda)$ (*resp.* $\tilde{\Psi}^*_{ij}(Q\,|\,P;\phi^*,\lambda^*)$) *have analytic continuations to meromorphic functions of* λ (*resp.* λ^*) *in* $\mathfrak{a}^*_{P,\boldsymbol{C}}$ (*resp.* $\mathfrak{a}^*_{wP,\boldsymbol{C}}$) *and satisfy the functional equation*

$$\tilde{\Psi}_{ij}(Q\,|\,P;\hat{\phi}^*,\lambda)=\sum_{1\le j^*\le\nu_i}\gamma^{(i)}_{jj^*}(\lambda)\tilde{\Psi}^*_{ij^*}(Q\,|\,P;\phi^*,w\lambda)\,,$$

where $\gamma^{(i)}_{jj^*}(\lambda)$ *are meromorphic functions independent of* ϕ^* *with elementary expression in terms of the gamma function and exponential functions.*

REMARK. The calculation of the gamma matrix $(\gamma^{(i)}_{jj^*}(\lambda))$ is reduced to the calculation of the gamma matrix of the local functional equation for the prehomogeneous vector space $(G_{P_Q}\times H_Q,\rho_{P_Q},V_{P_Q})$ (see [S1, §5.2]).

We rewrite the functional equation in Proposition 5.10 into the functional equation satisfied by Ψ_{ij} and Ψ^*_{ij}.

Let dh be the Haar measure on H^+ normalized by

$$\int_{G^+}f(x)d^\times x=\int_{X^{(0)}}d\omega_X(xH)\int_{H^+}f(xh)dh\,.$$

For $\phi\in\mathscr{F}_0(X_{Q|P})$ (resp. $\phi^*\in\mathscr{F}_0(X^*_{Q|P})$), take a $\tilde{\phi}\in\mathscr{S}(M(m;\boldsymbol{R})\oplus V_{P_Q}(\boldsymbol{R}))$ (resp. $\tilde{\phi}^*\in\mathscr{S}(M(m;\boldsymbol{R})\oplus V^*_{P_Q}(\boldsymbol{R}))$) such that

$$\int_{H^+} \tilde{\phi}(xh, v)dh = \phi(xH, v) \qquad \left(\text{resp.} \int_{H^+} \tilde{\phi}^*(xh, v^*)dh = \phi^*(xH, v^*)\right).$$

Then we have

$$\tilde{\Psi}_{ij}(Q\,|\,P; \tilde{\phi}; \lambda) = \Psi_{ij}(Q\,|\,P; \phi; \lambda), \quad \tilde{\Psi}^*_{ij}(Q\,|\,P; \tilde{\phi}^*; \lambda^*) = \Psi^*_{ij}(Q\,|\,P; \phi^*; \lambda^*).$$

Therefore, if we put for $\phi^* \in \mathscr{F}_0(X^*_{Q|P})$

$$\hat{\phi}^*(xH, v) = \int_{V^*_{P_Q}(\boldsymbol{R})} \phi^*(xH, v^*) \exp(2\pi i \langle v, v^* \rangle) dv^*,$$

then Proposition 5.10 gives the following functional equation satisfied by Ψ_{ij} and Ψ^*_{ij}.

PROPOSITION 5.11. *For $\phi \in \mathscr{F}_0(X_{Q|P})$ (resp. $\phi^* \in \mathscr{F}_0(X^*_{Q|P})$), the integrals $\Psi_{ij}(Q\,|\,P; \phi, \lambda)$ (resp. $\Psi^*_{ij}(Q\,|\,P; \phi^*, \lambda^*)$) have analytic continuations to meromorphic functions of λ (resp. λ^*) in $\mathfrak{a}^*_{P,C}$ (resp. $\mathfrak{a}^*_{{}^wP,C}$) and satisfy the functional equation*

$$\Psi_{ij}(Q\,|\,P; \hat{\phi}^*, \lambda) = \sum_{1 \le j^* \le \nu_i} \gamma^{(i)}_{jj^*}(\lambda) \Psi^*_{ij^*}(Q\,|\,P; \phi^*, w\lambda) \quad (\phi^* \in \mathscr{F}_0(X_{Q|P})),$$

where $\gamma^{(i)}_{jj^*}(\lambda)$ are meromorphic functions independent of ϕ^* with elementary expression in terms of the gamma function and exponential functions.

5.5. Now we are in a position to prove Theorems 3.5 and 3.6.

PROOF OF THEOREM 3.6. Let $\phi_1 \in C_0^\infty(X^{(0)})$, $\phi_2 \in \mathscr{S}(V_{P_Q}(\boldsymbol{R}))$ and $\phi_2^* \in \mathscr{S}(V^*_{P_Q}(\boldsymbol{R}))$. If ϕ_1 is K-invariant, then it follows from Propositions 5.8, 5.9 and 5.11 that the integrals $\Psi_{ij}(P; \phi_1, \lambda)$ (resp. $\Psi_{ij}({}^wP; \phi_1, \lambda^*)$) have analytic continuations to meromorphic functions of λ (resp. λ^*) in $\mathfrak{a}^*_{P,C}$ (resp. $\mathfrak{a}^*_{{}^wP,C}$) and satisfy the functional equation

$$\Psi_{ij}(P; \phi_1, \lambda) = \frac{\Gamma_{P_Q}(-\lambda)}{\Gamma_{P_Q}(\lambda)} \sum_{j^*} \gamma^{(i)}_{jj^*}(\lambda) \Psi_{ij^*}({}^wP; \phi_1, w\lambda).$$

Therefore, putting

$$C_{\mathrm{sph}}(w, \lambda)_{(i^*, j^*), (i, j)} = \begin{cases} \dfrac{\Gamma_{P_Q}(-\lambda)}{\Gamma_{P_Q}(\lambda)} \gamma^{(i)}_{jj^*}(\lambda) & \text{if } i = i^*, \\ 0 & \text{if } i \ne i^*, \end{cases} \tag{5.4}$$

we obtain Theorem 3.6. ∎

REMARK. Analytic continuations of the integrals $\Psi_{ij}(P; \phi_1, \lambda)$ and $\Psi_{ij}({}^wP; \phi_1, \lambda^*)$ can be proved for any $\phi_1 \in C_0^\infty(X^{(0)})$ without the assumption that ϕ_1 is K-invariant.

