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# Formulae for the Values of Zeta and $L$-functions at Half Integers

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## Introduction

Ramanujan's formula for the Riemann zeta function at half integers was formulated and established by Y. Matsuoka [3]. And he deduced interesting representations for $\zeta(1/2)\zeta(2a-1/2)$ and $\zeta(-1/2)\zeta(2a+1/2)$, where $\zeta(s)$ is the Riemann zeta function and $a$ is an integer greater than 1. Recently, the author [4] generalized the results of Y. Matsuoka [3]. The purpose of this paper is to derive similar results for the values of zeta and $L$-functions.

## § 1. Notations and results.

Throughout this paper, we assume that $a$ is a positive integer and $b$ is a non-negative integer. As usual, $\boldsymbol{N}$ and $\boldsymbol{Q}$ denote the set of natural numbers and the field of rational numbers, respectively. Let $\chi$ be a primitive non-principal character modulo $k$ and $L(s, \chi)$ the Dirichlet $L$-function associated with $\chi$. Denote by $\bar{\chi}$ the character conjugate to $\chi$. Let $B_{n,\chi}$ denote the $n$-th Bernoulli number corresponding to $\chi$ in the sense of Leopoldt. For any positive integer $n$, we set

$$g_1(a, b, \chi; n)=\sum_{\substack{n_1n_2n_3|n \\ (n_1,n_2,n_3)\in\boldsymbol{N}^3}} \chi(n_1)\bar{\chi}(n_2)n_1^{-b-1/2}n_2^{2a-1}n_3^{2a-b-3/2}$$

and

$$g_2(a, b, \chi; n)=\sum_{\substack{n_1n_2n_3|n \\ (n_1,n_2,n_3)\in\boldsymbol{N}^3}} \chi(n_1)\bar{\chi}(n_2)n_1^{-b-1/2}n_2^{2a}n_3^{2a-b-1/2} .$$

For brevity we will often write

$$g_i(n)=g_i(a, b, \chi; n) \quad (i=1, 2) .$$

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Further for any non-negative integer $m$, we put

$$\varepsilon_1(m)=(-1)^{m(m+1)/2}$$

and

$$\varepsilon_2(m)=(-1)^{m(m+3)/2}\ .$$

Then our formulae are formulated as follows.

THEOREM 1. *Let $\chi$ be an even character. Assume $2a\geqq b+1$, and define, for $x>0$,*

$$\begin{aligned} &G_1(a, b, \chi; x)\\ &\quad=x^{2a-b-1/2}\left\{\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_1(n)}{2^b(b-j)!\,j!}(4\sqrt{n/k}\,x)^j e^{-4\sqrt{n/k}x}\right.\\ &\qquad\left.+\frac{(-1)^{a-1}\varepsilon_1(b)(2b)!\,(4a-2b-2)!\,B_{2a,\bar{\chi}}L\left(b+\frac{1}{2},\chi\right)\zeta\left(2a-b-\frac{1}{2}\right)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}\right\}. \end{aligned}$$

*Then for arbitrary positive numbers $\alpha$, $\beta$ with $\alpha\beta=\pi^2$, we have*

$$G_1(a, b, \chi; \alpha)=G_1(a, b, \chi; \beta)\ . \tag{1}$$

THEOREM 2. *Let $\chi$ be an odd character. Assume $2a\geqq b$, and define, for $x>0$,*

$$\begin{aligned} &G_2(a, b, \chi; x)\\ &\quad=x^{2a-b+1/2}\left\{\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_2(n)}{2^b(b-j)!\,j!}(4\sqrt{n/k}\,x)^j e^{-4\sqrt{n/k}x}\right.\\ &\qquad\left.+\frac{(-1)^{a-1}\varepsilon_2(b)(2b)!\,(4a-2b)!\,B_{2a+1,\bar{\chi}}L\left(b+\frac{1}{2},\chi\right)\zeta\left(2a-b+\frac{1}{2}\right)}{(2a+1)\cdot b!\,(2a-b)!\,2^{6a-2b+1}\pi^{2a-b}}\right\}. \end{aligned}$$

*Then for arbitrary positive numbers $\alpha$, $\beta$ with $\alpha\beta=\pi^2$, we have*

$$G_2(a, b, \chi; \alpha)=G_2(a, b, \chi; \beta)\ . \tag{2}$$

For brevity, we put, for $x>0$,

$$\begin{aligned} &E_i(a, b, \chi; x)\\ &\quad=\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_i(n)}{2^b(b-j)!\,j!}(\pi\sqrt{n/k}\,x)^j e^{-\pi\sqrt{n/k}x}\quad(i=1, 2)\ . \end{aligned}$$

Then, by setting $(\alpha, \beta)=(2\pi, \pi/2)$ in (1), (2), we obtain the following corollaries.

