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Spectral Property of Goursat Problems

Masafumi YOSHINO

Tokyo Metropolitan University

Introduction

In the study of Goursat problems, a "spectral radius" plays an important role (see [1]). For example, let us take the Goursat problem studied by J. Leray in [4]

$$(1)\qquad \begin{aligned} &\varepsilon u_{x_1x_2}=u_{x_1x_1}+u_{x_2x_2}+h(x_1, x_2)\\ &u(x_1, 0)\equiv u(0, x_2)\equiv 0 \end{aligned}$$

where $(x_1, x_2)\in C^2$ and $h(x_1, x_2)$ is analytic in a neighborhood of the origin. The spectral radius of (1) is 2 and if the condition $|\varepsilon|>2$ is satisfied (1) has one and only one analytic solution in a neighborhood of the origin.

On the other hand the result when $|\varepsilon|<2$ is rare. As far as the author knows, the best work known hitherto is that of Leray's in [4]. He introduced the function

$$\rho(t)=\liminf_{k\to\infty}\ |\sin(k\pi t)|^{1/k}$$

with $\varepsilon=2\cos(\pi t)$ and showed that if the condition $\rho(t)>0$ is satisfied, (1) has one and only one analytic solution.

In this paper we shall extend Leray's results and give sufficient conditions for the existence and uniqueness of the solution when $\rho(t)=0$. We also give the eigenfunction expansion of the solution and using this we shall study the regularity of the solution with respect to ε.

§1. Notations and results.

Let $x=(x_1, \cdots, x_d)$ be the variable in complex d-dimensional space C^d with $d\geqq 2$. We use multi-indices $\alpha=(\alpha_1, \cdots, \alpha_d)\in Z^d$ with $|\alpha|=\alpha_1+\cdots+\alpha_d$ where Z denotes the set of integers. We denote by

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e_ν $(1\leqq\nu\leqq d)$ the ν-th unit vector $(0, \cdots, 0, 1, 0, \cdots, 0)$. We say that α is non-negative if all the components are non-negative and denote by $\alpha\geqq 0$. For non-negative multi-index α we set $\alpha!=\alpha_1!\cdots\alpha_d!$. For each ν, $1\leqq\nu\leqq d$ we denote by $(\partial/\partial x_\nu)^{-1}$ the integration with respect to the variable x_ν from the origin to x_ν and set

$$D^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_d)^{\alpha_d} .$$

Here, if some α_ν is negative we understand that

$$(\partial/\partial x_\nu)^{\alpha_\nu}=((\partial/\partial x_\nu)^{-1})^{-\alpha_\nu} .$$

We shall study the existence and uniqueness of the analytic solution of the Goursat problem with constant coefficients:

$$\text{(1.1)}\qquad \begin{aligned} &\varepsilon D^\beta v=aD^{\beta+e_i-e_j}v+bD^{\beta+e_j-e_i}v+\sum_{|\alpha|\leqq|\beta|-2}C_\alpha D^\alpha v+h(x) ,\\ &(\partial/\partial x_\nu)^m v|_{x_\nu=0}=0 \quad (0\leqq m<\beta_\nu,\ \nu=1, \cdots, d)\end{aligned}$$

where $1\leqq i<j\leqq d$, $\beta=(\beta_1, \cdots, \beta_d)$ and $h(x)$ is analytic in a neighborhood of the orign.

REMARK 1.1. The problem (1.1) is equivalent to the following one:

$$\text{(1.2)}\qquad \varepsilon u=aD^{e_i-e_j}u+bD^{e_j-e_i}u+\sum_{|\alpha|\leqq-2}C_\alpha D^\alpha u+h(x) .$$

For if the Goursat problem (1.2) has an analytic solution u, the function $v=D^{-\beta}u$ is an analytic solution of (1.1). Conversely, if (1.1) has an analytic solution v the function $u=D^\beta v$ is an analytic solution of (1.2).

REMARK 1.2. If the condition $ab=0$ is satisfied the problem (1.1) has one and only one analytic solution by the result of L. Gårding's in [1]. Hence we may assume that $ab\neq 0$.

