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MAXIMAL OPERATORS ASSOCIATED WITH COMMUTATORS OF SPHERICAL MEANS

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Abstract. In this paper, we prove that L^2 boundedness for the maximal operators associated with the commutators generated by BMO functions and some multiplier operators. And we also study the L^p boundedness for the maximal operator associated with the commutators of spherical means and a function in BMO or Lipschitz space.

1. Introduction. Coifman and Meyer observed that the L^p boundedness for the commutator $[b, T]$ defined by

$$[b, T]f(x)=b(x)Tf(x)-T(bf)(x)$$

could be obtained from the weighted L^p estimate for T with A_p weight when $b\in\mathrm{BMO}$ and T is a standard Calderón-Zygmund singular integral operator (see [4]), where A_p is the weight function class of Muckenhoupt (see [14, chapter V] for the definition and properties of A_p). In 1993, Alvarez, Babgy, Kurtz and Pérez [1] developed the idea of Coifman and Meyer, and established a general boundedness criterion for the commutators of linear operators. Their result can be stated as follows.

THEOREM A. *Let E be a Banach space, $1<p, q<\infty$. Suppose that the linear operator T: $C_0^\infty(\boldsymbol{R}^n)\to M(E)$ satisfies the weight estimates*

$$\|Tf\|_{L^p_w(E)}\le\bar{C}\|f\|_{p,w}$$

for all $w\in A_q$ and $\bar{C}$ depends only on n, p and $\tilde{C}_q(w)$ (the A_q constant of w), but not on the weight w. Then for any positive integer k and $b(x)\in\mathrm{BMO}(\boldsymbol{R}^n)$, the commutator

$$T_{b,k}f(x)=T((b(x)-b(\cdot))^kf)(x)$$

is bounded from $L^p_u(\boldsymbol{R}^n)$ to $L^p_u(E)$ for all $u\in A_q$ with norm $C(p, n, k, \tilde{C}_q(u))\|b\|^k_{\mathrm{BMO}}$.

This result is of great importance and is suitable for many classical operators in harmonic analysis. But for some important operators, the criterion of Alvarez-Babgy-Kurtz-Pérez breaks down. Let us consider the maximal operator of the spherical means defined by

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$$M_{*}f(x)=\sup_{t>0}|M_{t}f(x)| \qquad \text{for } \ f\in\mathscr{S} \tag{1.1}$$

with

$$M_{t}f(x)=\int_{S^{n-1}}f(x-ty')dy', \tag{1.2}$$

where S^{n-1} is the unit sphere in $\boldsymbol{R}^n$ and dy' is the rotationally invariant measure of total mass 1 on the unit sphere. This operator M_{*}, which is studied by Stein in [12], is of interest by itself and is very useful in the study of partial differential equations. In [12], Stein showed that the operator M_{*} is bounded on L^p provided that $n\geq 3$ and $p>n/(n-1)$. We do not know whether the operator M_{*} enjoys weighted L^p estimates with general A_q weights for some $q>1$. Thus Theorem A seems not to be well adapted to this operator.

Meanwhile, let $m\in L^{\infty}(\boldsymbol{R}^n)$ be a multiplier. Define the operator $\{T^t\}_{t>0}$ by

$$(T^{t}f)^{\wedge}(\xi)=m(t\xi)\hat{f}(\xi), \qquad f\in\mathscr{S} \tag{1.3}$$

and the associated maximal operator by

$$T^{*}f(x)=\sup_{t>0}|T^{t}f(x)|, \tag{1.4}$$

where $\hat{f}$ denotes the Fourier transform of f. It is well-known that the operator T^* plays a fundamental role in the study of the pointwise convergence of the averages along hypersurfaces (see [10] and [11]). A result of Rubio de Francia [10], Sogge and Stein [11] states that if $m\in C^{\infty}(\boldsymbol{R}^n)$ and

$$|m(\xi)|\leq C|\xi|^{-a_1}, \quad |\nabla m(\xi)|\leq C|\xi|^{-a_2} \tag{1.5}$$

for some positive constants C and a_1, a_2 with $a_1+a_2>1$, then T^* is bounded on $L^2(\boldsymbol{R}^n)$. If the multiplier m satisfies only the decay estimate (1.5), we do not know any weighted L^2 estimate with general A_q $(q>1)$ weights for T^*. Thus in this case the boundedness criterion for the commutators of linear operators does not apply to obtaining the L^2 boundedness of the maximal operator associated with commutators of T^t.

The purpose of this paper is to consider the L^p boundedness for the maximal operator associated to the commutator of the spherical means. Let k be a positive integer. For a function b in BMO, the k-th order commutators of spherical means, $M_{t;b,k}$ are defined to be

$$M_{t;b,k}f(x)=\int_{S^{n-1}}(b(x)-b(x-ty'))^{k}f(x-ty')dy' \tag{1.6}$$

and the maximal operator associated with them is defined by $M_{*;b,k}$,

$$M_{*;b,k}f(x)=\sup_{t>0}|M_{t;b,k}f(x)|. \tag{1.7}$$

We also consider the commutator generated by M_t and b in $\dot{\Lambda}_\beta$, the Lipschitz space. Denote by Δ_h^k the k-th difference operator, that is

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x)$$
$$\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x), \qquad k \geq 1 .$$

For $\beta > 0$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h \in \boldsymbol{R}^n, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty .$$

For b in $\dot{\Lambda}_\beta$, $0 < \beta < k \leq n/2$, as in [9], the k-th order commutator of spherical means, denoted by $\tilde{M}_{t;b,k}$, is defined by

$$\tilde{M}_{t;b,k} f(x) = \int_{S^{n-1}} \Delta_{ty'/k}^k b(x) f(x - ty') dy' \tag{1.8}$$

and $\tilde{M}_{*;b,k}$ is the maximal operator associated with $\tilde{M}_{t;b,k}$.