COROLLARY 1. *Let $a, b$ and $\chi$ be as in Theorem 1. Then we have*

$$L\left(b+\frac{1}{2}, \chi\right)\zeta\left(2a-b-\frac{1}{2}\right)$$
$$=C_1\pi^{2a-b-1}(E_1(a, b, \chi; 2)-2^{4a-2b-1}E_1(a, b, \chi; 8)) ,$$

*where*

$$C_1=\frac{(-1)^{a-1}\varepsilon_1(b)a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}}{(2b)!\,(4a-2b-2)!\,(2^{4a-2b-1}-1)B_{2a,\bar{\chi}}} .$$

COROLLARY 2. *Let $a, b$ and $\chi$ be as in Theorem 2. Then we have*

$$L\left(b+\frac{1}{2}, \chi\right)\zeta\left(2a-b+\frac{1}{2}\right)$$
$$=C_2\pi^{2a-b}(E_2(a, b, \chi; 2)-2^{4a-2b+1}E_2(a, b, \chi; 8)) ,$$

*where*

$$C_2=\frac{(-1)^{a-1}\varepsilon_2(b)(2a+1)\cdot b!\,(2a-b)!\,2^{6a-2b+1}}{(2b)!\,(4a-2b)!\,(2^{4a-2b+1}-1)B_{2a+1,\bar{\chi}}} .$$

Now, let $K$ be a quadratic field with the discriminant $d$ and $\chi_d=\left(\frac{d}{}\right)$ the Kronecker symbol. Then it is well known that $\chi_d$ is a real primitive character modulo $|d|$ and

$$L(s, \chi_d)\zeta(s)=\zeta_K(s) , \tag{3}$$

where $\zeta_K(s)$ is the Dedekind zeta function of $K$ (cf. [1]). Then from Corollaries 1 and 2, we obtain the following results.

COROLLARY 3. *Let $K$ be a real quadratic field, and assume $2a\geqq b+1$. Then we have*

$$\zeta_K\left(b+\frac{1}{2}\right)\zeta_K\left(2a-b-\frac{1}{2}\right)$$
$$=C_3\pi^{2a-1}(E_1(a, b, \chi_d; 2)-2^{4a-2b-1}E_1(a, b, \chi_d; 8))$$
$$\times(E_1(a, 2a-b-1, \chi_d; 2)-2^{2b+1}E_1(a, 2a-b-1, \chi_d; 8)) ,$$

*where*

$$C_3=\frac{(-1)^{a}}{(2^{4a-2b-1}-1)(2^{2b+1}-1)}\left(\frac{a\cdot b!\,(2a-b-1)!\,2^{4a}}{(2b)!\,(4a-2b-2)!\,B_{2a,\chi_d}}\right)^2 .$$

COROLLARY 4. *Let $K$ be an imaginary quadratic field, and assume $2a \geqq b$. Then we have*

$$\begin{aligned}\zeta_K\Big(b+\frac{1}{2}\Big)\zeta_K\Big(2a-b+\frac{1}{2}\Big) &= C_4\pi^{2a}(E_2(a, b, \chi_d; 2)-2^{4a-2b+1}E_2(a, b, \chi_d; 8)) \\ &\quad \times (E_2(a, 2a-b, \chi_d; 2)-2^{2b+1}E_2(a, 2a-b, \chi_d; 8)) ,\end{aligned}$$

*where*

$$C_4=\frac{(-1)^{a+b}}{(2^{4a-2b+1}-1)(2^{2b+1}-1)}\Big(\frac{(2a+1)b!\,(2a-b)!\,2^{4a+1}}{(2b)!\,(4a-2b)!\,B_{2a+1,\chi_d}}\Big)^2 .$$

Throughout the rest of this section, we assume that $K$ is an imaginary quadratic field. Setting $a=b$ in Corollary 2, we get the following corollary.