Then, by making use of the change of variables such as

$$x_i=a^{1/2}z_i,\ x_j=b^{1/2}z_j,\ z_\nu=x_\nu \text{ for } \nu\neq i, j$$

we may assume, in (1.1),

$$\text{(1.3)}\qquad a=b=1 .$$

Thus we assume the condition (1.3) from now on.

By the analyticity of $h(x)$ we have, for some constants M, $r>0$,

$$\text{(1.4)}\qquad |h_\alpha|\leqq Mr^{|\alpha|}\cdot|\alpha|!,\ \alpha\geqq 0,\ h(x)=\sum h_\alpha x^\alpha/\alpha! .$$

Moreover, by taking M sufficiently large we may assume

(1.5) $$|C_\alpha| \leqq M \text{ for all } C_\alpha \text{ in (1.1)}.$$

We define r_2 and N by

(1.6) $$r_2 = \max\{M, r, 1\}$$

(1.7) $$N = 1 + \{\text{the number of terms } C_\alpha D^\alpha u\text{'s appearing in the right-hand side of (1.1)}\}$$

THEOREM 1.1. *Suppose that* $\varepsilon \in C \backslash (-2, 2)$. *Then equation* (1.2) *has one and only one solution which is analytic in the domain* $\{x \in \boldsymbol{C}^d; |x_1| + \cdots + |x_d| < 1/(r_2 N)\}$.

Note that Theorem 1.1 says nothing about the existence or the uniqueness of the solution when $\varepsilon \in (-2, 2)$. So, in the following we shall study this case for the equation with constant coefficients:

(1.8) $$\varepsilon u = D^{e_1 - e_2} u + D^{e_2 - e_1} u + h(x)$$

where $x = (x_1, x_2) \in \boldsymbol{C}^2$.

We introduce the system of functions $\{f_l^k(x_1, x_2);\ l = 1, \cdots, k\}$ $(k = 1, 2, \cdots)$ by the following relations.

(1.9) $$[f_1^k(x_1, x_2), \cdots, f_k^k(x_1, x_2)] = \left[\frac{x_1^{k-1}}{(k-1)!}, \frac{x_1^{k-2} x_2}{(k-2)!}, \cdots, \frac{x_1 x_2^{k-2}}{(k-2)!}, \frac{x_2^{k-1}}{(k-1)!}\right] V_k$$

where the (p, q)-component of the matrix V_k is given by $c_k \sin(pq\pi/(k+1))$ with some constant c_k satisfying that $|c_k|^2 = 2/(k+1)$. Let

$$h(x) = \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} h_{k-l-1, l}\, x_1^{k-l-1} x_2^l / ((k-l-1)!\ l!)$$

be the Taylor expansion of an analytic function $h(x)$. Then, by using the relation (1.9) we rewrite this into

(1.10) $$h(x) = \sum_{k=1}^{\infty} \sum_{l=1}^{k} \tilde{h}_{k,l} f_l^k(x_1, x_2).$$

Note that the coefficients $\tilde{h}_{k,l}$'s are uniquely determined.

Let $\{r_m\}_{m=1}^{\infty}$ be the sequence of positive numbers such that $r_m \downarrow 0$ as $m \to \infty$ and let $k_0 (k_0 \gg 1)$ be an integer. For each $m (m \geqq 1)$ we define

(1.11) $$C_m = \{k \in \boldsymbol{Z};\ k \geqq k_0 \text{ and } \inf_{q \in \boldsymbol{Z}} |q - tk|^{1/k} \geqq r_m\}$$

and put

(1.12) $$H_0=\{h(x);\ h(x) \text{ is analytic at the origin}\}\ ,$$

(1.13) $$H_m=\{h(x)\in H_0;\ \tilde{h}_{k,l}=0 \text{ for all } k,\ l \text{ such that } 1\leqq l\leqq k,\ k\in Z\backslash C_m\}.$$

