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On the Units of Integral Group Rings

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§0. Let G be a finite group and let ZG be its integral group ring. Let $U(ZG)$ denote the unit group of ZG and define $V(ZG)=\{u\in U(ZG)\mid \varepsilon(u)=1\}$ where $\varepsilon: ZG\to Z$ is the augmentation map. In this paper we will study the following problems:

Problem 1. *How many conjugate classes are there in $V(ZG)$ of subgroups of $V(ZG)$ isomorphic to G?*

Problem 2. *Is there a torsion free normal subgroup F of $V(ZG)$ such that $V(ZG)=F\cdot G$?*

Let S_n(resp. A_n) denote the symmetric group (resp. alternating group) on n symbols, and let D_n denote the dihedral group of order $2n$.

Hughes and Pearson ([3]) raised Problem 1 with related problems and showed that there is only one conjugate class in $V(ZS_3)$ of subgroups of $V(ZS_3)$ isomorphic to S_3. Polcino ([6]) showed that there are two conjugate classes in $V(ZD_4)$ of subgroups of $V(ZD_4)$ isomorphic to D_4. On the other hand, Dennis ([2]) solved affirmatively Problem 2 in the case where $G=S_3$. Recently, Miyata ([5]) has solved Problem 2 in the case where $G=D_n$, n odd. He has also solved Problem 1 with an additional hypothesis that the class group of ZD_n is of odd order.

Our main results are as follows:

[I] *Let G be a finite metabelian group such that the exponent of G/G' is* 1, 2, 3, 4 *or* 6, *where G' is the commutator subgroup of G. Then there is a torsion free normal subgroup F of $V(ZG)$ such that $V(ZG)=F\cdot G$.*

[II] (1) *Let A_4 be the alternating group on* 4 *symbols. Then there are* 4 *conjugate classes in $V(ZA_4)$ of subgroups of $V(ZA_4)$ isomorphic to A_4.*

(2) *Let S_4 be the symmetric group on* 4 *symbols. Then there are* 16 *conjugate classes in $V(ZS_4)$ of subgroups of $V(ZS_4)$ isomorphic to S_4.*

§1. Let G be a finite group. For an ideal J of ZG, we write

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$U(1+J)=U(ZG)\cap(1+J)$, where $1+J$ is the set of all elements of the form $1+j$, $j\in J$. For $N\lhd G$, denote by $\varepsilon_{G.N}$ the natural map from ZG to $Z(G/N)$ and set $I(G, N)=\operatorname{Ker}\varepsilon_{G.N}$. Note that $\varepsilon_{G.G}$ is the augmentation map of ZG and $V(ZG)=U(1+I(G, G))$. Write $\varepsilon=\varepsilon_{G.G}$ and $I(G)=I(G, G)$.

We will use the following result.

PROPOSITION 1.1 ([11]). *Let G be a finite group and $N\lhd G$. Then*

$$N/N'\cong I(G, N)/I(G)I(G, N)$$

under the map $nN'\to n-1+I(G)I(G, N)$, $n\in N$.

Define the map $U(1+I(G, N))\to I(G, N)$ by $1+k\to k$, $k\in I(G, N)$. This map induces a group homomorphism

$$U(1+I(G, N))/U(1+I(G)I(G, N))\longrightarrow I(G, N)/I(G)I(G, N)\ .$$

It is easy to see that this is an isomorphism. Therefore we get

COROLLARY 1.2. *Let G be a finite group and $N\lhd G$. Then*

$$N/N'\cong U(1+I(G, N))/U(1+I(G)I(G, N))\ .$$

LEMMA 1.3 ([4]). *Let G be a finite group and let $g\in G$. Then $g-1\in I(G)^2$ if and only if $g\in G'$.*

PROPOSITION 1.4 ([4]). *Suppose that G is a finite metabelian group, then $U(1+I(G)I(G, G'))$ is a torsion free normal subgroup of $U(ZG)$.*

Suppose that H is a finite abelian group. Then, by the theorem of Higman, the only units of finite order in ZH are $\pm h(h\in H)$. It is also known that, if the exponent of H is 1, 2, 3, 4 or 6, the only units of ZH are $\pm h(h\in H)$.

THEOREM 1.5. *Let G be a finite metabelian group such that the exponent of G/G' is 1, 2, 3, 4 or 6. Then there is a torsion free normal subgroup F of $V(ZG)$ such that $V(ZG)=F\cdot G$.*

PROOF. By Corollary 1.2, $U(1+I(G))/U(1+I(G)^2)\cong G/G'$ and

$$U(1+I(G, G'))/U(1+I(G)I(G, G'))\cong G'\ .$$

It is clear that $U(1+I(G, G'))\subseteq U(1+I(G)^2)$. Since G/G' is an abelian group, $U(1+I(G/G')^2)\cap(G/G')=\{1\}$ by Lemma 1.3. Hence, by the assumption on the exponent of G/G', $U(1+I(G/G')^2)=\{1\}$. Let $\pi: U(ZG)\to U(Z(G/G'))$ be the natural map. Since $\pi(U(1+I(G)^2))\subseteq U(1+I(G/G')^2)=\{1\}$, $U(1+I(G)^2)\subseteq$

$\operatorname{Ker}\pi = U(1+I(G, G'))$, and therefore $U(1+I(G)^2) = U(1+I(G, G'))$. Hence

$$
\begin{aligned}
&|V(ZG)/U(1+I(G)I(G, G'))| \\
&\quad = |V(ZG)/U(1+I(G)^2)|\,|U(1+I(G, G'))/U(1+I(G)I(G, G'))| \\
&\quad = |G/G'|\,|G'| = |G| .
\end{aligned}
$$

On the other hand, by Proposition 1.4, $U(1+I(G)I(G, G'))$ is a torsion free normal subgroup of $V(ZG)$. Therefore it follows that $U(1+I(G)I(G, G'))\cdot G = V(ZG)$. Thus $U(1+I(G)I(G, G'))$ is a torsion free normal subgroup of $V(ZG)$, as desired.