The proof of Theorem 3.5 is based on the following lemma.

LEMMA 5.12. *Assume that $\phi^* \in \mathscr{F}_0(X^*_{Q|P})$ satisfies the condition that*

$$\phi^* \text{ vanishes outside } \Omega^*_{Q|P} \text{ and } \hat{\phi}^* \text{ vanishes outside } \Omega_{Q|P}.$$

Then the integral $Z_{Q|P}(\hat{\phi}^*; x, \lambda)$ *(resp.* $Z^*_{Q|P}(\phi^*; x, \lambda^*)$*) has an analytic continuation to a meromorphic function of* λ *(resp.* λ^**) in* $\mathfrak{a}^*_{P,C}$ *(resp.* $\mathfrak{a}^*_{{}^wP,C}$*) and is holomorphic in the convex hull of* $(\delta + \mathfrak{a}^{*+}_{P,C}) \cup (\delta^* + \mathfrak{a}^{*+}_{{}^wP,C})$*, where* δ *and* δ^* *are the same as in Proposition* 5.7. *Moreover they satisfy the functional equation*

$$Z_{Q|P}(\hat{\phi}^*; x, \lambda) = Z^*_{Q|P}(\phi^*; x, w\lambda).$$

By using Lemma 5.6, we can prove the lemma in the same manner as in the proof of [S1, Lemma 6.1]; hence we omit the proof.

PROOF OF THEOREM 3.5. Take a $\phi^*_0 \in C^\infty_0(\Omega^*_{Q|P,ij^*})$ and a $\chi \in \mathfrak{a}^{*+}_{P,C} \cap \mathfrak{X}(P)$. Let $\phi^* \in C^\infty_0(\Omega^*_{Q|P,ij^*})$ be the function satisfying

$$\hat{\phi}^*(x, v) = |F_{Q|P}(x, v)|^\chi \hat{\phi}^*_0(x, v).$$

Then the function ϕ^* satisfies the assumption in Lemma 5.12 (cf. [S1, Lemma 6.2]). Hence, by Proposition 5.7 and Lemma 5.12, we have

$$\begin{aligned}
&\zeta_{P_Q}(-\lambda) E_{ij^*}({}^wP; x, w\lambda) \Psi^*_{ij^*}(Q|P; \phi^*, w\lambda) \\
&= Z^*_{Q|P}(\phi^*; x, w\lambda) \\
&= Z_{Q|P}(\hat{\phi}^*; x, \lambda) \\
&= \zeta_{P_Q}(\lambda) \sum_{\substack{1 \le k \le \alpha \\ 1 \le j \le \nu_i}} E_{kj}(P; x, \lambda) \Psi_{kj}(Q|P; \hat{\phi}^*, \lambda).
\end{aligned}$$

Since the support of ϕ^* is assumed to be contained in $\Omega^*_{Q|P,ij^*}$, Proposition 5.11 and (5.4) yield the identity

$$\Psi_{kj}(Q|P; \hat{\phi}^*, \lambda) = \begin{cases} \dfrac{\Gamma_{P_Q}(\lambda)}{\Gamma_{P_Q}(-\lambda)} C_{\mathrm{sph}}(w, \lambda)_{(i,j^*),(i,j)} \Psi^*_{ij^*}(Q|P; \phi^*, w\lambda) & \text{if } k = i, \\ 0 & \text{if } k \ne i. \end{cases}$$

Note that we can choose ϕ^*_0 and χ so that $\Psi^*_{ij^*}(Q|P; w\lambda)$ does not vanish identically. Therefore, combining these two identities, we obtain

$$E_{ij^*}({}^wP; x, w\lambda) = \frac{\hat{\zeta}_{P_Q}(\lambda)}{\hat{\zeta}_{P_Q}(-\lambda)} \sum_{1 \le j \le \nu_i} C_{\mathrm{sph}}(w, \lambda)_{(i,j^*),(i,j)} E_{ij}(P; x, \lambda).$$

By (3.7), this proves Theorem 3.5. ∎

For $\chi \in \mathfrak{a}^{*+}_{{}^wP,C} \cap \mathfrak{X}({}^wP)$, let $F^{*\chi}_{Q|P}(x, v^*)$ be the relative invariant on $X^*_{Q|P}$ corresponding to χ. Then $F^{*\chi}_{Q|P}$ is regular on $X^*_{Q|P}$ and hence a polynomial function of v^*.

Let $F^{*\chi}_{Q|P}(x, \partial/\partial v)$ be the linear partial differential operator in $\boldsymbol{C}[X][\partial/\partial v]$ satisfying

$$F^{*\chi}_{Q|P}\left(x, \frac{\partial}{\partial v}\right)\exp(\langle v, v^*\rangle)=F^{*\chi}_{Q|P}(x, v^*)\,.$$

Then there exists a polynomial function $b_\chi(\lambda)$ on $\mathfrak{a}^*_{P,C}$ such that the identity

$$F^{*\chi}_{Q|P}\left(x, \frac{\partial}{\partial v}\right)|F_{Q|P}(x, v)|^{\lambda-\delta_P}=\pm b_\chi(\lambda)|F_{Q|P}(x, v)|^{\lambda-\delta_P+w^{-1}\chi}$$

holds on $\Omega_{Q|P,ij}$, where the sign in the right hand side of the identity depends on i, j. The polynomial $b_\chi(\lambda)$ is the b-function of the prehomogeneous vector space $(G_{P_Q}\times H_Q, \rho_{P_Q}, V_{P_Q})$.

Let $b_Q(\lambda)$ be the greatest common divisor of $b_\chi(\lambda)$ for all $\chi\in\mathfrak{a}^{*+}_{wP,C}\cap\mathfrak{X}({}^wP)$.