COROLLARY 5.

$$\zeta_K\Big(a+\frac{1}{2}\Big)=C_5\pi^a(E_2(a, a, \chi_d; 2)-2^{2a+1}E_2(a, a, \chi_d; 8)) ,$$

*where*

$$C_5=\frac{(-1)^{a-1}\varepsilon_2(a)(2a+1)(a!)^2 2^{4a+1}}{(2a)!^2(2^{2a+1}-1)B_{2a+1,\chi_d}} .$$

From Corollaries 4 and 5, we get the following.

COROLLARY 6. *For any positive integer $a$, we have*

$$\begin{aligned}\zeta_K\Big(\frac{1}{2}\Big) &= C_6(E_2(a, 0, \chi_d; 2)-2^{4a+1}E_2(a, 0, \chi_d; 8)) \\ &\quad \times (E_2(a, 2a, \chi_d; 2)-2E_2(a, 2a, \chi_d; 8)) \\ &\quad \div (E_2(2a, 2a, \chi_d; 2)-2^{4a+1}E_2(2a, 2a, \chi_d; 8)) ,\end{aligned}$$

*where*

$$C_6=-\frac{2(2a+1)^2B_{4a+1,\chi_d}}{(4a+1)B_{2a+1,\chi_d}^2} .$$

Setting $a=b$ in Theorem 2, we get the following theorem.

THEOREM 3. *Define, for $x>0$,*

$$G(a, K_d; x)$$
$$=x^{a+1/2}\left\{\sum_{j=0}^{a}\sum_{n=1}^{\infty}\frac{2^j(2a-j)!\,g(n)}{2^a(a-j)!\,j!}(4\sqrt{n/|d|}\,x)^j e^{-4\sqrt{n/|d|}\,x}\right.$$
$$\left.+\frac{(-1)^{a-1}\varepsilon_2(a)(2a)!\,{}^2B_{2a+1,\chi_d}\zeta_K\left(a+\dfrac{1}{2}\right)}{(2a+1)(a!)^2 2^{4a+1}\pi^a}\right\},$$

*where*

$$g(n)=g_2(a, a, \chi_d; n)\ .$$

*Then for arbitrary positive numbers* $\alpha$, $\beta$ *with* $\alpha\beta=\pi^2$, *we have*

$$G(a, K_d; \alpha)=G(a, K_d; \beta)\ .$$

## §2. Proof of Theorem 1.

The method of proof is similar to that used in [3] and [4]. So it is sufficient to sketch the argument.

Suppose that $\chi$ is even, and put

$$\varphi_1(a, b, \chi; s)$$
$$=\frac{2^b\Gamma(s)\Gamma\left(s+b+\dfrac{1}{2}\right)\zeta(s)L\left(s+b+\dfrac{1}{2}, \chi\right)L(s-2a+1, \bar{\chi})\zeta\left(s-2a+b+\dfrac{3}{2}\right)}{\pi^{1/2}\left(\dfrac{2\pi}{\sqrt{k}}\right)^{2s}}\ .$$

Using the functional equations of the gamma, zeta and $L$-functions, we obtain

$$\varphi_1(a, b, \chi; s)=\varphi_1\left(a, b, \chi; 2a-b-\frac{1}{2}-s\right)\ . \tag{4}$$

Now, we define, for $t>0$,

$$F_1(a, b, \chi; t)=\sum_{j=0}^{b}C_{b,j}\left\{\sum_{n=1}^{\infty}g_1(n)(4\pi\sqrt{nt/k})^j e^{-4\pi\sqrt{nt/k}}\right\},$$

where

$$C_{b,j}=\frac{2^j(2b-j)!}{2^b(b-j)!\,j!}\ .$$

By the same way as in [4], we know that the series

$$t^{s-1}\sum_{n=1}^{\infty} g_1(n)(4\pi\sqrt{nt/k})^j e^{-4\pi\sqrt{nt/k}} \quad (0\leqq j\leqq b)$$

converges absolutely in $t>0$ and uniformly in any interval $\delta\leqq t<\infty$ with $\delta>0$. Thus we get