REMARK 1. Let K be a finite group. Suppose that there is a torsion free normal subgroup F of $V(ZK)$ such that $V(ZK) = F\cdot K$. Let H be a finite abelian group. Put $G = K\times H$. Then there is a torsion free normal subgroup $\tilde{F}$ of $V(ZG)$ such that $V(ZG) = \tilde{F}\cdot G$.

SKETCH OF THE PROOF. First, we will show that $U(1+I(G, H)I(G))$ is torsion free. Take an element u of finite order in $U(1+I(G, H))$ and write $u = \sum a_i h_i + \sum_{k_m\neq 1} b_{jm} h_j k_m$, where h_i, h_j (resp. k_m) range over elements of H(resp. K) and a_i, $b_{jm}\in Z$. Since $\varepsilon_{G.H}(u) = \sum a_i + \sum_{k_m\neq 1} b_{jm} k_m = 1$ in $Z(G/H)$, there exists a_{i_0} such that $a_{i_0}\neq 0$. Then, by [1, (3.1)], $a_{i_0}=1$ and $u = h_{i_0}\in H$. For any $h\in H(h\neq 1)$, $h\notin U(1+I(G, H)I(G))$ by Lemma 1.3, hence $U(1+I(G, H)I(G))$ is torsion free. Next, set $\tilde{F} = U(1+I(G, H)I(G))\cdot F$. Then $\tilde{F}$ is torsion free and $[V(ZG)\colon \tilde{F}] = |G|$. Therefore $\tilde{F}$ is a torsion free normal subgroup as desired.

REMARK 2. For any finite group G and any integer $n\geqq 3$, $n\in Z$, consider the natural map $f_n\colon V(ZG)\to V((Z/nZ)G)$. Then, by [1, (3.1)], $\operatorname{Ker} f_n$ is a torsion free normal subgroup of $V(ZG)$ such that $[V(Z(G)\colon \operatorname{Ker} f_n] < \infty$. But $[V(ZG)\colon \operatorname{Ker} f_n] \neq |G|$ in general.

REMARK 3. Suppose that G is a finite metabelian group which is a semidirect product of G' by a subgroup H of G. Since H is an abelian group, by Proposition 1.1, Corollary 1.2 and Lemma 1.3, $U(1+I(H)^2)$ is torsion free and $V(ZH) = H\times U(1+I(H)^2)$. Set

$$F = U(1+I(G)I(G, G'))\cdot U(1+I(H)^2) .$$

Then F is torsion free and $[V(ZG)\colon F] = |G|$, but it is not a normal subgroup of $V(ZG)$ in general.

§ 2. Let A_4 be the alternating group on 4 symbols 1, 2, 3 and 4. Set $N = \{1, (12)(34), (13)(24), (14)(23)\} \lhd A_4$ and define $\bar{N} = 1+(12)(34)+(13)(24)+(14)(23)$ in ZA_4. Let ω be a generator of A_4/N.

Hereafter, the unit group of a ring R will be denoted by $U(R)$. Consider the pullback diagram

$$\begin{array}{ccc} ZA_4 & \xrightarrow{f_2} & Z(A_4/N)\cong Z[\omega] \\ {\scriptstyle f_1}\downarrow & & \downarrow \\ ZA_4/(\bar{N}) & \longrightarrow & (Z/4Z)[\omega] \,. \end{array}$$

From this diagram we get the Mayer-Vietoris exact sequence (e.g., [7]).

$$(1) \qquad 1\longrightarrow U(ZA_4)\longrightarrow U(ZA_4/(\bar{N}))\times U(Z[\omega])\longrightarrow U((Z/4Z)[\omega])\longrightarrow 1\,.$$

The exactness of the last map follows from the fact that $D(ZA_4)=0$ (e.g., [10]). Since $U(Z[\omega])=\{\pm 1, \pm\omega, \pm\omega^2\}$ we have an exact sequence

$$(2) \qquad 1\longrightarrow U(ZA_4)\longrightarrow U(ZA_4/(\bar{N}))\longrightarrow U((Z/4Z)[\omega])/\langle -1, \omega\rangle\longrightarrow 1\,.$$

Because $|U((Z/4Z)[\omega])|=24$, $|U((Z/4Z)[\omega])/\langle -1, \omega\rangle|=4$. Define a representation T of A_4 to $\mathrm{GL}(3, Z)$ by

$$(12)(34)\longrightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} \quad\text{and}\quad (123)\longrightarrow\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}.$$

Then we can extend T linearly to the map from ZA_4 to $M_3(Z)$ which is denoted by the same symbol T. By the map T, an arbitrary element $u=a_1+a_2(12)(34)+a_3(13)(24)+a_4(14)(23)+b_1(123)+b_2(243)+b_3(142)+b_4(134)+c_1(132)+c_2(234)+c_3(124)+c_4(143)$ of $ZA_4(a_i, b_j, c_k\in Z, 1\leqq i, j, k\leqq 4)$ is represented by the matrix

$$(3) \qquad \begin{bmatrix} a_1+a_2-a_3-a_4 & b_1+b_2-b_3-b_4 & c_1-c_2-c_3+c_4 \\ c_1+c_2-c_3-c_4 & a_1-a_2+a_3-a_4 & b_1-b_2+b_3-b_4 \\ b_1-b_2-b_3+b_4 & c_1-c_2+c_3-c_4 & a_1-a_2-a_3+a_4 \end{bmatrix}.$$

Putting this matrix to 0, we get $a_1=a_2=a_3=a_4$, $b_1=b_2=b_3=b_4$ and $c_1=c_2=c_3=c_4$. Therefore $\mathrm{Ker}\, T=\bar{N}\cdot ZA_4$. Thus T induces an injection from $ZA_4/(\bar{N})$ to $M_3(Z)$. For $x\in ZA_4$, denote by $\bar{x}$ the image of x under the natural map $ZA_4\rightarrow ZA_4/(\bar{N})$. An arbitrary element $p_1+p_2\overline{(12)(34)}+p_3\overline{(13)(24)}+q_1\overline{(123)}+q_2\overline{(12)(34)(123)}+q_3\overline{(13)(24)(123)}+r_1\overline{(132)}+r_2\overline{(12)(34)(132)}+r_3\overline{(13)(24)(132)}$ of $ZA_4/(\bar{N})(p_i, q_j, r_k\in Z, 1\leqq i, j, k\leqq 3)$ is represented by the matrix

$$(4)\qquad \begin{bmatrix} p_1+p_2-p_3 & q_1+q_2-q_3 & r_1+r_2-r_3 \\ r_1-r_2+r_3 & p_1-p_2+p_3 & q_1-q_2+q_3 \\ q_1-q_2-q_3 & r_1-r_2-r_3 & p_1-p_2-p_3 \end{bmatrix}.$$