PROPOSITION 5.13. *The functions $b_Q(\lambda)\zeta_{P_Q}(\lambda)E_{ij}(P; x, \lambda)$ are holomorphic in the convex hull of $(\delta+\mathfrak{a}^{*+}_{P,C})\cup(\delta^*+\mathfrak{a}^{*+}_{wP,C})$.*

The proof is quite the same as in the case of zeta functions associated with prehomogeneous vector spaces (see the proof of Theorem 2 in [S1, §6]) and is omitted. Applying the proposition to various Q, we often get fairly satisfactory information on the location of poles of the Eisenstein series (cf. Appendix, Proof of Theorem 3.8).

Appendix: The proof of the analytic continuation and the functional equations of the Eisenstein series on $GL(n)/O(n)$. In this appendix we give proofs of Theorems 3.8 and 3.9 in §3.4.

First, assuming the functional equations for $\sigma_i=(i, i+1)\in\mathfrak{S}_r$ $(i=1,\dots,r-1)$, we prove Theorem 3.8.

PROOF OF THEOREM 3.8. For $P\in\mathscr{P}$, we put $C_P=\delta_P+\mathfrak{a}^{*+}_{P,C}$. For $\sigma\in\mathfrak{S}_r$, let $l(\sigma)$ be the length of σ with respect to the generator system $\{\sigma_1,\dots,\sigma_{r-1}\}$. We denote by $\mathscr{D}^{(l)}_P$ the convex hull of the union of $\sigma^{-1}C_{\sigma_P}$ for all $\sigma\in\mathfrak{S}_r$ with $l(\sigma)\le l$. Here we consider σ as an element in $W_X(\mathfrak{a}^*_P, \mathfrak{a}^*_{\sigma P})$. It is easy to see that $\mathscr{D}^{(l)}_P$ coincides with $\mathfrak{a}^*_{P,C}$ for sufficiently large l. Therefore it is enough to prove the following:

The functions $\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ multiplied by

$$B^{(l)}_P(\lambda)=\prod_{\substack{1\le i<j\le r\\ j-i\le l}} b_{e_i+e_j,e_i}\left(\frac{2z_i-2z_j-e_i-e_j}{4}\right)$$

are holomorphic in $\mathscr{D}^{(l)}_P$.

We prove this assertion by induction on l. We put $P_i={}^{\sigma_i}P$. The case $l=1$ is an easy consequence of Proposition 5.13 and the fact that the product $\prod_{\mu=0}^{e_i-1}\zeta(z_i-z_j+(e_i+e_j)/2-\mu)$ $(i<j)$ is holomorphic in $\sigma_k^{-1}C_{P_k}$ unless $(i, j)=(k, k+1)$.

Now consider the case $l\ge2$. Note that $\mathscr{D}^{(l)}_P$ is the convex hull of

$$\mathscr{D}^{(l-1)}_P\cup\left(\bigcup_{i=1}^{r-1}\sigma_i^{-1}\mathscr{D}^{(l-1)}_{P_i}\right).$$

Hence it is sufficient to prove that $B_P^{(l)}\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ is holomorphic in $\mathscr{D}_P^{(l-1)} \cup \sigma_i^{-1}\mathscr{D}_{P_i}^{(l-1)}$ for any i. Since $B_P^{(l-1)}$ divides $B_P^{(l)}$, it is obvious that $B_P^{(l)}\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ is holomorphic in $\mathscr{D}_P^{(l-1)}$. To prove the holomorphy in $\sigma_i^{-1}\mathscr{D}_{P_i}^{(l-1)}$, one can use the functional equation for σ_i:

$$\zeta_P(\lambda)E_\varepsilon(P; x, \lambda) = \frac{\prod_{\mu=0}^{e_i-1}\zeta(z_i - z_{i+1} + (e_i + e_{i+1})/2 - \mu)}{\prod_{\nu=0}^{e_{i+1}-1}\zeta(z_{i+1} - z_i + (e_i + e_{i+1})/2 - \nu)}$$
$$\times C_{\mathrm{Eis}}(\sigma_i^{-1}; \sigma_i\lambda)\sum_\eta C_{\mathrm{sph}}(\sigma_i^{-1}; \sigma_i\lambda)_{\varepsilon,\eta}\zeta_{P_i}(\sigma_i\lambda)E_\eta(P_i; x, \sigma_i\lambda)\,.$$

By the induction hypothesis, the function

$$B_{P_i}^{(l-1)}(\sigma_i\lambda)\zeta_{P_i}(\sigma_i\lambda)E_\eta(P_i; x, \sigma_i\lambda)$$

viewed as a function of λ is holomorphic in $\sigma_i^{-1}\mathscr{D}_{P_i}^{(l-1)}$. Since $B_{P_i}^{(l-1)}(\sigma_i\lambda)$ is a divisor of $B_P^{(l)}(\lambda)$, the explicit form of the functional equation given in Theorem 3.9 implies that $B_P^{(l)}(\lambda)\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ is holomorphic in $\sigma_i^{-1}\mathscr{D}_{P_i}^{(l-1)}$ except at possible poles $z_i - z_{i+1} = c$ (c = some constant). Any hyperplane of this form intersects with $\mathscr{D}_P^{(1)}$, in which $B_P^{(l)}(\lambda)\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ is holomorphic. Hence $B_P^{(l)}(\lambda)\zeta_P(\lambda)E_\varepsilon(P; x, \lambda)$ is holomorphic in $\sigma_i^{-1}\mathscr{D}_{P_i}^{(l-1)}$. ∎

Proof of Theorem 3.9. By (5.4) and the remark to Proposition 5.10, the calculation of $C_{\mathrm{sph}}(\sigma_i, \lambda)$ can be reduced to the calculation of the gamma matrix of the local functional equation for the prehomogeneous vector space $(SO(e_i + e_{i+1}) \times GL(e_i), M(e_i + e_{i+1}, e_i))$. Namely, Theorem 3.9 follows from the functional equation (A.1) below together with Theorem A.1.