We will consider a general result for L^2 boundedness. Let $m \in L^\infty(\boldsymbol{R}^n)$ and the operators $\{T^t\}_{t>0}$ be as in (1.3). For a positive integer k and $b \in \mathrm{BMO}(\boldsymbol{R}^n)$. Define the k-th order commutator of T^t by

$$T_{b,k}^t f(x) = T^t((b(x) - b(\cdot))^k f)(x), \qquad f \in \mathcal{S} . \tag{1.9}$$

The maximal operator associated with $\{T_{b,k}^t\}_{t>0}$ is defined by

$$T_{b,k}^* f(x) = \sup_{t>0} |T_{b,k}^t f(x)| . \tag{1.10}$$

Now we state our main results in this paper.

THEOREM 1. *Let k, j $(j \geq 2)$ be positive integers and $b \in \mathrm{BMO}(\boldsymbol{R}^n)$. Suppose that the multiplier $m \in C^\infty(\boldsymbol{R}^n)$ enjoys the property* (1.5) *and*

$$\sum_{|\alpha|=j} |D^\alpha m(\xi)| \leq C(1 + |\xi|)^N ,$$

for some positive constants C and N. Then $T_{b,k}^*$ is bounded on $L^2(\boldsymbol{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}}^k$.

THEOREM 2. *Let k be a positive integer and b in $\mathrm{BMO}(\boldsymbol{R}^n)$. If $n \geq 3$ and $n/(n-1) < p < \infty$, then $M_{*;b,k}$ is bounded on L^p with norm $C\|b\|_{\mathrm{BMO}}^k$.*

THEOREM 3. *Let k be a positive integer. Suppose b in $\dot{\Lambda}_\beta$ with $0 < \beta < k \leq (n-2)/2$. Then $\tilde{M}_{*;b,k}$ is bounded from L^p into L^q with $1/q = 1/p - \beta/n$ provided that $n \geq 3$ and $n/(n-1) < p < n/\beta - n^2/((n-1)\beta(n-2\beta))$.*

The paper is arranged as follows. We give the proof of Theorem 1 and Theorem 2 in Section 2. In Section 3, we prove Theorem 3.

2. Estimates for commutators generated by a BMO function. In this section, we give the estimates for L^2 boundedness of the operator $T^*_{b,k}$. We begin with some preliminary lemmas.

LEMMA 2.1 (see [5]). *Let k be a positive integer and $b \in \mathrm{BMO}(\boldsymbol{R}^n)$. Denote by $M_{b,k}$ the k-th order commutator of the Hardy-Littlewood maximal operator, that is,*

$$M_{b,k}f(x) = \sup_{r>0} r^{-n} \int_{|x-y|<r} |b(x)-b(y)|^k |f(y)| dy .$$

Then for all $1<p<\infty$, $M_{b,k}$ is bounded on $L^p(\boldsymbol{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}}$.

LEMMA 2.2. *Let $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ be a radial function such that* $\operatorname{supp}\varphi \subset \{1/4 \le |x| \le 4\}$ *and*

$$\sum_{l\in\boldsymbol{Z}} \varphi(2^{-l}x) = 1 , \qquad |x|>0 .$$

Denote by g_l the multiplier operator

$$(g_l f)^\wedge(\xi) = \varphi(2^{-l}\xi)\hat{f}(\xi) .$$

Then for any positive integer k and $b \in \mathrm{BMO}(\boldsymbol{R}^n)$, the k-th order commutator of g_l defined by

$$g_{l;b,k}f(x) = g_l((b(x)-b(\cdot))^k f)(x)$$

satisfies

$$\left\| \left(\sum_{l\in\boldsymbol{Z}} |g_{l;b,k}f|^2 \right)^{1/2} \right\|_p \le C\|b\|^k_{\mathrm{BMO}} \|f\|_p$$

for all $1<p<\infty$.

PROOF. Let $1<p<\infty$ and $w \in A_p$. The weighted Littlewood-Paley theory (see [4]) shows that the estimate

$$\left\| \left(\sum_{l\in\boldsymbol{Z}} |g_l f|^2 \right)^{1/2} \right\|_{p,w} \le C\|f\|_{p,w}$$

holds for some constant C independent of w. Note that the mapping

$$f \to \{g_l f\}_{l\in\boldsymbol{Z}}$$

is linear, the boundedness criterion for the commutators of linear operators of Alvarez-Babgy-Kurtz-Pérez (see [1, Theorem 2.13]) yields the desired estimate.