Viewing p_i as variables, the diophantine equation

$$\begin{cases} p_1+p_2-p_3=x_1 \\ p_1-p_2+p_3=x_2 \\ p_1-p_2-p_3=x_3 \end{cases}$$

has a solution in $\boldsymbol{Z}$ if and only if $x_1\equiv x_2\equiv x_3 \pmod 2$. Applying the same way to q_j and r_k, we get

$$U(\boldsymbol{Z}A_4/(\bar{N}))\cong\left\{A\in \mathrm{GL}(3, \boldsymbol{Z})\mid A\equiv E \pmod 2, \quad\text{or}\quad A\equiv\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\pmod 2,\right.$$

$$\left.\text{or}\quad A\equiv\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\pmod 2\right\}.$$

By (1), an element $u=a_1+a_2(12)(34)+a_3(13)(24)+a_4(14)(23)+b_1(123)+b_2(243)+b_3(142)+b_4(134)+c_1(132)+c_2(234)+c_3(124)+c_4(143)$ of $\boldsymbol{Z}A_4$ is in $U(\boldsymbol{Z}A_4)$ if and only if $f_1(u)\in U(\boldsymbol{Z}A_4/(\bar{N}))$ and $f_2(u)\in U(\boldsymbol{Z}[\omega])$. Since $U(\boldsymbol{Z}[\omega])=\{\pm 1, \pm\omega, \pm\omega^2\}$ and $f_2(u)=\sum_{i=1}^4 a_i+(\sum_{j=1}^4 b_j)\omega+(\sum_{k=1}^4 c_k)\omega^2$, u is in $U(\boldsymbol{Z}A_4)$ if and only if the matrix of (3) is in $\mathrm{GL}(3, \boldsymbol{Z})$ and $(\sum_{i=1}^4 a_i, \sum_{j=1}^4 b_j, \sum_{k=1}^4 c_k)=(\pm 1, 0, 0), (0, \pm 1, 0)$, or $(0, 0, \pm 1)$.

By the same way as in $U(\boldsymbol{Z}A_4/(\bar{N}))$, we get

$$U(\boldsymbol{Z}A_4)\cong\left\{\begin{bmatrix} x_1 & y_1 & z_1 \\ z_2 & x_2 & y_2 \\ y_3 & z_3 & x_3 \end{bmatrix}\in \mathrm{GL}(3, \boldsymbol{Z})\;\middle|\; \begin{matrix} x_i \text{ odd, } y_j,\, z_k \text{ even and} \\ \sum_{j=1}^3 y_j\equiv\sum_{k=1}^3 z_k\equiv 0 \pmod 4 \end{matrix}, \text{ or}\right.$$

$$\left. \begin{matrix} y_j \text{ odd, } x_i,\, z_k \text{ even and} \\ \sum_{i=1}^3 x_i\equiv\sum_{k=1}^3 z_k\equiv 0 \pmod 4 \end{matrix}, \text{ or } \begin{matrix} z_k \text{ odd, } y_j,\, x_i \text{ even and} \\ \sum_{j=1}^3 y_j\equiv\sum_{i=1}^3 x_i\equiv 0 \pmod 4 \end{matrix} \quad 1\leqq i, j, k\leqq 3\right\}.$$

Let $\psi\colon \mathrm{GL}(3, \boldsymbol{Z})\to \mathrm{GL}(3, \boldsymbol{Z}/2\boldsymbol{Z})$ be the natural map. Since

$$\psi(U(\boldsymbol{Z}A_4/(\bar{N})))=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right\},$$

we have $N_{GL(3,Z/2Z)}(\psi(U(ZA_4/(\bar{N})))) = \psi(U(ZA_4/(\bar{N}))) \cdot \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$. Hence $N_{GL(3,Z)}(U(ZA_4/(\bar{N}))) = U(ZA_4/(\bar{N})) \cdot \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$. Similarly we see that $N_{GL(3,Z)}(U(ZA_4)) = U(ZA_4/(\bar{N})) \cdot \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$.

Lemma 2.1. $N_{GL(3,Z)}(A_4) = \pm A_4 \cdot \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$.

Proof. First we will show that $U(\mathrm{cent}(ZA_4/(\bar{N}))) = \{\pm 1\}$ where $\mathrm{cent}(ZA_4/(\bar{N}))$ is the center of $ZA_4/(\bar{N})$. Let $u = p_1 + p_2\overline{(12)(34)} + p_3\overline{(13)(24)} + q_1\overline{(123)} + q_2\overline{(12)(34)(123)} + q_3\overline{(13)(24)(123)} + r_1\overline{(132)} + r_2\overline{(12)(34)(132)} + r_3\overline{(13)(24)(132)}$ be an element of $U(\mathrm{cent}(ZA_4/(\bar{N})))$. Since $T(u) \in T(\mathrm{cent}(Q \otimes_Z ZA_4/(\bar{N}))) = \mathrm{cent}\, M_3(Q)$, $T(u)$ is a diagonal matrix. Hence, in (4), $q_1 = q_2 = q_3 = r_1 = r_2 = r_3 = p_2 = p_3 = 0$. Therefore $U(\mathrm{cent}(ZA_4/(\bar{N}))) = \{\pm 1\}$. Every element of $ZA_4/(\bar{N})$ can be written as a Z-linear combination of elements in A_4. So, if an element of GL(3, Z) normalizes A_4, then it also normalizes $U(ZA_4/(\bar{N}))$. Thus $N_{GL(3,Z)}(A_4) \subseteq N_{GL(3,Z)}(U(ZA_4/(\bar{N})))$. Clearly $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in N_{GL(3,Z)}(A_4)$. Let $X \in N_{GL(3,Z)}(A_4) \cap U(ZA_4/(\bar{N}))$. Then $X \equiv E$, or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (mod 2). Since $T((123)) \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (mod 2) and $T((132)) \equiv \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (mod 2), X acts trivially on A_4/N. On the other hand, the automorphism group of A_4 is isomorphic to S_4. Hence there exists $Y \in \mathrm{Aut}\, A_4 = S_4$ such that $X^{-1}Y$ acts trivially on A_4. Therefore Y acts trivially on A_4/N, and hence $Y \in A_4$. Since $X^{-1}Y \in (\mathrm{cent}(ZA_4/(\bar{N}))) = \{\pm 1\}$, $X^{-1}Y = \pm 1$. Consequently $X = \pm Y$. This completes the proof.