For irrational number $t(0<t<1)$ we expand t in the continued fraction $t=[a_1, a_2, \cdots]$ with

$$a_1=[1/t],\ \alpha_2=1/t-a_1,\ \cdots,\ a_n=[1/\alpha_n],\ \alpha_{n+1}=1/\alpha_n-a_n,\ \cdots ;$$

where $[\mu]$ denotes the largest integer $\leqq\mu$. Then we determine the relatively prime integers p_{n+2} and q_{n+2} by

(1.14) $$\frac{p_{n+2}}{q_{n+2}}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\ \cfrac{1}{a_n}}}}$$

and define the set of irrational numbers $J_\gamma(\gamma\geqq 0)$ by

(1.15) $$J_\gamma=\{t;\ 0<t<1,\ t \text{ is irrational and satisfies } a_n\leqq R(q_{n+1})(q_{n+1}!)^\gamma,\ n=1, 2, \cdots\}.$$

Here $R(s)$ is the function of one variable such that, for some constants $K, N>0$,

$$0\leqq R(s)\leqq Ks^{(\log s)^N} \text{ for all } s>0\ .$$

NOTE. For example, we can take $R(s)$ as the polynomial of s.

Let an analytic function $h(x)$ be expanded in (1.4). Then we define the class of entire functions $B_\eta(\eta\geqq 0)$ of order $1/\eta$ by

(1.16) $$B_\eta=\{h(x)\in H_0;\ |h_\alpha|\leqq M_0 r_1^{|\alpha|}(\alpha!)^{1-\eta} \text{ for some constants } M_0,\ r_1>0 \text{ independent of } \alpha\}.$$

Note that $H_0=B_0$.

For each $\varepsilon\in(-2, 2)$ we determine $t(0<t<1)$ by

(1.17) $$\varepsilon=2\cos(\pi t)$$

and define Leray's auxiliary function $\rho(t)$ by

(1.18) $$\rho(t)=\liminf_{k\to\infty}\ (\inf_{l\in Z}|l-tk|^{1/k})\ .$$

Then we classify the open interval $(-2, 2)$ following Leray [4]:

$$I\colon \rho(t)>0\,,$$

$$II\colon\ \rho(t)=0\begin{cases} II_a;\ t \text{ is rational.}\\ II_b;\ t \text{ is irrational.}\end{cases}$$

REMARK 1.3. It was proved by J. Leray and C. Pisot in [5] that though the Lebesgue measure of the set of t satisfying $\rho(t)=0$ is zero it has the density of continuum.

THEOREM 1.2. *Suppose that* $\rho(t)>0$ *and that* $h(x)$ *be expanded in* (1.10). *Then equation* (1.8) *has one and only one analytic solution* $u(x, \varepsilon)$ *in a neighborhood of the origin. Moreover we have the expression*

$$u(x, \varepsilon)=\sum_{k=1}^{\infty}\sum_{l=1}^{k}(\varepsilon-\varepsilon_l^k)^{-1}\tilde{h}_{k,l}f_l^k(x_1, x_2) \tag{1.19}$$

with $\varepsilon_l^k=2\cos(l\pi/(k+1))$ $(l=1, 2\cdots, k,\ k=1, 2, \cdots)$.

REMARK 1.4. In proving Theorem 1.2 we can also prove that the expression (1.19) is valid when $\varepsilon\notin[-2, 2]$.

Next we shall study the case II_a. Write $t=q_0/p_0$ with relatively prime positive integers p_0, q_0 and let $h(x)$ be expanded in (1.10). Then

THEOREM 1.3. *Suppose that* $t=q_0/p_0$. *Then equation* (1.8) *admits an analytic solution if and only if* $h(x)$ *satisfies*

$$\sum\tilde{h}_{k,l}f_l^k(x_1, x_2)\equiv 0 \tag{1.20}$$

where the summation is taken for all k *and* l *such that* $l/(k+1)=q_0/p_0$.