LEMMA 2.3. *Let $1 \le \delta < \infty$, j be a positive integer, c and N be real numbers. Suppose that $m_\delta \in C^j(\boldsymbol{R}^n)$ is a multiplier such that* $\operatorname{supp} m_\delta \subset \{\delta/2 \le |x| \le 2\delta\}$ *and*

$$\|m_\delta\|_\infty \le C\delta^c , \quad \sum_{|\alpha|=j} \|D^\alpha m_\delta\|_\infty \le C\delta^N$$

for some positive constant C which is independent of δ. *Let* T_δ^t *be the multiplier operator defined by*

$$(T_\delta^t f)^\wedge(\xi) = m_\delta(t\xi)\hat{f}(\xi)\,.$$

For a positive integer k and $b \in \mathrm{BMO}(\boldsymbol{R}^n)$, *denote by* $T_{\delta;b,k}^t$ *the k-th order commutator of* T_δ^t, *which is defined as in* (1.9). *Then for any* $\varepsilon > 0$, *there exists a positive constant* $C = C(n, k, c, \varepsilon, N)$ *such that*

$$\int_1^2 \int_{\boldsymbol{R}^n} |T_{\delta;b,k}^t f(x)|^2 dx \frac{dt}{t} \le C\delta^{2(c+\varepsilon)} \|b\|_{\mathrm{BMO}}^{2k} \|f\|_2^2\,.$$

PROOF. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}} = 1$. Obviously, it suffices to show that

$$\|T_{\delta;b,k}^1 f\|_2 \le C\delta^{c+\varepsilon}\|f\|_2\,.$$

Let ψ_0, ψ be radial functions such that

$$\operatorname{supp}\psi \subset \{1/4 \le |x| \le 4\}$$

and

$$\psi_0(x) + \sum_{l=1}^{\infty} \psi(2^{-l}x) = 1\,, \qquad \text{if} \quad |x| > 0\,.$$

Set $\psi_l(x) = \psi(2^{-l}x)$ for $l \ge 1$ and $K_\delta(x) = m_\delta^\vee(x)$, the inverse Fourier transform of m_δ. Split K_δ as

$$K_\delta(x) = K_\delta(x)\psi_0(x) + \sum_{l=1}^{\infty} K_\delta(x)\psi_l(x) = \sum_{l=0}^{\infty} K_\delta^l(x)\,.$$

Recall that $1 \le \delta < \infty$ and $\operatorname{supp} m_\delta \subset \{\delta/2 \le |x| \le 2\delta\}$. A straightforward computation shows that

$$\|K_\delta^l\|_\infty \le C\|K_\delta\|_\infty \le C\delta^{n+c}\,.$$

Let $T_\delta^{1,l}$ be the convolution operator whose kernel is K_δ^l. Young's inequality now says that

$$\|T_\delta^{1,l} f\|_\infty \le C\delta^{n+c}\|f\|_1\,. \tag{2.1}$$

Write

$$(K_\delta^l)^\wedge(\xi) = \int_{\boldsymbol{R}^n} m_\delta(\xi - 2^{-l}\eta)\hat{\psi}(\eta)d\eta\,.$$

Since ψ is null in a neighborhood of the origin and a Schwarz function, we have

$$\int_{\boldsymbol{R}^n} \eta^\alpha \hat{\psi}(\eta)d\eta = 0$$

for any multi-index α, and

$$\int_{\boldsymbol{R}^n} |\eta|^j |\hat{\psi}(\eta)| d\eta < \infty .$$

Expanding m_δ into a Tayloy series around ξ gives

$$|(K_\delta^l)^\wedge(\xi)| \le \sum_{|\alpha|=j} \|D^\alpha m_\delta\|_\infty 2^{-jl} \int_{\boldsymbol{R}^n} |\eta|^j |\hat{\psi}(\eta)| d\eta \le C2^{-l}\delta^N .$$

Thus,

$$\|T_\delta^{1,l} f\|_2 \le C2^{-l}\delta^N \|f\|_2 . \tag{2.2}$$

On the other hand, another application of Young's inequality gives that

$$\|(K_\delta^l)^\wedge\|_\infty \le \|(K_\delta)^\wedge\|_\infty \|\hat{\psi}_l\|_1 \le C\delta^c ,$$

which in turn implies

$$\|T_\delta^{1,l} f\|_2 \le C\delta^c \|f\|_2 . \tag{2.3}$$

Therefore, for each fixed v, $0<v<1$,

$$\|T_\delta^{1,l} f\|_2 \le C\delta^{c+v(N-c)} 2^{-vl} \|f\|_2 . \tag{2.4}$$

Interpolation between the inequalities (2.1) and (2.4) tells us that for each q with $2 \le q < \infty$,

$$\|T_\delta^{1,l} f\|_q \le C2^{-2vl/q} \delta^{n+c+[v(N-c)-n]2/q} \|f\|_{q'} , \tag{2.5}$$

where q' is the dual exponent of q, i.e., $q' = q/(q-1)$.