To simplify the notation, we write m and n for $e_i + e_{i+1}$ and e_i, respectively. We also put

$$e[z] = \exp(2\pi iz),$$

$$I_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}, \quad p+q=m\,,$$

$$V_{(i,j)}^{(p,q)} = \{x \in M(m, n; \boldsymbol{R}) \mid \mathrm{sgn}(I_{p,q}[x]) = (i, j)\}\,, \quad i+j=n\,,$$

$$V_{(i,j)}^{(p,q)\pm} = \{x \in V_{(i,j)}^{(p,q)} \mid \mathrm{sgn}(I_{p,q}[x_1]) = \pm\}\,,$$

$$V_{(i,j)}^{(p,q)*\pm} = \{x \in V_{(i,j)}^{(p,q)} \mid \mathrm{sgn}(I_{p,q}[x']) = (i-1, j) \text{ or } (i, j-1) \text{ according as } + \text{ or } -\}\,,$$

where, for $x \in M(m, n; \boldsymbol{R})$, $I_{p,q}[x] = {}^txI_{p,q}x$, x_1 is the first row vector of x and x' is the m by $n-1$ matrix obtained from x by removing x_1. The set $V_{(i,j)}^{(p,q)}$ is not empty if and only if $\max\{0, n-q\} \le i \le \min\{p, n\}$. If m, n, p, q are fixed and there exists no fear of confusion, we write V_i $V_i^\pm$, $V_i^{*\pm}$ for $V_{(i,j)}^{(p,q)}$, $V_{(i,j)}^{(p,q)\pm}$, $V_{(i,j)}^{(p,q)*\pm}$.

For $f \in \mathscr{S}(M(m, n; \boldsymbol{R}))$, we put

$$\Phi_i(f;s)=\int_{V_i}|\det I_{p,q}[x]|^s f(x)dx\,,$$

$$\Phi_i^{\pm}(f;s)=\int_{V_i^{\pm}}|\det I_{p,q}[x]|^s f(x)dx\,,$$

$$\Phi_i^{*\pm}(f;s)=\int_{V_i^{*\pm}}|\det I_{p,q}[x]|^s f(x)dx\,.$$

Then we have

$$\Phi_i(f;s)=\Phi_i^{+}(f;s)+\Phi_i^{-}(f;s)=\Phi_i^{*+}(f;s)+\Phi_i^{*-}(f;s)\,.$$

We define the Fourier transform of $f\in\mathscr{S}(M(m,n;\boldsymbol{R}))$ by setting

$$\hat{f}(x)=\int_{M(m,n;\boldsymbol{R})}f(y)\boldsymbol{e}[\operatorname{tr}({}^t xy)]dy\,.$$

Then, by the general theory of prehomogeneous vector spaces ([SS]), we have the following functional equation:

$$\text{(A.1)}\qquad \Phi_i(\hat{f};s)=\sum_{\max\{0,n-q\}\le i^*\le\min\{p,n\}}\gamma_{m,n}(s)C_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)\Phi_{i^*}\left(f;-\frac{m}{2}-s\right),$$

where

$$\gamma_{m,n}(s)=\prod_{\mu=1}^{n}\Gamma_{\boldsymbol{R}}(2s+\mu+1)\Gamma_{\boldsymbol{R}}(2s+m-\mu+1)\,,\quad \Gamma_{\boldsymbol{R}}(z)=\pi^{-z/2}\Gamma(z/2)\,.$$

Our problem is to calculate $C_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)$ explicitly.

THEOREM A.1. *The coefficients* $C_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)$ $(i+j=i^*+j^*=n)$ *vanishes unless* $|i-i^*|\le 1$. *In the case* $|i-i^*|\le 1$, *we have the following explicit formulas*:

$$C_{(i,j)}^{(i,j)}(I_{p,q};s)=\begin{cases}(-1)^{ij}\displaystyle\prod_{\mu=1}^{i}\cos\pi\left(s+\frac{q+\mu}{2}\right)\prod_{\mu=i+1}^{n}\cos\pi\left(s+\frac{p+\mu}{2}\right) \\ \qquad\qquad \textit{if}\quad q\equiv j\pmod 2\,, \\ (-1)^{ij}\displaystyle\prod_{\mu=1}^{j}\cos\pi\left(s+\frac{p+\mu}{2}\right)\prod_{\mu=j+1}^{n}\cos\pi\left(s+\frac{q+\mu}{2}\right) \\ \qquad\qquad \textit{if}\quad q\not\equiv j\pmod 2\,,\end{cases}$$

$$C_{(i,j)}^{(i+1,j-1)}(I_{p,q};s)=\begin{cases}\sin\pi\left(\dfrac{q-j+1}{2}\right)\displaystyle\prod_{\mu=1}^{j-1}\cos\pi\left(s+\frac{p+\mu}{2}\right)\prod_{\mu=j+1}^{n}\cos\pi\left(s+\frac{q+\mu}{2}\right) \\ \qquad\qquad \textit{if}\quad i\equiv 0\pmod 2\,, \\ 0\qquad \textit{if}\quad i\equiv 1\pmod 2\,,\end{cases}$$

$$C_{(i,j)}^{(i-1,j+1)}(I_{p,q};s)=\begin{cases}\sin\pi\left(\dfrac{p-i+1}{2}\right)\displaystyle\prod_{\mu=1}^{i-1}\cos\pi\left(s+\frac{q+\mu}{2}\right)\prod_{\mu=i+1}^{n}\cos\pi\left(s+\frac{p+\mu}{2}\right) \\ \qquad\qquad\qquad\qquad \text{if } j\equiv 0 \pmod 2\,, \\ 0 \qquad\qquad\qquad\quad \text{if } j\equiv 1 \pmod 2\,. \end{cases}$$

The theorem was obtained by T. Suzuki in his master thesis except the case where $\max\{p,q\}>n>\min\{p,q\}$. His calculation is based on the method of the microlocal calculus. Here we give an elementary proof.

PROOF. To determine $C_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)$, let us calculate the integral $\Phi_i(\hat{f},s)$ under the assumption that the support of f is contained in $\{x\in M(m,n;\boldsymbol{R})\mid \det I_{p,q}[s]\neq 0\}$ and $\mathrm{Re}(s)>0$.