The solutions of (1.8) *for* $h=0$ *are given by*

$$K=\Big\{\sum_{\substack{k,l\\ l/(k+1)=q_0/p_0}} b_{k,l}f_l^k(x_1, x_2);\ b_{k,l}\text{'s are arbitrary if this sum converges}\Big\}\,. \tag{1.21}$$

REMARK 1.5. In proving Theorem 1.3 we can prove that the formal solution $u(x, \varepsilon)$ of (1.8) has the expression

$$u(x, \varepsilon)=\sum(\varepsilon-\varepsilon_l^k)^{-1}\tilde{h}_{k,l}f_l^k(x_1, x_2)+\sum b_{k,l}f_l^k(x_1, x_2) \tag{1.22}$$

where the summations of the first and second terms are taken for all positive integers k and l such that $l/(k+1)\neq q_0/p_0$, $1\leqq l\leqq k$ and $l/(k+1)=q_0/p_0$, $1\leqq l\leqq k$ respectively. Here the number $b_{k,l}$'s are arbitrary constants and the first term of the right-hand side of (1.22) converges and represents an analytic function of x.

REMARK 1.6. It follows from Theorem 1.3 and Remark 1.5 that if equation (1.8) has a formal solution, it has infinitely many analytic solutions.

Next we shall study the case II_b. Let H_0 and H_m be defined by (1.12) and (1.13) respectively and define the operator P by $P=\varepsilon-D^{e_1-e_2}-D^{e_2-e_1}$. Then we have

THEOREM 1.4. *The analytic solution of* (1.8) *is unique and has the representation* (1.19). *Moreover we have*

$$H_0 \supsetneqq PH_0 \supsetneqq H_m, \quad m=1, 2, \cdots . \tag{1.23}$$

If $t \in J_\gamma$ *for some* $\gamma \geqq 0$, *we have*

$$PH_0 \supset B_\eta \text{ for all } \eta \geqq \gamma . \tag{1.24}$$

§ 2. Proof of Theorems 1.1–1.4.

If ε is not contained in $[-2, 2]$ Theorem 1.1 reduces to the special case of the result of the preceeding paper [7]. So we may restrict ourselves to the case

$$\varepsilon = \pm 2 . \tag{2.1}$$

For the sake of simplicity we assume, for the equation (1.1),

$$i=1, \quad j=2 . \tag{2.2}$$

PROOF OF THEOREM 1.1. We devide the proof into three steps.

Step 1. Substituting the expansions $u(x)=\sum u_\beta x^\beta/\beta!$ and (1.4) into (1.2) and comparing the coefficients of x^β we have

$$\varepsilon u = u_{\beta+e_1-e_2} + u_{\beta+e_2-e_1} + \sum_{|\alpha| \leq -2} C_\alpha u_{\alpha+\beta} + h_\beta . \tag{2.3}$$

Here we used the assumption (1.3). On the other hand the holomorphy of $u(x)$ implies

$$u_\gamma = 0 \quad \text{if } \gamma \ngeqq 0 . \tag{2.4}$$

For each k ($k=1, 2, \cdots$) we wish to determine u_β with $|\beta|=k-1$ by (2.3). For this we take a multi-index $\tilde{\beta}$ such that $|\tilde{\beta}|=k-p(1 \leqq p \leqq k)$, $\tilde{\beta}=(0, 0, \tilde{\beta}_2, \cdots, \tilde{\beta}_d)=(0, 0, \tilde{\beta}'')$ and set

$$
\begin{aligned}
&U_{p,\widetilde{\beta}} = {}^t(u_{p-1,0,\widetilde{\beta}''}, u_{p-2,1,\widetilde{\beta}''}, \cdots, u_{1,p-2,\widetilde{\beta}''}, u_{0,p-1,\widetilde{\beta}''}) , \\
&H_{p,\widetilde{\beta}} = {}^t(h_{p-1,0,\widetilde{\beta}''}, h_{p-2,1,\widetilde{\beta}''}, \cdots, h_{1,p-2,\widetilde{\beta}''}, h_{0,p-1,\widetilde{\beta}''}) , \\
&U_{p,\alpha,\widetilde{\beta}} = {}^t(u_{\alpha_1+p-1,\alpha_2,\alpha''+\widetilde{\beta}''}, u_{\alpha_1+p-2,\alpha_2+1,\alpha''+\widetilde{\beta}''}, \cdots, \\
&\qquad u_{\alpha_1+1,\alpha_2+p-2,\alpha''+\widetilde{\beta}''}, u_{\alpha_1,\alpha_2+p-1,\alpha''+\widetilde{\beta}''}) ,
\end{aligned}
\tag{2.5}
$$