$$(5)\qquad \begin{aligned}&\int_0^\infty F_1(a, b, \chi; t)t^{s-1}dt\\ &=\sum_{j=0}^{b} C_{b,j}\left\{\sum_{n=1}^{\infty} g_1(n)(4\pi\sqrt{n/k})^j \int_0^\infty t^{s+j/2-1}e^{-4\pi\sqrt{nt/k}}dt\right\}\\ &=2\left(\frac{4\pi}{\sqrt{k}}\right)^{-2s}\sum_{j=0}^{b} C_{b,j}\Gamma(2s+j)\sum_{n=1}^{\infty} n^{-s}g_1(n) .\end{aligned}$$

Since

$$|g_1(n)|\leqq n^{4a+1} ,$$

it follows that

$$(6)\qquad \sum_{n=1}^{\infty} n^{-s}g_1(n)=\zeta(s)L\left(s+b+\frac{1}{2}, \chi\right)L(s-2a+1, \bar{\chi})\zeta\left(s-2a+b+\frac{3}{2}\right)$$

for $\mathrm{Re}(s)>4a+2$, and so for all $s$ (by the theorem of identity). Thus we get, from (5), (6) and the proposition of [4],

$$\varphi_1(a, b, \chi; s)=\int_0^\infty F_1(a, b, \chi; t)t^{s-1}dt .$$

Since $\varphi_1(a, b, \chi; s)$ is regular in $\mathrm{Re}(s)>2a$, Mellin's inversion formula permits us to write

$$(7)\qquad F_1(a, b, \chi; t)=\frac{1}{2\pi i}\int_{2a+1/2-i\infty}^{2a+1/2+i\infty}\varphi_1(a, b, \chi; s)t^{-s}ds .$$

Then it is easily verified that

$$(8)\qquad \begin{aligned}F_1(a, b, \chi; t)=&\frac{1}{2\pi i}\int_{-(b+1)-i\infty}^{-(b+1)+i\infty}\varphi_1(a, b, \chi; s)t^{-s}ds\\ &+\left\{\text{sum of residues of integrand at poles } s=0,\ 2a-b-\frac{1}{2}\right\} .\end{aligned}$$

Substituting $s=2a-b-1/2-S$, it follows from (4) and (7) that

$$(9)\qquad \frac{1}{2\pi i}\int_{-(b+1)-i\infty}^{-(b+1)+i\infty}\varphi_1(a, b, \chi; s)t^{-s}ds=t^{-2a+b+1/2}F_1\left(a, b, \chi; \frac{1}{t}\right) .$$

The residues in the sum are as follows:

$$\operatorname*{Res}_{s=0}(\varphi_1(a, b, \chi; s)t^{-s})$$
$$=\frac{(-1)^a\varepsilon_1(b)(2b)!\,(4a-2b-2)!\,B_{2a,\bar\chi}L\left(b+\frac{1}{2}, \chi\right)\zeta\left(2a-b-\frac{1}{2}\right)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}\,.$$

$$\operatorname*{Res}_{s=2a-b-1/2}(\varphi_1(a, b, \chi; s)t^{-s})$$
$$=\frac{(-1)^{a-1}\varepsilon_1(b)(2b)!\,(4a-2b-2)!\,B_{2a,\bar\chi}L\left(b+\frac{1}{2}, \chi\right)\zeta\left(2a-b-\frac{1}{2}\right)t^{-2a+b+1/2}}{a\cdot b!(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}\,.$$

Using (8) and (9), these calculations give the equality

$$\begin{aligned}&F_1(a, b, \chi; t)+(-1)^{a-1}A_1\\&\quad=t^{-2a+b+1/2}\left(F_1\left(a, b, \chi; \frac{1}{t}\right)+(-1)^{a-1}A_1\right),\end{aligned}$$

where

$$A_1=\frac{\varepsilon_1(b)(2b)!\,(4a-2b-2)!\,B_{2a,\bar\chi}L\left(b+\frac{1}{2}, \chi\right)\zeta\left(2a-b-\frac{1}{2}\right)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}\,.$$

Setting $t=(\alpha/\pi)^2$ and $1/t=(\beta/\pi)^2$, the above equality immediately yields Theorem 1.