Our calculation is done by induction on m. For simplicity we put

$$u_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)=\gamma_{m,n}(s)C_{(i,j)}^{(i^*,j^*)}(I_{p,q};s)\,.$$

We introduce a parametrization of $V_i^{\pm}$. For $x\in V_i^{\pm}$, we write $x=(\overset{1}{\overbrace{x_1}},\overset{n-1}{\overbrace{x'}})$. Put $W=\{v\in\boldsymbol{R}^m\mid\langle v,x_1\rangle:={}^t vI_{p,q}x_1=0\}$. Since $I_{p,q}[x_1]\neq 0$, we have $\boldsymbol{R}^m=\boldsymbol{R}x_1\oplus W$. We can choose a basis $w_1,\dots,w_{m-1}$ such that

$$I_{p,q}[x_1,w]=\begin{cases}\begin{pmatrix}I_{p,q}[x_1] & 0\\ 0 & I_{p-1,q}\end{pmatrix} & \text{if } +\,,\\[2ex] \begin{pmatrix}I_{p,q}[x_1] & 0\\ 0 & I_{p,q-1}\end{pmatrix} & \text{if } -\,,\end{cases}$$

where $w=(w_1,\dots,w_{m-1})\in M(m,n-1;\boldsymbol{R})$. It is obvious that $\det(x_1,w)^2=|I_{p,q}[x_1]|$. Writing

$$x=(x_1,w)\begin{pmatrix}1 & {}^t a\\ 0 & A\end{pmatrix}=(x_1,w)\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\begin{pmatrix}1 & {}^t a\\ 0 & 1_{n-1}\end{pmatrix},\quad a\in\boldsymbol{R}^{n-1}\,,\quad A\in M(m-1,n-1;\boldsymbol{R})\,,$$

we take x_1, a, A as a coordinate system on $V_i^{\pm}$. Then

$$\begin{cases}x_1\in V_{(1,0)}^{(p,q)}\,, & A\in V_{(i-1,j)}^{(p-1,q)} & \text{if } +\,,\\ x_1\in V_{(0,1)}^{(p,q)}\,, & A\in V_{(i,j-1)}^{(p,q-1)} & \text{if } -\,.\end{cases}$$

For simplicity we put

$$I'=\begin{cases}I_{p-1,q} & \text{if } +\,,\\ I_{p,q-1} & \text{if } -\,.\end{cases}$$

Since

$$I_{p,q}[x]=\begin{pmatrix}I_{p,q}[x_1] & 0\\ 0 & I'\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\right]\left[\begin{pmatrix}1 & {}^t a\\ 0 & 1_{n-1}\end{pmatrix}\right],$$

we have

$$dx=|I_{p,q}[x_1]|^{(n-1)/2}dx_1 da dA$$
$$\det I_{p,q}[x]=I_{p,q}[x_1]\cdot\det I'[A]\,.$$

We change the variable y on $M(m, n; \boldsymbol{R})$ into

$$B=\begin{pmatrix} b_1 & {}^t b_2 \\ b_3 & B_4 \end{pmatrix}=(x_1, w)^{-1}y\,,\ b_1\in\boldsymbol{R}\,,\ b_2\in\boldsymbol{R}^{n-1}\,,\ b_3\in\boldsymbol{R}^{m-1}\,,\ B_4\in M(m-1, n-1; \boldsymbol{R})\,.$$

Then we have

$$dy=|I_{p,q}[x_1]|^{n/2}dB\,.$$

We also have

$$\mathrm{tr}({}^t xI_{p,q}y)=b_1\cdot I_{p,q}[x_1]+\mathrm{tr}({}^t AI'B_4)+I_{p,q}[x_1]\cdot{}^t b_2 a\,.$$

Put

$$F(x_1; B)=f((x_1, w)B)\,.$$

Then, using the parameter introduced above, we have

$$\begin{aligned}
\Phi_i^+(\hat{f}; s)=&\int_{I_{p,q}[x_1]>0}|I_{p,q}[x_1]|^{s+n-1/2}dx_1\int_{V^{(p-1,q)}_{(i-1,j)}}|\det I_{p-1,q}[A]|^s dA\int_{\boldsymbol{R}^{n-1}}da\\
&\times\int_{M(m,n;\boldsymbol{R})}F(x_1; B)e[b_1\cdot I_{p,q}[x_1]+\mathrm{tr}({}^t AI'B_4)+I_{p,q}[x_1]\cdot{}^t b_2 a]dB\\
=&\int_{I_{p,q}[x_1]>0}|I_{p,q}[x_1]|^{s+1/2}dx_1\int_{V^{(p-1,q)}_{(i-1,j)}}|\det I_{p-1,q}[A]|^s dA\\
&\times\int F\left(x_1; \begin{pmatrix} b_1 & 0 \\ b_3 & B_4 \end{pmatrix}\right)e[b_1\cdot I_{p,q}[x_1]+\mathrm{tr}({}^t AI_{p-1,q}B_4)]db_1 db_3 dB_4\\
=&\sum_{i^*}u^{(i^*-1,j^*)}_{(i-1,j)}(I_{p-1,j}; s)\int_{I_{p,q}[x_1]>0}|I_{p,q}[x_1]|^{s+1/2}dx_1\\
&\times\int_{V^{(p-1,q)}_{(i^*-1,j^*)}}|\det I_{p-1,q}[B_4]|^{-(m-1)/2-s}dB_4\\
&\times\int F\left(x_1; \begin{pmatrix} b_1 & 0 \\ b_3 & B_4 \end{pmatrix}\right)e[b_1\cdot I_{p,q}[x_1]]db_1 db_3\,.
\end{aligned}$$

Let D be the subset of $\boldsymbol{R}^m\times M(m, n; \boldsymbol{R})$ of elements $(x_1, y)=(x_1, (\overbrace{y_1}^{1}, \overbrace{y'}^{n-1}))$ satisfying

$$\begin{cases} I_{p,q}[x_1]>0\,, \\ {}^t y' I_{p,q}x_1=0 \quad \text{in} \quad \boldsymbol{R}^{n-1}\,, \\ \mathrm{sgn}(I_{p,q}[y'])=(i^*-1, j^*)\,. \end{cases}$$

The last integral above can be viewed as an integral on D with respect to the measure $\omega = dx_1 db_1 db_3 dB_4$. In fact, since $b_1 \cdot I_{p,q}[x_1] = {}^t x_1 I_{p,q} y_1$ and $I_{p-1,q}[B_4] = I_{p,q}[y']$, the integral is rewritten as follows:

$$\text{(A.2)} \qquad \int_D |I_{p,q}[x_1]|^{s+1/2} |\det I_{p,q}[y']|^{-(m-1)/2-s} f(y) e[{}^t x_1 I_{p,q} y_1] \omega(x_1, y) .$$