By the result of [8], there are 3 conjugate classes in GL(3, Z) of subgroups of GL(3, Z) isomorphic to A_4. The representatives are as follows:

$$W_1 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}, \quad W_2 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \right\},$$

$$W_3=\left\{\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ \begin{bmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{bmatrix}\right\}.$$

Since, for any $X\in \mathrm{GL}(3, \boldsymbol{Z})$, $X^{-1}W_iX \not\subseteq U(\boldsymbol{Z}A_4/(\bar{N}))$, $i=2, 3$, a subgroup G of $V(\boldsymbol{Z}A_4)$ isomorphic to A_4 is conjugate to A_4 in $\mathrm{GL}(3, \boldsymbol{Z})$. Further G and A_4 are conjugate in $N_{GL(3,\boldsymbol{Z})}(U(\boldsymbol{Z}A_4))=U(\boldsymbol{Z}A_4/(\bar{N}))\cdot\left\langle\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle$.

THEOREM 2.2. *There are* 4 *conjugate classes in* $V(\boldsymbol{Z}A_4)$ *of subgroups of* $V(\boldsymbol{Z}A_4)$ *isomorphic to* A_4.

PROOF. Let X, Y be elements of $U(\boldsymbol{Z}A_4/(\bar{N}))\cdot\left\langle\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle$. Then XA_4X^{-1} and YA_4Y^{-1} are conjugate in $V(\boldsymbol{Z}A_4)$ if and only if there exists $Z\in V(\boldsymbol{Z}A_4)$ such that $Y^{-1}ZX\in N_{GL(3,\boldsymbol{Z})}(A_4)=A_4\cdot\left\langle\pm\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle$ (Lemma 2.1), i.e., $Y^{-1}V(\boldsymbol{Z}A_4)X\cap A_4\cdot\left\langle\pm\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle\neq\phi$. This condition is equivalent to $X^{-1}Y\in U(\boldsymbol{Z}A_4)\cdot\left\langle\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle$. Therefore the number of conjugate classes in $V(\boldsymbol{Z}A_4)$ of subgroups of $V(\boldsymbol{Z}A_4)$ isomorphic to A_4 is $\left[U(\boldsymbol{Z}A_4/(\bar{N}))\cdot\left\langle\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle : U(\boldsymbol{Z}A_4)\cdot\left\langle\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right\rangle\right]=4$.

§ 3. In this section, we will consider S_4, the symmetric group on 4 symbols 1, 2, 3 and 4. We now write $S_3=\langle\sigma,\ \tau\,|\,\sigma^3=\tau^2=1,\ \tau\sigma\tau^{-1}=\sigma^{-1}\rangle$. Then we have

LEMMA 3.1. $\sigma-\sigma^{-1}\pm\tau-\sigma\tau+\sigma^2\tau$, $-5+2\sigma+2\sigma^2-4\sigma\tau+4\sigma^2\tau$ *and* $1-2\sigma+2\sigma^2+2\sigma^{i+1}\tau-2\sigma^{i+2}\tau$, $0\leq i\leq 2$, *are units of* $\boldsymbol{Z}S_3$.

PROOF. By direct calculations, we have $(\sigma-\sigma^{-1}\pm\tau-\sigma\tau+\sigma^2\tau)^{-1}=\sigma-\sigma^{-1}\pm\tau-\sigma\tau+\sigma^2\tau$, $(-5+2\sigma+2\sigma^2-4\sigma\tau+4\sigma^2\tau)^{-1}=-5+2\sigma+2\sigma^2+4\sigma\tau-4\sigma^2\tau$ and $(1-2\sigma+2\sigma^2+2\sigma^{i+1}-2\sigma^{i+2}\tau)^{-1}=1+2\sigma-2\sigma^2-2\sigma^{i+1}\tau+2\sigma^{i+2}\tau$, $0\leq i\leq 2$.

We note that some of the units in Lemma 3.1 were obtained by Taussky ([9]). Consider the pullback diagram

$$\begin{array}{ccc}\boldsymbol{Z}S_3 & \longrightarrow & \boldsymbol{Z}S_3/(\sigma-1)\cong\boldsymbol{Z}[\tau]\\ \downarrow & & \downarrow\\ \boldsymbol{Z}S_3/(\sigma^2+\sigma+1) & \longrightarrow & (\boldsymbol{Z}/3\boldsymbol{Z})[\tau]\,.\end{array}$$

From this diagram we get the exact sequence

$$(5)\quad 1\longrightarrow U(\boldsymbol{Z}S_3)\longrightarrow U(\boldsymbol{Z}S_3/(\sigma^2+\sigma+1))\times U(\boldsymbol{Z}[\tau])\longrightarrow U((\boldsymbol{Z}/3\boldsymbol{Z})[\tau])\longrightarrow 1\,.$$

The exactness of the last map follows from the fact that $D(\boldsymbol{Z}S_3)=0$.