where $\alpha=(\alpha_1, \alpha_2, \alpha'')=(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_d)$. Then we write the equations (2.4) and (2.3) with $|\beta|=k-1$ in

$$A_p U_{p,\widetilde{\beta}} = \sum_{|\alpha|\leq -2} C_\alpha U_{p,\alpha,\widetilde{\beta}} + H_{p,\widetilde{\beta}} . \tag{2.6}$$

Here the p by p matrix A_p is given by

$$A_p=(a^p_{ij}) \quad a^p_{ij}=\begin{cases} \varepsilon, & i=j \\ -1, & i-j=\pm 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Step 2. We shall prove the existence of the formal solution. To determine u_β ($|\beta|=k-1$, $k\geqq 1$) by (2.6) we shall study A_p. We set

$$I_p=\det A_p , \tag{2.8}$$

then, by (2.7) and simple computations we have

$$I_1=\varepsilon, \ I_2=\varepsilon^2-1 . \tag{2.9}$$

By expanding $\det A_p$ with respect to the first row we obtain the difference equation:

$$I_{p+1}=\varepsilon I_p - I_{p-1} \quad (p=2, 3, \cdots) . \tag{2.10}$$

By solving (2.10) under the condition (2.9) we have

$$
\begin{aligned}
I_p &= (\varepsilon^2-1)^{-1/2}\{((\varepsilon+\sqrt{\varepsilon^2-4})/2)^{p+1} \\
&\qquad -((\varepsilon-\sqrt{\varepsilon^2-4})/2)^{p+1}\} \quad \text{when } \varepsilon\neq\pm 2 . \\
I_p &= (\pm 1)^p(p+1) \qquad\qquad \text{when } \varepsilon=\pm 2 .
\end{aligned}
\tag{2.11}
$$

By (2.11) we can see that the zeros of $I_p\equiv I_p(\varepsilon)$ are contained in the open interval $(-2, 2)$. Thus, by the assumption $\varepsilon\notin(-2, 2)$ we can uniquely determine $u_\beta(|\beta|=k-1)$ by (2.6). Hence equation (1.2) has the unique formal solution.

Step 3. In this step we use the condition (2.1) and prove the convergence of the formal solution.

By (2.11) and the assumption $\varepsilon=\pm 2$, A_p^{-1} exists. Hence, by setting $A_p^{-1}=(b^p_{ij})$ and by computing the cofactor of $\det A_p$ we have

$$b_{ij}^{p}=\begin{cases} I_{j-1}I_{p-i}/I_{p} & \text{when } i\geqq j \\ I_{i-1}I_{p-j}/I_{p} & \text{when } i<j \end{cases}$$

where $I_\nu=\det A_\nu(\nu\geqq 1)$ and we consider $I_0=1$. Hence, it follows from (2.11) that the absolute value of b_{ij}^p is equal to $j(p-i+1)/(p+1)$ if $i\geqq j$ and equal to $i(p-j+1)/(p+1)$ if $i<j$. Hence, if we write $x=(p+1-i)/(p+1)$ for $i\geqq j$ and $x=(p+1-j)/(p+1)$ for $i<j$ we have

$$|b_{ij}^p|\leqq x(1-x)(p+1)\leqq(p+1)/4 \quad (0\leqq x\leqq 1) \tag{2.12}$$

By (1.4), (2.6), Lemma 2 and the assumption $|\alpha|\leqq -2$ we see that the absolute value of u_β is smaller than $M/2$ if $|\beta|=0$ and than $3\cdot 2rM/4$ if $|\beta|=1$. For the estimate of u_β with $|\beta|=2$, by using (2.4) and the assumption $|\alpha|\leqq -2$ we see that any component of $U_{p,\alpha,\tilde{\beta}}$ in (2.5) has the form $u_\gamma(|\gamma|=0)$. So, by using (2.4), (2.12) and the estimates of h_α, C_α in (1.4), (1.5) we get, from (2.6),