We introduce another coordinate system on D. For $(x_1, y) = (x_1, (y_1, y')) \in D$, fix $y' \in M(m, n-1; \boldsymbol{R})$ and choose $z \in M(m, m-n+1; \boldsymbol{R})$ such that

$$I_{p,q}[(z, y')] = \begin{pmatrix} I_{p-i^*+1,q-j^*} & 0 \\ 0 & I_{p,q}[y'] \end{pmatrix} .$$

The row vectors of z forms a basis of the orthogonal complement of the space spaned by the row vectors of y' with respect to the inner product $\langle v, v^* \rangle = {}^t v I_{p,q} v^*$. Hence we can write

$$x_1 = zu \qquad \text{for some} \quad u \in \boldsymbol{R}^{m-n+1}$$

and

$$y = (z, y') \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & 1_{n-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \boldsymbol{R}^m .$$

Since

$$I_{p,q}[x_1] = I_{p-i^*+1,q-j^*}[u] > 0 ,$$

we see that

$$D \ni (x, y) \Longleftrightarrow \begin{cases} I_{p-i^*+1,q-j^*}[u] > 0 , \\ I_{p,q}[y'] \in V^{(p,q)}_{(i^*-1,j^*)} . \end{cases}$$

LEMMA A.2. *We have*

$$\omega(x_1, y) = |I_{p-i^*+1,q-j^*}[u]|^{n/2-1} du dy' d\beta .$$

PROOF. Consider the mapping $\boldsymbol{R}^m \times M(m, n; \boldsymbol{R}) \to \boldsymbol{R}^{n-1}$ defined by $(x_1, (y_1, y')) \mapsto {}^t y' I_{p,q} x_1$. The image ${}^t y' I_{p,q} x_1$ can be written as

$${}^t y' I_{p,q} x_1 = \begin{cases} I_{p,q}[x_1] \cdot b^2 & \text{for the parameter system } x_1, B , \\ I_{p,q}[y'] v & \text{for the parameter system } \begin{pmatrix} u \\ v \end{pmatrix} = (z, y')^{-1} x_1, \beta, y' . \end{cases}$$

Hence we have

$$\frac{dx_1 dy}{d({}^t y' I_{p,q} x_1)} = |I_{p,q}[x_1]|^{-n/2+1} dx_1 db_1 db_3 dB_4 = du dy' d\beta . \qquad \blacksquare$$

Using the parameter u, β, y', we can rewrite the integral (A.2) as follows:

$$\begin{aligned}
\text{Integral (A.2)} &= \int_{I_{p-i^*+1,q-j^*}[u]>0} |I_{p-i^*+1,q-j^*}[u]|^{s+(n-1)/2} du \\
&\quad \times \int_{V^{(p,q)}_{(i^*-1,j^*)}} |\det I_{p,q}[y']|^{-(m-1)/2-s} dy' \\
&\quad \times \int_{\boldsymbol{R}^m} f(y) e[{}^t u I_{p-i^*+1,q-j^*}\beta_1] d\beta_1 d\beta_2 \\
&= \sum_{\pm} u^{\sigma(\pm)}_{(1,0)}\left(I_{p-i^*+1,q-j^*};\, s+\frac{n-1}{2}\right) \\
&\quad \times \int_{\operatorname{sgn}(I_{p-i^*+1,q-j^*}[\beta_1])=\pm} |I_{p-i^*+1,q-j^*}[\beta_1]|^{-m/2-s} d\beta_1 d\beta_2 \\
&\quad \times \int_{V^{(p,q)}_{(i^*-1,j^*)}} |\det I_{p,q}[y']|^{-m/2-s} f(y) |\det I_{p,q}[y']|^{1/2} dy',
\end{aligned}$$

where $\sigma(+)=(1,0)$ and $\sigma(-)=(0,1)$. The domain of integration can be identified with $V^{(p,q)*+}_{(i^*,j^*)}$ or $V^{(p,q)*-}_{(i^*-1,j^*+1)}$ according as $+$ or $-$. Thus we obtain

$$\begin{aligned}
\Phi_i^+(\hat{f};s) &= \sum_{i^*} u^{(i^*-1,j^*)}_{(i-1,j)}(I_{p-1,q};s)\, u^{(1,0)}_{(1,0)}\left(I_{p-i^*+1,q-j^*};\, s+\frac{n-1}{2}\right)\Phi^{*+}_{i^*}\left(f;\, -\frac{m}{2}-s\right) \\
&\quad + \sum_{i^*} u^{(i^*,j^*-1)}_{(i-1,j)}(I_{p-1,q};s)\, u^{(0,1)}_{(1,0)}\left(I_{p-i^*,q-j^*+1};\, s+\frac{n-1}{2}\right)\Phi^{*-}_{i^*}\left(f;\, -\frac{m}{2}-s\right).
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\Phi_i^-(\hat{f};s) &= \sum_{i^*} u^{(i^*-1,j^*)}_{(i,j-1)}(I_{p,q-1};s)\, u^{(1,0)}_{(0,1)}\left(I_{p-i^*+1,q-j^*};\, s+\frac{n-1}{2}\right)\Phi^{*+}_{i^*}\left(f;\, -\frac{m}{2}-s\right) \\
&\quad + \sum_{i^*} u^{(i^*,j^*-1)}_{(i,j-1)}(I_{p,q-1};s)\, u^{(0,1)}_{(0,1)}\left(I_{p-i^*,q-j^*+1};\, s+\frac{n-1}{2}\right)\Phi^{*-}_{i^*}\left(f;\, -\frac{m}{2}-s\right).
\end{aligned}$$