By [3], there exists a monomorphism $\boldsymbol{Z}S_3/(\sigma^2+\sigma+1)\longrightarrow M_2(\boldsymbol{Z})$. An arbitrary element $a_1+a_2\bar{\sigma}+b_1\bar{\tau}+b_2\bar{\sigma}\bar{\tau}$ of $\boldsymbol{Z}S_3/(\sigma^2+\sigma+1)(a_i,\ b_j\in\boldsymbol{Z},\ 1\leqq i,\ j\leqq 2)$ is represented by the matrix

$$\begin{bmatrix} a_1+b_1 & -a_2-b_1+b_2 \\ a_2+b_2 & a_1-a_2-b_1 \end{bmatrix}.$$

Therefore $U(\boldsymbol{Z}S_3/(\sigma^2+\sigma+1))\cong\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in \mathrm{GL}(2,\ \boldsymbol{Z})\,\middle|\,a+c\equiv b+d (\mathrm{mod}\ 3)\right\}$. Set $F=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in \mathrm{GL}(2,\ \boldsymbol{Z})\,\middle|\,a+c\equiv b+d (\mathrm{mod}\ 3)\right\}$. By the above discussion we can define a monomorphism $\Psi: U(\boldsymbol{Z}S_3)\rightarrow F\times U(\boldsymbol{Z}[\tau])$.

Consider the commutative diagram

$$\begin{array}{ccc} U(\boldsymbol{Z}S_3) & \xrightarrow{\ \Psi\ } & F\times U(\boldsymbol{Z}[\tau]) \\ \varphi\downarrow & & \downarrow\tilde{\varphi} \\ U(F_4S_3) & \xrightarrow{\ \tilde{\Psi}\ } & \mathrm{GL}(2,\ F_4)\times U(F_4[\tau])\,, \end{array}$$

where $F_4=\boldsymbol{Z}/4\boldsymbol{Z}$, φ and $\tilde{\varphi}$ are natural maps and $\tilde{\Psi}$ is induced by Ψ. Then $\tilde{\Psi}$ is an isomorphism.

LEMMA 3.2. $|\mathrm{Coker}\,\varphi|=4$.

PROOF. It is easy to see that the natural map $\mathrm{GL}(2,\ \boldsymbol{Z})\rightarrow\mathrm{GL}(2,\ F_4)$ is surjective. Set $H=\mathrm{Ker}\{\mathrm{GL}(2,\ \boldsymbol{Z})\rightarrow\mathrm{GL}(2,\ F_4)\}$. Since $FH/H\cong F/F\cap H$, $[FH:F]=[H:F\cap H]$ is a divisor of 4. Because $\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix}\in H\backslash(F\cap H)$ and $\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix}\notin F\cap H$, $[H:F\cap H]\geqq 3$. Therefore $[FH:F]=4$ and hence $FH=\mathrm{GL}(2,\ \boldsymbol{Z})$. It follows that $F\rightarrow\mathrm{GL}(2,\ F_4)$ is surjective. Since $U(F_4[\tau])=\{\pm 1,\ \pm\tau,\ \pm 1+2\tau,\ 2\pm\tau\}$, $|\mathrm{Coker}\,\tilde{\varphi}|=2$. By (5), $|\mathrm{Coker}\,\Psi|=4$ and $F\times U(\boldsymbol{Z}[\tau])=\mathrm{Im}\,\Psi\times U(\boldsymbol{Z}[\tau])$. To prove $|\mathrm{Coker}\,\varphi|=4$, it suffices to show that $[\tilde{\Psi}^{-1}(\mathrm{Im}\,\tilde{\varphi}):\mathrm{Im}\,\varphi]=2$. Let $x=\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ -1\right)\in F\times U(\boldsymbol{Z}[\tau])$, then $\tilde{\Psi}^{-1}\circ\tilde{\varphi}(x)=-1+2\sigma+2\sigma^2\in U(F_4S_3)$. If we put $u=-5+2\sigma+2\sigma^2-4\sigma\tau+4\sigma^2\tau$, then $u\in U(\boldsymbol{Z}S_3)$ by Lemma 3.1 and $\varphi(u)=-1+2\sigma+2\sigma^2$. Next, we will show that $\tilde{\Psi}^{-1}\circ\tilde{\varphi}\left(\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \tau\right)\right)=2+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau\notin\mathrm{Im}\,\varphi$. Suppose conversely that there exists $u\in U(\boldsymbol{Z}S_3)$ such that $\varphi(u)=2+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau$. Then we may write $u=2+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau+4f+4g\tau(f,\ g\in\boldsymbol{Z}[\sigma])$. Since $\varepsilon_{S_3,\langle\sigma\rangle}(u)\in$

$U(\boldsymbol{Z}[\tau])=\{\pm 1, \pm\tau\}$, f and g must be written as $f=a_0+a_1\sigma+(-1-a_0-a_1)\sigma^2$, $g=b_0+b_1\sigma+(1-b_0-b_1)\sigma^2$ for some $a_i, b_j \in \boldsymbol{Z}$. Therefore $u=2+4a_0+(1+4a_1)\sigma+(-3-4a_0-4a_1)\sigma^2+(-1+4b_0)\tau+(-1+4b_1)\sigma\tau+(3-4b_0-4b_1)\sigma^2\tau$. Hence we have

$$\Psi(u)=\left(\begin{bmatrix} 1+4(2a_0+a_1+2b_0+b_1) & -4(1+a_0+2a_1+b_0-b_1) \\ 4(a_0+2a_1+b_0+2b_1) & 1+4(1+a_0-a_1-2b_0-b_1) \end{bmatrix}, \tau\right).$$

Since the determinant of this matrix is $1+4(1+3a_0)+16(2a_0+a_1+2b_0+b_1)(1+a_0-a_1-b_1)+16(1+a_0+2a_1+b_0-b_1)(a_0+2a_1+b_0+2b_1)$, we get $1+3a_0+4(2a_0+a_1+2b_0+b_1)(1+a_0-a_1-2b_0-b_1)+4(1+a_0+2a_1+b_0-b_1)(a_0+2a_1+b_0+2b_1)=0$. But this is equal to $1+3a_0+12(a_0^2+a_0+a_0a_1+a_1+b_0+b_1+a_1^2-b_0^2-b_0b_1-b_1^2)$ which is a contradiction. Therefore $[\tilde{\Psi}^{-1}(\operatorname{Im}\tilde{\varphi})\colon \operatorname{Im}\varphi]=2$.