$$|u_\beta|\leqq 3(Mr^2 2!+M^2(N-1)/2), \ |\beta|=2 . \tag{2.13}$$

By (1.6) and (1.7), the right-hand side of (2.13) is smaller than $(4!)\times(r_2N)^3$. By induction on $k=|\beta|+1$, we can generally prove

$$|u_\beta|\leqq((|\beta|+2)!)(r_2N)^{|\beta|+1}$$

for any $\beta\geqq 0$. Hence

$$\begin{aligned}|\sum u_\beta x^\beta/\beta!|&\leqq\sum((|\beta|+2)!)(r_2N)^{|\beta|+1}|x_1|^{\beta_1}\cdots|x_d|^{\beta_d}/\beta!\\ &=\sum_{\nu=0}^{\infty}\sum_{|\beta|=\nu}(\nu+2)(\nu+1)(r_2N)^{\nu+1}|\beta|!\ \ |x_1|^{\beta_1}\cdots|x_d|^{\beta_d}/\beta!\\ &=\sum_{\nu=0}^{\infty}(\nu+2)(\nu+1)(r_2N)^{\nu+1}(|x_1|+\cdots+|x_d|)^\nu .\end{aligned}$$

The last term converges if $|x_1|+\cdots+|x_d|<1/(r_2N)$. This ends the proof.

REMARK 2.1. The function $\rho(t)$ given in (1.18) was first introduced by J. Leray in [4] but its meaning was not clear. By using (2.11) we can interpret $\rho(t)$ as follows.

By the definition of $\rho(t)$ and by using the inequality $m\cdot\inf_{p\in Z}|kt-p|\leqq|\sin(k\pi t)|\leqq M\cdot\inf_{p\in Z}|kt-p|(k=1, 2, \cdots)$ for m, $M>0$ we have

$$\rho(t)=\liminf_{k\to\infty}|\sin(k\pi t)|^{1/k} .$$

On the other hand, by (1.17) and the first identity of (2.11) we get

$$\mathrm{I}_k=\det A_k=\sin((k+1)t\pi)/(\sin t\pi) .$$

Hence we have

$$\rho(t)=\liminf_{k\to\infty} |\det A_k|^{1/(k+1)}$$

This expression is important in the investigation of our subsequent studies. To prove Theorems 1.2–1.4 we prepare three lemmas.

LEMMA 2.1. *Let V_k be the matrix whose (p, l)-component is given by $c_k \sin(pl\pi/(k+1))$ where c_k is a constant satisfying that $|c_k|^2=2/(k+1)$. Then we have, for $k=1, 2, \cdots$,*

$$V_k^{-1}A_k V_k=D_k=\operatorname{diag}[\varepsilon-\varepsilon_1^k, \cdots, \varepsilon-\varepsilon_k^k] \tag{2.14}$$

where $\varepsilon_l^k(1\leqq l\leqq k)$ is given by

$$\varepsilon_l^k=2\cos(l\pi/(k+1)) . \tag{2.15}$$

Lemma 2.1 is an elementary consequence of the symmetry of the matrix A_k. The constant c_k is necessary for V_k to be a unitary matrix.

LEMMA 2.2. *Let $\varepsilon=2\cos\pi t(0<t<1)$ and let ε_l^k be given by* (2.15). *Then we have, for $\gamma\geqq 0$,*

$$\liminf_{k\to\infty}\big(\inf_{1\leqq l\leqq k} k^\gamma|\varepsilon-\varepsilon_l^k|^{1/(k+1)}\big)=\liminf_{k\to\infty}\big(\inf_{l\in Z} k^r|l-tk|^{1/k}\big) . \tag{2.16}$$

If $t=q_0/p_0$ for some integers q_0 and p_0 we have

$$\liminf_{k\to\infty}(\inf|\varepsilon-\varepsilon_l^k|^{1/(k+1)})=1 \tag{2.17}$$

where the infimum is taken with respect to l such that $1\leqq l\leqq k$ and $q_0/p_0\neq l/(k+1)$.