From these two formulas, it follows that

$$\text{(A.3)} \qquad \Phi_i(\hat{f};s) = \sum_{i^*}\left\{ u^+_{ii^*}(s)\Phi^{*+}_{i^*}\left(f;\, -\frac{m}{2}-s\right) + u^-_{ii^*}(s)\Phi^{*-}_{i^*}\left(f;\, -\frac{m}{2}-s\right)\right\},$$

where

$$\begin{aligned}
u^+_{ii^*}(s) &= u^{(i^*-1,j^*)}_{(i-1,j)}(I_{p-1,q};s)\, u^{(1,0)}_{(1,0)}\left(I_{p-i^*+1,q-j^*};\, s+\frac{n-1}{2}\right) \\
&\quad + u^{(i^*-1,j^*)}_{(i,j-1)}(I_{p,q-1};s)\, u^{(1,0)}_{(0,1)}\left(I_{p-i^*+1,q-j^*};\, s+\frac{n-1}{2}\right),
\end{aligned}$$

$$u^{-}_{ii^*}(s)=u^{(i^*,j^*-1)}_{(i-1,j)}(I_{p-1,q};\,s)u^{(0,1)}_{(1,0)}\left(I_{p-i^*,q-j^*+1};\,s+\frac{n-1}{2}\right)$$
$$+u^{(i^*,j^*-1)}_{(i,j-1)}(I_{p,q-1};\,s)u^{(0,1)}_{(0,1)}\left(I_{p-i^*,q-j^*+1};\,s+\frac{n-1}{2}\right).$$

Choose the test function f so that its support is contained in $V^{(p,q)*\pm}_{i^*,j^*}$ and compare (A.3) with (A.1). Then we have the following recursion formula.

LEMMA A.3. *We have*

$$C^{(i^*,0)}_{(i,j)}(I_{p,q};\,s)=C^{(i^*-1,0)}_{(i-1,j)}(I_{p-1,q};\,s)C^{(1,0)}_{(1,0)}\left(I_{p-i^*+1,q};\,s+\frac{n-1}{2}\right)\tag{A.4}$$
$$+C^{(i^*-1,0)}_{(i,j-1)}(I_{p,q-1};\,s)C^{(1,0)}_{(0,1)}\left(I_{p-i^*+1,q};\,s+\frac{n-1}{2}\right),$$

$$C^{(0,j^*)}_{(i,j)}(I_{p,q};\,s)=C^{(0,j^*-1)}_{(i-1,j)}(I_{p-1,q};\,s)C^{(0,1)}_{(1,0)}\left(I_{p,q-j^*+1};\,s+\frac{n-1}{2}\right)\tag{A.5}$$
$$+C^{(0,j^*-1)}_{(i,j-1)}(I_{p,q-1};\,s)C^{(0,1)}_{(0,1)}\left(I_{p,q-j^*+1};\,s+\frac{n-1}{2}\right).$$

If $i^*,j^*\neq 0$, *then*

$$C^{(i^*,j^*)}_{(i,j)}(I_{p,q};\,s)=C^{(i^*-1,j^*)}_{(i-1,j)}(I_{p-1,q};\,s)C^{(1,0)}_{(1,0)}\left(I_{p-i^*+1,q-j^*};\,s+\frac{n-1}{2}\right)\tag{A.6}$$
$$+C^{(i^*-1,j^*)}_{(i,j-1)}(I_{p,q-1};\,s)C^{(1,0)}_{(0,1)}\left(I_{p-i^*+1,q-j^*};\,s+\frac{n-1}{2}\right),$$

$$=C^{(i^*,j^*-1)}_{(i-1,j)}(I_{p-1,q};\,s)C^{(0,1)}_{(1,0)}\left(I_{p-i^*,q-j^*+1};\,s+\frac{n-1}{2}\right)\tag{A.7}$$
$$+C^{(i^*,j^*-1)}_{(i,j-1)}(I_{p,q-1};\,s)C^{(0,1)}_{(0,1)}\left(I_{p-i^*,q-j^*+1};\,s+\frac{n-1}{2}\right).$$

If $n=1$, then it is known (cf. [GS]) that

$$\begin{pmatrix} C^{(1,0)}_{(1,0)} & C^{(0,1)}_{(1,0)} \\ C^{(1,0)}_{(0,1)} & C^{(0,1)}_{(0,1)} \end{pmatrix}(I_{p,q};\,s)=\begin{pmatrix} \cos\pi(s+(q+1)/2) & \sin(\pi p/2) \\ \sin(\pi q/2) & \cos\pi(s+(p+1)/2) \end{pmatrix}.\tag{A.8}$$

We note that

$$C^{(j^*,i^*)}_{(j,i)}(I_{q,p};\,s)=C^{(i^*,j^*)}_{(i,j)}(I_{p,q};\,s)\,.\tag{A.9}$$

Using Lemma A.3 and (A.8), we prove Theorem A.1. We begin by calculating some special cases.

LEMMA A.4. *If $p \geq n$, then we have*

$$C^{(n,0)}_{(n,0)}(I_{p,q};\, s) = \prod_{\mu=1}^{n} \cos\pi\left(s + \frac{q+\mu}{2}\right).$$

PROOF. In the formula (A.4), substitute j by 0. Then the first factor of the second term vanishes. Hence

$$\begin{aligned} C^{(n,0)}_{(n,0)}(I_{p,q};\, s) &= C^{(n-1,0)}_{(n-1,0)}(I_{p-1,q};\, s)C^{(1,0)}_{(1,0)}\left(I_{p-n+1,q};\, s + \frac{n-1}{2}\right) \\ &= \prod_{\mu=1}^{n} C^{(1,0)}_{(1,0)}\left(I_{p-n+1,q};\, s + \frac{n-\mu}{2}\right). \end{aligned}$$

By (A.8), this implies the required identity. ∎

LEMMA A.5. *If $p \geq n$, then we have*

$$C^{(n,0)}_{(n-1,1)}(I_{p,q};\, s) = \begin{cases} \sin\dfrac{\pi q}{2} \displaystyle\prod_{\mu=2}^{n} \cos\pi\left(s + \frac{q+\mu}{2}\right), & \textit{if} \quad n \textit{ is odd} \\ 0 & \textit{if} \quad n \textit{ is even}. \end{cases}$$