Now we turn our attention to $T_{\delta;b,k}^{1,l}$, the k-th order commutator of the operator $T_\delta^{1,l}$. We decompose $\boldsymbol{R}^n$ into a grid of non-overlapping cubes with side length 2^l, i.e., $\boldsymbol{R}^n = \bigcup_i Q_i$. Denote by χ_{Q_i} the characteristic function of Q_i. Set $f_i = f\chi_{Q_i}$. Then

$$f(x) = \sum_i f_i(x) , \quad \text{a.e. } x \in \boldsymbol{R}^n .$$

Since $\operatorname{supp} K_\delta^l \subset \{|x| \le C2^l\}$, it is obvious that the support of $T_\delta^{1,l} f_i$ is contained in a fixed multiple of Q_i, and that the supports of various terms $T_{\delta;b,k}^{1,l} f_i$ have bounded overlaps. So we have the following almost orthogonality property:

$$\|T_{\delta;b,k}^{1,l} f\|_2^2 \le C \sum_i \|T_{\delta;b,k}^{1,l} f_i\|_2^2 .$$

Thus we may assume that $\operatorname{supp} f \subset Q$ for some cube Q with side length 2^l. Choose $\phi \in C_0^\infty(\boldsymbol{R}^n)$, $0 \le \phi \le 1$, ϕ is identically one on $50nQ$ and vanishes outside $100nQ$. Set $\tilde{Q} = 200nQ$, and $\tilde{b} = (b(x) - b_{\tilde{Q}})\phi(x)$, where $b_{\tilde{Q}}$ is the mean value of b on $\tilde{Q}$. Let $2 < q_1$, $q_2 < \infty$ such that $1/q_1 + 1/q_2 = 1/2$. By Hölder's inequality and (2.5), we deduce

$$\begin{aligned}\|\tilde{b}^m T_\delta^{1,l}(\tilde{b}^{k-m}f)\|_2 &\le \|\tilde{b}^m\|_{q_1}\|T_\delta^{1,l}(\tilde{b}^{k-m}f)\|_{q_2}\\ &\le C2^{-2vl/q_2}\delta^{n+c+[v(N-c)-n]2/q_2}\|\tilde{b}^m\|_{q_1}\|\tilde{b}^{k-m}f\|_{q_2'}\\ &\le C2^{-2vl/q_2}\delta^{n+c+[v(N-c)-n]2/q_2}\|\tilde{b}^m\|_{q_1}\|\tilde{b}^{k-m}\|_{2q_2/(q_2-2)}\|f\|_2\\ &\le C2^{-2vl/q_2}\delta^{n+c+[v(N-c)-n]2/q_2}2^{ln(1-2/q_2)}\|f\|_2\,,\end{aligned}$$

where in the last inequality we have invoked the fact

$$\|\tilde{b}^m\|_{q_1}\le C\|b\|_{\mathrm{BMO}}^m|Q|^{1/q_1}\,.$$

For each fixed $\varepsilon>0$, we choose q_2 larger than and sufficiently close to 2, v larger than zero but sufficiently close to zero so that

$$2v/q_2>n(1-2/q_2)\,,\quad n+[v(N-c)-n]2/q_2<\varepsilon\,.$$

We then have that for some positive constant γ,

$$\|\tilde{b}^m T_\delta^{1,l}(\tilde{b}^{k-m}f)\|_2\le C2^{-\gamma l}\delta^{c+\varepsilon}\|f\|_2\,.$$

Observing that

$$|T_{\delta;b,k}^{1,l}f(x)|\le\sum_{m=0}^{k}C_k^m|\tilde{b}^m(x)T_\delta^{1,l}(\tilde{b}^{k-m}f)(x)|\,,$$

we have

$$\|T_{\delta;b,k}^{1,l}f\|_2\le C2^{-\gamma l}\delta^{c+\varepsilon}\|f\|_2\,.$$

Summing over the last inequality for all $l\ge 0$ then completes the proof of Lemma 2.3.

PROOF OF THEOREM 1. As in the proof of Lemma 2.3, we may assume that $\|b\|_{\mathrm{BMO}}=1$. Let ψ_0, ψ be the same as in the proof of Lemma 2.3. Decompose the multiplier m as

$$m(\xi)=m(\xi)\psi_0(\xi)+\sum_{l=1}^{\infty}m(\xi)\psi(2^{-l}\xi)=\sum_{l=0}^{\infty}m_l(\xi)\,.$$

Define the operator T_l^t by

$$(T_l^t f)^\wedge(\xi)=m_l(t\xi)\hat{f}(\xi)\,.$$

Let $T_{l;b,k}^t$ be the k-th order commutator of T_l^t defined analogously to (1.9) and let $T_{l;b,k}^*$ be the maximal operator associated with $T_{l;b,k}^t$ as in (1.10). Then

$$T_{b,k}^*f(x)\le\sum_{l=0}^{\infty}T_{l;b,k}^*f(x)\,.$$

Since $m_0\in C_0^\infty(\boldsymbol{R}^n)$, a trivial computation shows that

$$T_{0;b,k}^*f(x)\le CM_{b,k}f(x)\,,$$

with $M_{b,k}$ the k-th order commutator of the Hardy-Littlewood maximal operator (see

Lemma 2.1). Thus by Lemma 2.1 we need only to care about $T^{*}_{l;b,k}$ for $l \geq 1$. Let $\tilde{m}_l(\xi) = \nabla m_l(\xi) \cdot \xi$. Define the operator $\tilde{T}^t_l$ by

$$(\tilde{T}^t_l f)^\wedge(\xi) = \tilde{m}_l(t\xi)\hat{f}(\xi)\,.$$

We introduce the quadratic operators

$$G_l f(x) = \left(\int_0^\infty |T^t_{l;b,k} f(x)|^2 \frac{dt}{t} \right)^{1/2}$$

and

$$\tilde{G}_l f(x) = \left(\int_0^\infty |\tilde{T}^t_{l;b,k} f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