Set $N_1=\{1, 1+2\sigma^i(\sigma+\sigma^2)\tau, 1+2(\sigma+\sigma^2), 1+2(\sigma+\sigma^2)+2\sigma^i(\sigma+\sigma^2)\tau, 0\leqq i\leqq 2\}\subseteqq U(F_4S_3)$ and $N_2=\langle\sigma-\sigma^2+\tau-\sigma\tau+\sigma^2\tau\rangle$. Then N_1(resp. N_2) is a subgroup of $U(F_4S_3)$ of order 8 (resp. 2). A direct calculation shows that $\sigma-\sigma^2+\tau-\sigma\tau+\sigma^2\tau$ is commutative with each element of N_1.

COROLLARY 3.3. $\operatorname{Im}\varphi=(\pm N_1\times N_2)\cdot S_3$.

PROOF. By Lemma 3.1, $(\pm N_1\times N_2)\cdot S_3\subseteqq\operatorname{Im}\varphi$. But $[U(F_4S_3)\colon(\pm N_1\times N_2)\cdot S_3]=4$. Therefore, by Lemma 3.2, $\operatorname{Im}\varphi=(\pm N_1\times N_2)\cdot S_3$.

Let $S_4\triangleright N=\{1, (12)(34), (13)(24), (14)(23)\}$. Consider the pullback diagram

$$\begin{array}{ccc} \boldsymbol{Z}S_4 & \longrightarrow & \boldsymbol{Z}(S_4/N)\cong\boldsymbol{Z}S_3 \\ \downarrow & & \downarrow{\scriptstyle g_2} \\ \boldsymbol{Z}S_4/(\bar{N}) & \xrightarrow[g_1]{} & F_4S_3 \end{array}$$

where $F_4=\boldsymbol{Z}/4\boldsymbol{Z}$. From this diagram we get the exact sequence

$$U^*(\boldsymbol{Z}S_4/(\bar{N}))\times U^*(\boldsymbol{Z}S_3)\xrightarrow{\ \rho\ } U(F_4S_3)\longrightarrow 1,$$

where $U^*(\boldsymbol{Z}S_4/(\bar{N}))$(resp. $U^*(\boldsymbol{Z}S_3)$) denotes the image of $U(\boldsymbol{Z}S_4/(\bar{N}))$(resp. $U(\boldsymbol{Z}S_3)$) in $U(F_4S_3)$. The exactness of ρ follows from the fact that $D(\boldsymbol{Z}S_4)=0$ (e.g., [10]). Since $U^*(\boldsymbol{Z}S_4/(\bar{N}))\supseteqq U^*(\boldsymbol{Z}S_3)$, $\rho(U^*(\boldsymbol{Z}S_4/(\bar{N})))=U(F_4S_3)$. We also have an exact sequence

$$1\longrightarrow U(\boldsymbol{Z}S_4)\longrightarrow U(\boldsymbol{Z}S_4/(\bar{N}))\times U(\boldsymbol{Z}S_3).$$

Define a representation T_1 of $\boldsymbol{Z}S_4$ to $M_3(\boldsymbol{Z})\oplus M_3(\boldsymbol{Z})$ by

$$T_1((12))=\left(\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\right),$$

$$T_1((1234))=\left(\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\right).$$

Then we see that Ker $T_1=\bar{N}\cdot ZS_4$. Thus T_1 induces an injection from $ZS_4/(\bar{N})$ to $M_3(Z)\oplus M_3(Z)$. For $x\in ZS_4$, denote by $\bar{x}$ the image of x under the natural map $ZS_4\to ZS_4/(\bar{N})$. Then an arbitrary element

$$(6)\quad u=a_1+a_2\overline{(12)(34)}+a_3\overline{(13)(24)}+b_1\overline{(123)}+b_2\overline{(12)(34)(123)}+b_3\overline{(13)(24)(123)}$$
$$+c_1\overline{(132)}+c_2\overline{(12)(34)(132)}+c_3\overline{(13)(24)(132)}+d_1\overline{(12)}+d_2\overline{(12)(34)(12)}$$
$$+d_3\overline{(13)(24)(12)}+e_1\overline{(13)}+e_2\overline{(12)(34)(13)}+e_3\overline{(13)(24)(13)}+f_1\overline{(23)}$$
$$+f_2\overline{(12)(34)(23)}+f_3\overline{(13)(24)(23)}$$

of $ZS_4/(\bar{N})$ ($a_i, b_j, c_k, d_l, e_m, f_n\in Z$, $1\leqq i, j, k, l, m, n\leqq 3$) is represented by the matrices

$$\left(\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}\right),$$

where

$$a_{11}=a_1+a_2-a_3+d_1+d_2-d_3,\ a_{12}=b_1+b_2-b_3+f_1+f_2-f_3,$$
$$a_{13}=c_1+c_2-c_3+e_1+e_2-e_3,\ a_{21}=c_1-c_2+c_3+f_1-f_2+f_3,$$
$$a_{22}=a_1-a_2+a_3+e_1-e_2+e_3,\ a_{23}=b_1-b_2+b_3+d_1-d_2+d_3,$$
$$a_{31}=b_1-b_2-b_3+e_1-e_2-e_3,\ a_{32}=c_1-c_2-c_3+d_1-d_2-d_3,$$
$$a_{33}=a_1-a_2-a_3+f_1-f_2-f_3,\ b_{11}=a_1+a_2-a_3-d_1-d_2+d_3,$$
$$b_{12}=b_1+b_2-b_3-f_1-f_2+f_3,\ b_{13}=c_1+c_2-c_3-e_1-e_2+e_3,$$
$$b_{21}=c_1-c_2+c_3-f_1+f_2-f_3,\ b_{22}=a_1-a_2+a_3-e_1+e_2-e_3,$$
$$b_{23}=b_1-b_2+b_3-d_1+d_2-d_3,\ b_{31}=b_1-b_2-b_3-e_1+e_2+e_3,$$
$$b_{32}=c_1-c_2-c_3-d_1+d_2+d_3 \text{ and } b_{33}=a_1-a_2-a_3-f_1+f_2+f_3.$$