## §3. Proof of Theorem 2.

Suppose that $\chi$ is odd, and put

$$\varphi_2(a, b, \chi; s)$$
$$=\frac{2^b\Gamma(s)\Gamma\left(s+b+\frac{1}{2}\right)\zeta(s)L\left(s+b+\frac{1}{2}, \chi\right)L(s-2a, \bar\chi)\zeta\left(s-2a+b+\frac{1}{2}\right)}{\pi^{1/2}\left(\frac{2\pi}{\sqrt{k}}\right)^{2s}}\,.$$

Then we obtain

$$\varphi_2(a, b, \chi; s)=\varphi_2\left(a, b, \chi; 2a-b+\frac{1}{2}-s\right). \tag{10}$$

Next, we difine, for $t>0$,

$$F_2(a, b, \chi; t)=\sum_{j=0}^{b} C_{b,j}\left\{\sum_{n=1}^{\infty} g_2(n)(4\pi\sqrt{nt/k})^j e^{-4\pi\sqrt{nt/k}}\right\},$$

where

$$C_{b,j}=\frac{2^j(2b-j)!}{2^b(b-j)!\,j!}\,.$$

By the same argument as in the proof of Theorem 1, we see that

$$F_2(a,b,\chi;t)=\frac{1}{2\pi i}\int_{2a+1-i\infty}^{2a+1+i\infty}\varphi_2(a,b,\chi;s)t^{-s}ds\,. \tag{11}$$

Then it is easily verified that

$$\begin{aligned}F_2(a,b,\chi;t)&=\frac{1}{2\pi i}\int_{-(b+1/2)-i\infty}^{-(b+1/2)+i\infty}\varphi_2(a,b,\chi;s)t^{-s}ds \\ &\quad+\left\{\text{sum of residues of integrand at poles } s=0,\ 2a-b+\frac{1}{2}\right\}.\end{aligned} \tag{12}$$

Substituting $s=2a-b+1/2-S$, it follows from (10) and (11) that

$$\frac{1}{2\pi i}\int_{-(b+1/2)-i\infty}^{-(b+1/2)+i\infty}\varphi_2(a,b,\chi;s)t^{-s}ds=t^{-2a+b-1/2}F_2\left(a,b,\chi;\frac{1}{t}\right). \tag{13}$$

The residues in the sum are as follows:

$$\operatorname*{Res}_{s=0}(\varphi_2(a,b,\chi;s)t^{-s})$$
$$=\frac{(-1)^a\varepsilon_2(b)(2b)!\,(4a-2b)!\,B_{2a+1,\bar\chi}L\left(b+\frac{1}{2},\chi\right)\zeta\left(2a-b+\frac{1}{2}\right)}{(2a+1)\cdot b!\,(2a-b)!\,2^{6a-2b+1}\pi^{2a-b}}\,.$$

$$\operatorname*{Res}_{s=2a-b+1/2}(\varphi_2(a,b,\chi;s)t^{-s})$$
$$=\frac{(-1)^{a-1}\varepsilon_2(b)(2b)!\,(4a-2b)!\,B_{2a+1,\bar\chi}L\left(b+\frac{1}{2},\chi\right)\zeta\left(2a-b+\frac{1}{2}\right)t^{-2a+b-1/2}}{(2a+1)\cdot b!\,(2a-b)!\,2^{6a-2b+1}\pi^{2a-b}}\,.$$

Using (12) and (13), these calculations give the equality

$$\begin{aligned}&F_2(a,b,\chi;t)+(-1)^{a-1}A_2\\ &\quad=t^{-2a+b-1/2}\left(F_2\left(a,b,\chi;\frac{1}{t}\right)+(-1)^{a-1}A_2\right),\end{aligned}$$

where

$$A_2=\frac{\varepsilon_2(b)(2b)!\,(4a-2b)!\,B_{2a+1,\bar\chi}L\left(b+\frac{1}{2},\chi\right)\zeta\left(2a-b+\frac{1}{2}\right)}{(2a+1)\cdot b!\,(2a-b)!\,2^{6a-2b+1}\pi^{2a-b}}\,.$$

Setting $t=(\alpha/\pi)^2$ and $1/t=(\beta/\pi)^2$ in the above equality, we can easily deduce Theorem 2.

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