As in [10, page 308], it is easy to check that

$$|T^{*}_{l;b,k} f(x)|^2 \leq 2 G_l f(x) \tilde{G}_l f(x)\,.$$

We now estimate $\|G_l f\|_2$. We claim that for each fixed $\varepsilon > 0$,

$$\|G_l f\|_2 \leq C(n, k, \varepsilon, a_1) 2^{-l(a_1 - \varepsilon)} \|f\|_2\,. \tag{2.6}$$

Indeed, by (1.5) we see that m_l is supported in the spherical shell $2^{l-1} \leq |\xi| \leq 2^{l+1}$ and $\|m_l\|_\infty \leq C 2^{-la_1}$, $\|\nabla m_l\|_\infty \leq C(2^{-la_2} + 2^{-l(a_1+1)})$. Thus by Lemma 2.3, we see that for each fixed $\varepsilon > 0$ and non-negative integer k, there exists a positive constant $C = C(n, k, \varepsilon, a_1, a_2)$ such that

$$\int_{\boldsymbol{R}^n} \int_1^2 |T^t_{l;b,k} f(x)|^2 \frac{dt}{t}\, dx \leq C 2^{-2l(a_1 - \varepsilon)} \|f\|_2^2\,. \tag{2.7}$$

Observe that if $b \in \mathrm{BMO}(\boldsymbol{R}^n)$, then for any $t > 0$, $b_t(x) = b(tx)$ also belongs to $\mathrm{BMO}(\boldsymbol{R}^n)$ and $\|b_t\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}}$. By dilation-invariance, it follows from (2.7) that for any $d \in \boldsymbol{Z}$,

$$\int_{\boldsymbol{R}^n} \int_{2^{-d}}^{2^{-d+1}} |T^t_{l;b,k} f(x)|^2 \frac{dt}{t}\, dx \leq C 2^{-2l(a_1 - \varepsilon)} \|f\|_2^2\,. \tag{2.8}$$

Let $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ as in Lemma 2.2. Set

$$T^{d,t}_{l;b,k} f(x) = \int_{\boldsymbol{R}^n} (\varphi(2^{-d-l} \cdot) m_l(t \cdot))^\vee (x-y)(b(x) - b(y))^k f(y) dy\,.$$

Then

$$\begin{aligned} T^t_{l;b,k} f(x) &= \sum_{d \in \boldsymbol{Z}} \int_{\boldsymbol{R}^n} (\varphi(2^{-d-l} \cdot) m_l(t \cdot))^\vee (x-y)(b(x) - b(y))^k f(y) dy \\ &= \sum_{d \in \boldsymbol{Z}} T^{d,t}_{l;b,k} f(x)\,. \end{aligned}$$

With the aid of the formula

$$(b(x)-b(y))^k=\sum_{i=0}^{k} C_k^i(b(x)-b(z))^i(b(z)-b(y))^{k-i}, \qquad z\in \boldsymbol{R}^n,$$

we have

$$T_{l;b,k}^{d,t}f(x)=\sum_{i=0}^{k} C_k^i T_{l;b,i}^t(g_{l+d;b,k-i}f)(x),$$

where g_d is the multiplier operator associated with $\varphi(2^{-d}\cdot)$ defined in Lemma 2.2. Note that for each fixed t and l, the number of d's for which $\operatorname{supp}\varphi(2^{-d-l}\cdot)\cap\operatorname{supp} m_l(t\cdot)$ is non-empty is at most 100. Hence,

$$\int_0^\infty |T_{l;b,k}^t f(x)|^2\frac{dt}{t}\le C\sum_{d\in \boldsymbol{Z}}\int_0^\infty |T_{l;b,k}^{d,t}f(x)|^2\frac{dt}{t}\le C\sum_{d\in \boldsymbol{Z}}\int_{2^{-d}}^{2^{-d+1}} |T_{l;b,k}^{d,t}f(x)|^2\frac{dt}{t}$$

$$\le C\sum_{i=0}^{k}\sum_{d\in \boldsymbol{Z}}\int_{2^{-d}}^{2^{-d+1}} |T_{l;b,i}^t(g_{l+d;b,k-i}f)(x)|^2\frac{dt}{t}.$$

By the inequality (2.8) and Lemma 2.2, we finally obtain

$$\|G_lf\|_2^2\le C2^{-2l(a_1-\varepsilon)}\sum_{i=0}^{k}\sum_{d\in \boldsymbol{Z}}\|g_{l+d;b,k-i}f\|_2^2\le C2^{-2l(a_1-\varepsilon)}\|f\|_2^2,$$

which establishes our assertion.