Put $\tilde{E}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$. Then we have

$$U(ZS_4/(\bar{N}))\cong\left\{A_1\times A_2\in \mathrm{GL}(3, Z)^2\ \middle|\ \begin{array}{l}A_1\equiv A_2\equiv E(\mathrm{mod}\ 2)\\ \text{and}\\ A_1-A_2\equiv 0 \quad\text{or}\quad 2\tilde{E}(\mathrm{mod}\ 4)\end{array}\right.,$$

$$\text{or}\quad A_1\equiv A_2\equiv\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}(\bmod 2)\,,\ \text{and}\ A_1-A_2\equiv 0\ \text{or}\ 2\tilde{E}(\bmod 4)$$

$$\text{or}\quad A_1\equiv A_2\equiv\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}(\bmod 2)\,,\ \text{and}\ A_1-A_2\equiv 0\ \text{or}\ 2\tilde{E}(\bmod 4)\qquad \text{or}\quad A_1\equiv A_2\equiv\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}(\bmod 2)\ \text{and}\ A_1+A_2\equiv 0\ \text{or}\ 2\tilde{E}(\bmod 4)$$

$$\left.\begin{array}{l}\text{or}\quad A_1\equiv A_2\equiv\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}(\bmod 2)\,,\ \text{and}\ A_1+A_2\equiv 0\ \text{or}\ 2\tilde{E}(\bmod 4)\qquad \text{or}\quad A_1\equiv A_2\equiv\begin{bmatrix}1&0&0\\1&0&0\\0&0&1\end{bmatrix}(\bmod 2)\ \text{and}\ A_1+A_2\equiv 0\ \text{or}\ 2\tilde{E}(\bmod 4)\end{array}\right\}.$$

Set $U_o(ZS_4/(\bar{N}))=g_1^{-1}(g_2(U(ZS_3)))\cap U(ZS_4/(\bar{N}))$ and $V_o(ZS_4/(\bar{N}))=g_1^{-1}(g_2(V(ZS_3)))\cap U(ZS_4/(\bar{N}))$. Then $[U(ZS_4/(\bar{N})): V_0(ZS_4/(\bar{N}))]=8$ by Lemma 3.2 and $V(ZS_4)=\{(v_1, v_2)\in V_0(ZS_4/(\bar{N}))\times V(ZS_3)\,|\,g_1(v_1)=g_2(v_2)\}$.

Since every element of $U(ZS_4/(\bar{N}))$ is a Z-linear combination of elements of $V_o(ZS_4/(\bar{N}))$, $N_{GL(3,Z)^2}(V_o(ZS_4/(\bar{N})))\subseteq N_{GL(3,Z)^2}(U(ZS_4/(\bar{N})))$ and we get $N_{GL(3,Z)^2}(U(ZS_4/(\bar{N})))=U(ZS_4/(\bar{N}))\cdot\left\langle\left(E,\begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}\right)\right\rangle$. We will show that $N_{GL(3,Z)^2}(V_0(ZS_4/(\bar{N})))=N_{GL(3,Z)^2}(U(ZS_4/(\bar{N})))$. Since $g_2(V(ZS_3))\lhd U(F_4S_3)$, $U(ZS_4/(\bar{N}))\subseteq N_{GL(3,Z)^2}(V_o(ZS_4/(\bar{N})))$. To show that $\left\langle\left(E,\begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}\right)\right\rangle\in N_{GL(3,Z)^2}(V_0(ZS_4/(\bar{N})))$, let u be an element of $U(ZS_4/(\bar{N}))$ which is represented as in (6). By a direct calculation, we have

$$\left(E,\begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}\right)T_1(u)\left(E,\begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}^{-1}\right)=T_1(u)+(0,\,(\alpha_{ij}))$$

for some $(\alpha_{ij})\in M_3(Z)$ such that

$$\begin{aligned}&\alpha_{11}+\alpha_{22}+\alpha_{33}\equiv\alpha_{12}+\alpha_{23}+\alpha_{31}\equiv\alpha_{13}+\alpha_{21}+\alpha_{32}\\&\equiv\alpha_{11}+\alpha_{23}+\alpha_{32}\equiv\alpha_{12}+\alpha_{21}+\alpha_{33}\equiv\alpha_{13}+\alpha_{22}+\alpha_{31}\equiv 0(\bmod 8)\,.\end{aligned}$$

Let $v=p_{11}+p_{12}\overline{(12)(34)}+p_{13}\overline{(13)(24)}+p_{21}\overline{(123)}+p_{22}\overline{(12)(34)}\overline{(123)}+p_{23}\overline{(13)(24)}\overline{(123)}+p_{31}\overline{(132)}+p_{32}\overline{(12)(34)}\overline{(132)}+p_{33}\overline{(13)(24)}\overline{(132)}+p_{41}\overline{(12)}+p_{42}\overline{(12)(34)}\overline{(12)}+p_{43}\overline{(13)(24)}\overline{(12)}+p_{51}\overline{(13)}+p_{52}\overline{(12)(34)}\overline{(13)}+p_{53}\overline{(13)(24)}\overline{(13)}+p_{61}\overline{(23)}+$

$p_{62}\overline{(12)(34)}\overline{(23)}+p_{63}\overline{(13)(24)}\overline{(23)}$ $(p_{ij}\in \boldsymbol{Z},\ 1\leq i\leq 6,\ 1\leq j\leq 3)$ be the element of $U(ZS_4/(\bar{N}))$ such that $T_1(v)=(0,\ (\alpha_{ij}))$.