PROOF. We shall prove (2.16). It follows from the conditions $0<t<1$ and $0<l/(k+1)<1$ that

$$t/2<(t+l/(k+1))/2<(t+1)/2,\quad -1/2<(t-l/(k+1))/2<1/2 .$$

Hence there exist constants κ, $\delta>0$ depending only on t such that

$$\begin{aligned}&\left|\sin\frac{\pi}{2}\Big(t+\frac{l}{k+1}\Big)\right|\geqq\delta>0 ,\\ &\kappa^{-1}\left|\frac{\pi}{2}\Big(t-\frac{l}{k+1}\Big)\right|\leqq\left|\sin\frac{\pi}{2}\Big(t-\frac{l}{k+1}\Big)\right|\leqq\kappa\left|\frac{\pi}{2}\Big(t-\frac{l}{k+1}\Big)\right|\end{aligned} \tag{2.18}$$

for $1\leqq l\leqq k$, $k=1, 2, \cdots$.

On the other hand we have

$$|\varepsilon-\varepsilon_l^k|=\left|2\cos(\pi t)-2\cos\frac{l\pi}{k+1}\right|=4\left|\sin\frac{\pi}{2}\left(t-\frac{l}{k+1}\right)\right|\left|\sin\frac{\pi}{2}\left(t+\frac{l}{k+1}\right)\right| .$$

Therefore, by (2.18) we have

$$\begin{aligned}\liminf_{k\to\infty}\left(\inf_{1\leq l\leq k} k^\gamma|\varepsilon-\varepsilon_l^k|^{1/(k+1)}\right)&=\liminf_{k\to\infty}\left(\inf k^\gamma\left|\sin\frac{\pi}{2}\left(t-\frac{l}{k+1}\right)\right|^{1/(k+1)}\right)\\&=\liminf_{k\to\infty}\left(\inf k^\gamma\left|t-\frac{l}{k+1}\right|^{1/(k+1)}\right)=\liminf_{k\to\infty}\left(\inf_{l\in Z} k^\gamma|l-tk|^{1/k}\right)\end{aligned}$$

where we used $\lim_{k\to\infty} k^{1/k}=1$. This ends the proof of (2.16). We can prove (2.17) similarly. Thus we have proved Lemma 2.2.

LEMMA 2.3. *Let $J_\gamma(\gamma\geqq 0)$ be given by* (1.15) *and suppose that t is contained in J_γ. Then*

$$\liminf_{k\to\infty}\left(\inf_{l\in Z} k^\gamma|l-tk|^{1/k}\right)>0 .$$

This lemma is proved in [5] when $\gamma=0$. But the proof there is also applicable to the case $\gamma>0$ after slight modifications. So we omit the proof.

PROOF OF THEOREM 1.2. We prove Theorem 1.2 in two steps.

Step 1. Substituting the expansions $u(x)=\sum u_{\nu,\mu}x_1^\nu x_2^\mu/(\nu!\mu!)$ and $h(x)=\sum h_{\nu,\mu}x_1^\nu x_2^\mu/(\nu!\mu!)$ into (1.8) we get

$$\varepsilon u_{\nu,\mu}-u_{\nu+1,\mu-1}-u_{\nu-1,\mu+1}=h_{\nu,\mu},\ \nu,\mu\geqq 0 . \tag{2.19}$$

On the other hand the holomorphy of $u(x)$ implies

$$u_{\nu,\mu}=0 \quad\text{if } \nu<0 \text{ or } \mu<0 . \tag{2.20}$$

For each $k(k=1,2\cdots)$ we define the vectors U_k and H_k by

$$U_k={}^t(u_{k-1,0}, u_{k-2,1}, \cdots, u_{1,k-2}, u_{0,k-1}),\ H_k={}^t(h_{k-1,0}, h_{k-2,1}, \cdots, h_{1,k-2}, h_{0,k-1})$$

and write (2.19) in the matrix notation:

$$A_kU_k=H_k \tag{2.21}$$

where the k by k matrix A_k is given by (2.7).