The L^2 boundedness of $T_{b,k}^*$ follows immediately. Indeed, without loss of generality, one may assume that $a_1\ge a_2-1$; otherwise, if $a_1<a_2-1$ and $a_1+a_2>1$, then $a_2>1$ so that $\lim_{|\xi|\to\infty} m(\xi)=\alpha$ exists and

$$|m(\xi)-\alpha|\le C|\xi|^{-a_2+1}.$$

Thus we may replace $m(\xi)$ by $m(\xi)-\alpha$ and a_1 by a_2-1. As in the proof of (2.7), we have that for each given $\mu>0$, there exists a positive constant $C=C(n,k,\mu,a_2,N)$ such that

$$\|\tilde{G}_lf\|_2\le C2^{-l(a_2-1-\mu)}\|f\|_2.$$

So

$$\|T_{l;b,k}^*f\|_2\le C\|G_lf\|_2^{1/2}\|\tilde{G}_lf\|_2^{1/2}\le C2^{-l(a_1+a_2-1-\mu-\varepsilon)/2}\|f\|_2.$$

For each fixed pair a_1 and a_2 with $a_1+a_2>1$, we can choose positive numbers ε, μ so small that $\varepsilon+\mu<a_1+a_2-1$. Then for some positive constant θ independent of l,

$$\|T_{l;b,k}^*f\|_2\le C2^{-\theta l}\|f\|_2.$$

This leads to the conclusion of our Theorem 1.

Now we turn our attention to the proof of Theorem 2. Let us introduce additional operators M_t^α, which is defined by

$$(M_t^\alpha f)^\wedge(\xi)=m_\alpha(t\xi)\hat{f}(\xi),$$

for $f\in\mathscr{S}$, where

$$m_\alpha(\xi)=2^{n/2+\alpha-1}\Gamma\left(\frac{n}{2}+\alpha\right)(2\pi|\xi|)^{-n/2-\alpha+1}J_{n/2+\alpha-1}(2\pi|\xi|)\,. \tag{2.9}$$

For a complex number α, put

$$M_{t;b,k}^\alpha f(x)=M_t^\alpha((b(x)-b(\cdot))^k f)(x)$$

and

$$M_{*;b,k}^\alpha f(x)=\sup_{t>0}|M_{t;b,k}^\alpha f(x)|\,.$$

In view of the method of the proof in [12], the conclusion of Theorem 2 can be deduced from the following results.

LEMMA 2.4. *If* $\operatorname{Re}\alpha>1-n/2$, *then*

$$\|M_{*;b,k}^\alpha f\|_2\le C_1 e^{C_1|\operatorname{Im}\alpha|}\|b\|_{\mathrm{BMO}}^k\|f\|_2\,, \tag{3.2}$$

where C_1 *is a bounded constant when* $\operatorname{Re}\alpha$ *is in any compact subinterval of* $(1-n/2,\infty)$.

By the asymptotic property of the Bessel function J_ν, Lemma 2.4 is a consequence of Theorem 1 with $a_1=n/2+\operatorname{Re}\alpha-1/2$ and $a_2=n/2+\operatorname{Re}\alpha-1/2$. Now we turn to give the estimates for $M_{*;b,k}^\alpha$ on L^p.

THEOREM 2.5. *Let* f *be in* $\mathscr{S}$. *The inequality*

$$\|M_{*;b,k}^\alpha f\|_p\le C_\alpha\|b\|_{\mathrm{BMO}}^k\|f\|_p$$

holds provided that

(a) $1<p\le 2$, *when* $\alpha>1-n+n/p$

(b) $2\le p<\infty$, *when* $\alpha>(2-n)/p$.

If $\alpha=0$, *this means* $n\ge 3$ *and* $n/(n-1)<p<\infty$.

PROOF. If $\operatorname{Re}\alpha\ge 1$, then $M_*^\alpha f(x)\le C\mathrm{HL}f(x)$, where $\mathrm{HL}f$ is the Hardy-Littlewood maximal function of f. By Lemma 2.1, we see that

$$\|M_{*;b,k}^\alpha f\|_p\le C\|b\|_{\mathrm{BMO}}^k\|f\|_p$$

for all $1<p\le 2$. For the case of $2\le p<\infty$, we claim that if $\operatorname{Re}\alpha>0$, then for p large enough,

$$\|M_{*;b,k}^\alpha f\|_p\le C\|b\|_{\mathrm{BMO}}^k\|f\|_p\,. \tag{2.11}$$

Indeed, since

$$\begin{aligned} M^{\alpha}_{*;b,k}f(x) &= \sup_{t>0} t^{-n}\left|\int_{|x-y|<t}\left(1-\frac{|x-y|^2}{t^2}\right)^{\alpha-1}(b(x)-b(y))^k f(y)dy\right| \\ &\le \left(\sup_{t>0} t^{-n}\int_{|x-y|<t} |b(x)-b(y)|^{pk}|f(y)|dy\right)^{1/p} \\ &\quad\times\left(\sup_{t>0} t^{-n}\int_{|x-y|<t}\left(1-\frac{|x-y|^2}{t^2}\right)^{(\mathrm{Re}\,\alpha-1)p'}|f(y)|dy\right)^{1/p'} \\ &:= \mathrm{I}_1^{1/p}\mathrm{I}_2^{1/p'}, \end{aligned}$$

and I_1 which is the commutator of Hardy-Littelwood maximal operator is bounded on L^p with $1<p<\infty$ (see Lemma 2.1), it is sufficient to consider the operator

$$\sup_{t>0} t^{-n}\left|\int_{|x-y|<t}\left(1-\frac{|x-y|^2}{t^2}\right)^{\beta-1} f(y)dy\right|$$