Comparing the diagonal entries, we have

$$p_{11}+p_{12}-p_{13}+p_{41}+p_{42}-p_{43}=0$$
$$p_{11}-p_{12}+p_{13}+p_{51}-p_{52}+p_{53}=0$$
$$p_{11}-p_{12}-p_{13}+p_{61}-p_{62}-p_{63}=0$$
$$3p_{11}-p_{12}-p_{13}-p_{41}-p_{42}+p_{43}-p_{51}+p_{52}-p_{53}-p_{61}+p_{62}+p_{63}\equiv 0 \pmod 8)\ .$$

From this equation it is easy to see that $p_{11}+p_{12}+p_{13}\equiv 0 \pmod 4)$. Similarly we get $p_{i1}+p_{i2}+p_{i3}\equiv 0 \pmod 4)(2\leq i\leq 6)$, hence $g_1(T_1^{-1}(0,\ (\alpha_{ij})))=0\in U(F_4S_3)$. Therefore $\left\langle\left(E,\ \begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}\right)\right\rangle\in N_{GL(3,Z)^2}(V_o(ZS_4/(\bar{N})))$. Thus we get

$$N_{GL(3,Z)^2}(V_o(ZS_4/(\bar{N})))=U(ZS_4/(\bar{N}))\cdot\left\langle\left(E,\ \begin{bmatrix}1&2&2\\-2&-3&-2\\2&2&1\end{bmatrix}\right)\right\rangle.$$

By the result of [8], there are 6 conjugate classes in GL(3, $\boldsymbol{Z}$) of subgroups of GL(3, $\boldsymbol{Z}$) isomorphic to S_4. The representatives are as follows:

$$W_1=\left\{\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix},\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\right\},\quad W_2=\left\{\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\},$$

$$W_3=\left\{\begin{bmatrix}0&-1&0\\1&1&1\\-1&0&0\end{bmatrix},\begin{bmatrix}-1&-1&0\\0&1&0\\0&0&-1\end{bmatrix}\right\},\quad W_4=\left\{\begin{bmatrix}0&1&0\\-1&-1&-1\\1&0&0\end{bmatrix},\begin{bmatrix}1&1&0\\0&-1&0\\0&0&1\end{bmatrix}\right\},$$

$$W_5=\left\{\begin{bmatrix}1&1&0\\-2&-1&-1\\0&0&1\end{bmatrix},\begin{bmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{bmatrix}\right\},\quad W_6=\left\{\begin{bmatrix}-1&-1&0\\2&1&1\\0&0&-1\end{bmatrix},\begin{bmatrix}1&1&1\\0&0&-1\\0&-1&0\end{bmatrix}\right\}.$$

We see that there are two conjugate classes in GL(3, $\boldsymbol{Z}$)^2 of subgroups of GL(3, $\boldsymbol{Z}$)^2 isomorphic to S_4 which are not trivially intersected with $U(ZS_4/(\bar{N}))$. The representatives are as follows:

$$L_1=\left\{\left(\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\right),\ \left(\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\right)\right\},$$

$$L_2=\left\{\left(\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix},\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix}\right),\left(\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\right)\right\}.$$

Hence the conjugate class containing L_1 is the only one class which is not trivially intersected with $V_0(ZS_4/(\bar{N}))$. Therefore any subgroup G of $V_o(ZS_4/(\bar{N}))$ isomorphic to S_4 is conjugate to S_4 in $GL(3, Z)^2$. Further, G is conjugate to S_4 in $U(ZS_4/(\bar{N}))\cdot\left\langle\left(E, \begin{bmatrix} 1 & 2 & 2 \\ -2 & -3 & -2 \\ 2 & 2 & 1 \end{bmatrix}\right)\right\rangle$.

Lemma 3.4. *There are* 8 *conjugate classes in* $V_o(ZS_4/(\bar{N}))$ *of subgroups of* $V_0(ZS_4/(\bar{N}))$ *isomorphic to* S_4.

Proof. In the same way as in the proof of Lemma 2.1, we see that $U(\text{cent}(ZS_4/(\bar{N})))=\{\pm 1\}$. Since the automorphism group of S_4 is isomorphic to S_4, $N_{N_{GL(3,Z)^2(U(ZS_4/(\bar{N})))}}(S_4)=\pm S_4$. Let X, Y be elements of $U(ZS_4/(\bar{N}))\cdot\left\langle\left(E, \begin{bmatrix} 1 & 2 & 2 \\ -2 & -3 & -2 \\ 2 & 2 & 1 \end{bmatrix}\right)\right\rangle$. Then XS_4X^{-1} and YS_4Y^{-1} are conjugate in $V_o(ZS_4/(\bar{N}))$ if and only if $Y^{-1}V_o(ZS_4/(\bar{N}))X\cap(\pm S_4)\neq\phi$. This condition is equivalent to $X^{-1}Y\in U_o(ZS_4/(\bar{N}))$. Therefore the number of conjugate classes in $V_o(ZS_4/(\bar{N}))$ of subgroups of $V_o(ZS_4/(\bar{N}))$ isomorphic to S_4 is $\left[U(ZS_4/(\bar{N}))\cdot\left\langle\left(E, \begin{bmatrix} 1 & 2 & 2 \\ -2 & -3 & -2 \\ 2 & 2 & 1 \end{bmatrix}\right)\right\rangle : U_0(ZS_4/(\bar{N}))\right]=8$.

Theorem 3.5. *There are* 16 *conjugate classes in* $V(ZS_4)$ *of subgroups of* $V(ZS_4)$ *isomorphic to* S_4.

Proof. Recall that $V(ZS_4)=\{(v_1, v_2)\in V_o(ZS_4/(\bar{N}))\times V(ZS_3)\mid g_1(v_1)=g_2(v_2)\}$. Set $W=V_o(ZS_4/(\bar{N}))\times V(ZS_3)$. Then $N_W(S_4)=S_4$ By Corollary 3.3, $\text{cent}(g_1(V(ZS_3)))=1+2\sigma+2\sigma^2$, and hence $N_W(V(ZS_4))=V(ZS_4)\langle 1, 5-2\sigma-2\sigma^2+4\sigma\tau-4\sigma^2\tau)\rangle$. Since there is 1 conjugate class in $V(ZS_3)$ of subgroups of $V(ZS_3)$ isomorphic to $S_3([3])$, the number of conjugate classes in W of subgroups of $V(ZS_4)$ isomorphic to S_4 is 8. Therefore the number of conjugate classes in $V(ZS_4)$ of subgroups of $V(ZS_4)$ isomorphic to S_4 is $8\times[V(ZS_4)\cdot\langle(1, \langle 5-2\sigma-2\sigma^2+4\sigma\tau-4\sigma^2\tau\rangle)\rangle : V(ZS_4)]=16$.

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