PROOF. By repeated use of (A.4), we have

$$\begin{aligned} C^{(n,0)}_{(n-1,1)}(I_{p,q};\, s) &= C^{(n-1,0)}_{(n-2,1)}(I_{p-1,q};\, s)C^{(1,0)}_{(1,0)}\left(I_{p-n+1,q};\, s + \frac{n-1}{2}\right) + C^{(n-1,0)}_{(n-1,0)}(I_{p,q-1};\, s)\sin\frac{\pi q}{2} \\ &= \sin\frac{\pi q}{2}\sum_{k=1}^{n}\prod_{\mu=1}^{k-1} C^{(1,0)}_{(1,0)}\left(I_{p-n+1,q};\, s + \frac{n-\mu}{2}\right)C^{(n-k,0)}_{(n-k,0)}(I_{p-k+1,q-1};\, s)\,. \end{aligned}$$

Hence, by Lemma A.4 and (A.8), we obtain

$$C^{(n,0)}_{(n-1,1)}(I_{p,q};\, s) = \sin\frac{\pi q}{2}\left\{\sum_{k=1}^{n}(-1)^{n-k}\right\}\prod_{\mu=0}^{n-2}\cos\pi\left(s + \frac{n+q-\mu}{2}\right).$$

Since $\sum_{k=1}^{n}(-1)^{n-k} = 1$ or 0 according as n is odd or even, this proves the lemma. ∎

The following lemma proves the first assertion of Theorem A.1.

LEMMA A.6.

$$C^{(i^*,j^*)}_{(i,j)}(I_{p,q};\, s) = 0 \quad \textit{unless} \ |i - i^*| \leq 1\,.$$

PROOF. We prove the lemma by induction on n. The lemma is obvious for $n=1$. Assume that $n \geq 2$. First consider the case where $i^*, j^* \neq 0$. Then, by the induction hypothesis, if the right hand side of (A.6) (resp. (A.7)) does not vanish, then we have $|i-(i^*-1)| \leq 1$ (resp. $|i^*-(i-1)| \leq 1$). Since (A.6) and (A.7) give the same value, this implies that $i = i^*$. If $j^* = 0$ and $C^{(i^*,j^*)}_{(i,j)}(I_{p,q};\, s) \neq 0$, then, by (A.4) and the induction hypothesis, we have $j = 0$, 1 or 2. If $j = 2$, then

$$C^{(n,0)}_{(n-2,2)}(I_{p,q};s)=C^{(n-1,0)}_{(n-2,1)}(I_{p,q-1};s)C^{(1,0)}_{(0,1)}\left(I_{p-i^*+1,q};s+\frac{n-1}{2}\right).$$

The identity (A.8) implies that the second factor of the right hand side vanishes if q is even. On the other hand, Lemma A.5 implies that the first factor of the right hand side vanishes if q is odd. Hence we obtain $C^{(n,0)}_{(n-2,2)}(I_{p,q};s)=0$. The case $i^*=0$ reduces to the case $j^*=0$ by the identity (A.9). ∎

By the lemma above, what remains to be done is the calculation of $C^{(i^*,j^*)}_{(i,j)}$ for the cases $i^*=i+1$ and $i^*=i$. The case $i^*=i-1$ reduces to the case $i^*=i+1$ by (A.9).

LEMMA A.7.

$$C^{(i+1,j-1)}_{(i,j)}(I_{p,q};s)=\begin{cases}\sin\pi\left(\dfrac{q-j+1}{2}\right)\displaystyle\prod_{\mu=1}^{j-1}\cos\pi\left(s+\frac{p+\mu}{2}\right)\prod_{\mu=j+1}^{n}\cos\pi\left(s+\frac{q+\mu}{2}\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } i\equiv 0 \pmod 2, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } i\equiv 1 \pmod 2. \end{cases}$$

PROOF. Since the calculation for the case $j=1$ has been done in Lemma A.5, we assume here that $j>1$. In the present case the first factor of the first term of (A.7) vanishes by Lemma A.6. Hence (A.7) yields the identity

$$\begin{aligned}C^{(i+1,j-1)}_{(i,j)}(I_{p,q};s)&=C^{(i+1,j-2)}_{(i,j-1)}(I_{p,q-1};s)C^{(0,1)}_{(0,1)}\left(I_{p-i-1,q-j+2};s+\frac{n-1}{2}\right)\\&=C^{(i+1,0)}_{(i,1)}(I_{p,q-j+1};s)\prod_{\mu=1}^{j-1}C^{(0,1)}_{(0,1)}\left(I_{p-i-1,q-j+2};s+\frac{n-\mu}{2}\right).\end{aligned}$$

The identity together with Lemma A.5 and (A.8) implies the lemma. ∎

LEMMA A.8.

$$C^{(i,j)}_{(i,j)}(I_{p,q};s)=\begin{cases}(-1)^{ij}\displaystyle\prod_{\mu=1}^{i}\cos\pi\left(s+\frac{q+\mu}{2}\right)\prod_{\mu=i+1}^{n}\cos\pi\left(s+\frac{p+\mu}{2}\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q\equiv j \pmod 2, \\ (-1)^{ij}\displaystyle\prod_{\mu=1}^{j}\cos\pi\left(s+\frac{p+\mu}{2}\right)\prod_{\mu=j+1}^{n}\cos\pi\left(s+\frac{q+\mu}{2}\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q\not\equiv j \pmod 2. \end{cases}$$

PROOF. Since the calculation for the case $j=0$ has been done in Lemma A.4, we assume that $j>0$. First consider the case where $q-j$ is odd. Then, by Lemma A.7, the first factor of the first term of (A.7) vanishes; hence we have

$$C^{(i,j)}_{(i,j)}(I_{p,q};s)=C^{(i,j-1)}_{(i,j-1)}(I_{p,q-1};s)C^{(0,1)}_{(0,1)}\left(I_{p-i,q-j+1};s+\frac{n-1}{2}\right)$$
$$=C^{(i,0)}_{(i,0)}(I_{p,q-j};s)\prod_{\mu=1}^{j}C^{(0,1)}_{(0,1)}\left(I_{p-i,q-j+1};s+\frac{n-\mu}{2}\right).$$

The identity together with Lemma A.4 and (A.8) implies the lemma in the case where $q-j$ is odd. Using the recursion formula (A.6), we can prove quite similarly the lemma in the case where $q-j$ is even. ∎

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