for $f\ge 0$ and $\beta\in \boldsymbol{R}$. It is well-known by Stein in [12] that this operator is bounded on L^p when $\beta\ge(2-n)/p$ with $2\le p<\infty$. Choosing p so large that $(\mathrm{Re}\,\alpha-1)p'+1>(2-n)/p$, i.e., $p>(-(n-3)+\sqrt{(n-3)^2+4\,\mathrm{Re}\,\alpha(n-2)})/2\mathrm{Re}\,\alpha$, we conclude that I_2 is bounded on L^p. Since

$$\begin{aligned} \int_{\boldsymbol{R}^n}(\mathrm{I}_1^{1/p}\mathrm{I}_2^{1/p'})^p dx &\le \left(\int_{\boldsymbol{R}^n}\mathrm{I}_1^p dx\right)^{1/p}\left(\int_{\boldsymbol{R}^n}\mathrm{I}_2^p dx\right)^{1/p'} \\ &\le C\|b\|^{kp}_{\mathrm{BMO}}\|f\|^p_p, \end{aligned}$$

(2.11) holds and the conclusion of Theorem 2.5 follows from the complex interpolation theorem (see [15]).

3. Estimates for commutators generated by a Lipschitz function. We first consider a maximal operator N^{β}_{*} defined by

$$N^{\beta}_{*}f(x)=\sup_{t>0} t^{\beta}\left|\int_{|y'|=1} f(x-ty')d\sigma(y')\right|,$$

with $0<\beta<(n-2)/2$. The maximal operator is interesting by itself. With the notation M_t and M^{α}_t the same as in the previous section, we can rewrite N^{β}_{*} as

$$N^{\beta}_{*}f(x)=\sup_{t>0} t^{\beta}|M_t f(x)|.$$

Let $N^{\alpha,\beta}_{*}f(x)=\sup_{t>0} t^{\beta}|M^{\alpha}_t f(x)|$. The estimates for N^{β}_{*} follows that of $N^{\alpha,\beta}_{*}$ at $\alpha=0$.

THEOREM 3.1. *Suppose* $0<\beta<(n-2)/2$ *and* $\mathrm{Re}\,\alpha>1+\beta-n/2$. *Let* f *be in* $\mathscr{S}$. *The following inequality*

$$\|N^{\alpha,\beta}_{*}f\|_2\le Ce^{C|\mathrm{Im}\,\alpha|}\|f\|_{2n/(n+2\beta)} \tag{3.1}$$

holds with the constant C depending on n, β and $\operatorname{Re}\alpha$, *which is bounded when* $\operatorname{Re}\alpha$ *is in a subinterval of* $(1+\beta-n/2, \infty)$.

To prove Theorem 3.1, write $\mathscr{M}^{\alpha,\beta}f(x)=\sup_{t>0}\{t^{-1}\int_0^t|s^\beta M_s^\alpha f(x)|^2ds\}^{1/2}$. Assuming that $\operatorname{Re}\alpha>\operatorname{Re}\alpha'>-n/2$ and $C_{n,\alpha}=2\Gamma(n/2+\alpha)/\Gamma(\alpha-\alpha')\Gamma(n/2+\alpha')$, by the formula in [12, p. 2174],

$$t^\beta M_t^\alpha f(x)=C_{n,\alpha}\int_0^1(ts)^\beta M_{st}^{\alpha'}f(x)(1-s^2)^{\alpha-\alpha'-1}s^{n+2\alpha'-\beta-1}ds\,. \tag{3.2}$$

Hence, if $\operatorname{Re}\alpha>\operatorname{Re}\alpha'+1/2$ and $\operatorname{Re}\alpha'>\beta/2-n/2+1/4$, then an application of Schwarz inequality shows that $N_*^{\alpha,\beta}f(x)\le C_{n,\alpha}\mathscr{M}^{\alpha',\beta}f(x)$, and (3.1) is a consequence of the following result for $\mathscr{M}^{\alpha',\beta}$.

LEMMA 3.2. *Suppose that f is in $\mathscr{S}$ and $0<\beta<(n-2)/2$. If* $\operatorname{Re}\alpha>1/2+\beta-n/2$, *then*

$$\|\mathscr{M}^{\alpha,\beta}f\|_2\le Ce^{C|\operatorname{Im}\alpha|}\|f\|_{2n/(n+2\beta)}\,, \tag{3.3}$$

where C is a constant depending on n, $\operatorname{Re}\alpha$, *and β.*

PROOF. Since

$$\begin{aligned}(t^\beta M_t^\alpha f)^\wedge(\xi)&=t^\beta m^\alpha(t|\xi|)\hat{f}(\xi)\\&=(t|\xi|)^\beta m^\alpha(t|\xi|)(I_\beta f)^\wedge(\xi)\\&=(W_t^{\alpha,\beta}*I_\beta f)^\wedge(\xi)\,,\end{aligned} \tag{3.4}$$

where $(W^{\alpha,\beta})^\wedge(\xi)=|\xi|^\beta m^\alpha(|\xi|)$ and I_β is the Riesz potential operator. By the boundedness of I_β, for the inequality (3.3), it is sufficient to show that if $\operatorname{Re}\alpha>1/2+\beta-n/2$, then for $f\in\mathscr{S}$

$$\left\|\left(\sup_{t>0}\frac{1}{t}\int_0^t|W^{\alpha,\beta}*f|^2ds\right)^{1/2}\right\|_2\le C\|f\|_2\,. \tag{3.5}$$

Obviously, (3.5) follows from the estimate

$$\left\|\left(\int_0^\infty|W_t^{\alpha,\beta}*f|^2\frac{dt}{t}\right)^{1/2}\right\|_2\le C\|f\|_2\,. \tag{3.6}$$

We claim that (3.6) holds with the assumptions in Lemma 3.2. Indeed, by Parseval's theorem, the proof of (3.6) comes down to the estimate

$$\int_0^\infty|(t|\xi|)^\beta m^\alpha(t\xi)|^2\frac{dt}{t}\le C \tag{3.7}$$

for $|\xi|=1$. Since $m^\alpha(0)=1$ and $\beta>0$, the portion of the integral $t\le1$ in (3.7) is easily seen to be bounded. To deal with the contribution for large t, we note

$$(t|\xi|)^{\beta}M^{\alpha}(t|\xi|)\leq C_{\alpha}t^{-n/2-\mathrm{Re}\,\alpha+1/2+\beta}\,.$$

If $\mathrm{Re}\,\alpha>1/2+\beta-n/2$, then the integral (3.7) is bounded. This completes the proof of Lemma 3.2.

Then estimate for $N_{*}^{\alpha,\beta}$ on L^{p} is the following statement.

THEOREM 3.3. *Suppose* $0<\beta<(n-2)/2$ *and* f *is in* $\mathcal{S}$. *The inequality*

$$\|N_{*}^{\alpha,\beta}f\|_{q}\leq C\|F\|_{p}$$

holds with $1/q=1/p-\beta/n$ *in the following circumstances*:

(a) $1<p\leq 2n/(n+2\beta)$, *when* $\mathrm{Re}\,\alpha>1-n+n/p$.

(b) $2n/(n+2\beta)<p<n/\beta$, *when*

$$\mathrm{Re}\,\alpha>(2-n)/p+2(n-1)\beta/np+(n-1)\beta/n-2(n-1)\beta^{2}/n^{2}\,.$$

If $\alpha=0$, *this means* $n\geq 3$, $n/(n-1)<p<n/\beta-n^{2}/(n-1)\beta(n-2\beta)$.

PROOF. If $\mathrm{Re}\,\alpha\geq 1$, by the definition of M_{t}^{α} in Section 2, we have

$$\begin{aligned}N_{*}^{\alpha,\beta}f(x)&=C\sup_{t>0}t^{-n+\beta}\left|\int_{|y|<t}(1-|y|^{2}/t^{2})^{\alpha-1}f(x-y)dy\right|\\&\leq C\sup_{t>0}t^{-n+\beta}\int_{|y|<t}|f(x-y)|dy\\&:=Cf_{\beta}^{*}(x)\,,\end{aligned}$$

where f_{β}^{*} is the maximal fractional integral operator introduced by Muckenhoupt and Wheeden in [8], in which it was proved that f_{β}^{*} is of type (p,q) with $1/q=1/p-\beta/n$ and of weak type $(1,n/(n-\beta))$. Using (3.1) as an endpoint estimate, the first result in Theorem 3.3 will follow from the analytic interpolation theorem.

Now we turn to the proof of the second result. Let $1<r<\infty$ and $1/r+1/r'=1$. Using the Hölder inequality,

$$\begin{aligned}N_{*}^{\alpha,\beta}f(x)\leq&\sup_{t>0}\left(t^{-n}\int_{|y|<t}\left(1-\frac{|y|^{2}}{t^{2}}\right)^{(\mathrm{Re}\,\alpha-1)r'}dy\right)^{1/r'}\\&\times\sup_{t>0}\left(t^{-n+r\beta}\int_{|y|<t}|f(x-y)|^{r}dy\right)^{1/r}.\end{aligned}$$

When $\mathrm{Re}\,\alpha>\beta/n$, letting $r<n/\beta$ and r be close to n/β yields $\mathrm{Re}\,\alpha>(r'-1)/r$. Thus

$$\left(t^{-n}\int_{|y|<t}\left(1-\frac{|y|^{2}}{t^{2}}\right)^{(\mathrm{Re}\,\alpha-1)r'}dy\right)^{1/r'}<\infty$$

and this implies

$$N_*^{\alpha,\beta}f(x) \le C \sup_{t>0} \left(t^{-n+r\beta} \int_{|y|<t} |f(x-y)|^r dy \right)^{1/r}$$

$$:= Cf^*_{\beta,r}(x) .$$

The result in [3, Lemma 2] shows that if $r<p<n/p$ and $1/q = 1/p - \beta/n$ then

$$\|f_{\beta,r}\|_q \le C\|f\|_p .$$

Therefore, if $\mathrm{Re}\,\alpha > \beta/n$, p is less than n/β but is close to n/β, and $1/q = 1/p - \beta/n$, then

$$\|N_*^{\alpha,\beta}f\|_q \le C\|f\|_p .$$

The analytic interpolation yields the result (b).

To prove Theorem 3, we first assume $f \in L^2 \cap L^p$ and $f \ge 0$. By the definition of Lipschitz space, we have

$$|\Delta^k_{ty'/k} b(x)| \le Ct^\beta .$$

Thus,

$$\tilde{M}_{*;b,k} f(x) \le CN_*^{0,\beta} f(x) .$$

Theorem 3 follows obviously from Theorem 3.3.

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