Now we introduce the system of functions $\{f_l^k(x_1,x_2);\ 1\leqq l\leqq k\}$ $(k=1,2,\cdots)$ by (1.9) and expand a holomorphic function $g(x)$ into

$$g(x)=\sum_{k=1}^\infty\sum_{l=1}^k\tilde g_{k,l}f_l^k(x_1,x_2)=\sum_{k=1}^\infty\sum_{l=0}^{k-1}g_{k-l-1,l}\frac{x_1^{k-l-1}x_2^l}{(k-l-1)!l!} .$$

If we set

$$\tilde{G}_k = {}^t(\tilde{g}_{k,1}, \cdots, \tilde{g}_{k,k}), \ G_k = {}^t(g_{k-1,0}, \cdots, g_{0,k-1})$$

we have, for $k=1, 2, \cdots$,

$$\tilde{G}_k = V_k G_k . \tag{2.22}$$

We expand $h(x)$ and $u(x)$ in (1.10) and in

$$u(x) = \sum_{k=1}^{\infty} \sum_{l=1}^{k} \tilde{u}_{k,l} f_l^k(x_1, x_2) . \tag{2.23}$$

Then, noting (2.14) we rewrite (2.21) into

$$D_k \tilde{U}_k = \tilde{H}_k \tag{2.24}$$

where the matrix D_k is given by (2.14) and the vectors $\tilde{U}_k$ and $\tilde{H}_k$ are given by $\tilde{U}_k = {}^t(\tilde{u}_{k,1}, \cdots, \tilde{u}_{k,k})$ and $\tilde{H}_k = {}^t(\tilde{h}_{k,1}, \cdots, \tilde{h}_{k,k})$ respectively. Since we have $\varepsilon \neq \varepsilon_l^k (1 \leqq l \leqq k, \ k=1, 2, \cdots)$ by the assumption of Theorem 1.2 we get, from (2.24),

$$\tilde{u}_{k,l} = (\varepsilon - \varepsilon_l^k)^{-1} \tilde{h}_{k,l}, \ l=1, \cdots, k . \tag{2.25}$$

Therefore the formal solution exists and is unique. Moreover, by (2.23) and (2.25) we get the expression (1.19).

Step 2. We shall prove the convergence of (1.19). By remembering the coefficients of the $(k-1)$-th homogeneous part of $h(x)$ and the fact that V_k is unitary, we obtain

$$|\tilde{h}_{k,l}| \leqq M r_1^{k-1} k! , \ l=1, \cdots, k, \ k=1, 2, \cdots . \tag{2.26}$$

On the other hand the definition of $f_l^k(x_1, x_2)$ implies, for $1 \leqq l \leqq k$, $k=1, 2, \cdots$,

$$\begin{aligned} |f_l^k(x_1, x_2)| &\leqq \sum_{p=1}^{k} |c_k| \, |x_1|^{k-p} |x_2|^{p-1} / ((k-p)! \, (p-1)!) \\ &\leqq ((k-1)!)^{-1} \sum \binom{k-1}{p-1} |x_1|^{k-p} |x_2|^{p-1} = (|x_1| + |x_2|)^{k-1} / ((k-1)!) . \end{aligned} \tag{2.27}$$

Since $\rho(t) > 0$, it follows from (1.18) and (2.16) with $\gamma = 0$ that

$$|\varepsilon - \varepsilon_l^k| \geqq \rho^{k+1}, \ 1 \leqq l \leqq k, \ k \geqq k_1 \tag{2.28}$$

for some number $\rho > 0$ and integer k_1. Hence, by (2.26)–(2.28) we get

$$\sum_{k \geqq k_1} \sum_{l=1}^{k} |\varepsilon - \varepsilon_l^k|^{-1} |\tilde{h}_{k,l}| \, |f_l^k(x_1, x_2)| \leqq \sum_{k \geqq k_1} k^2 M \rho^{-2} ((|x_1| + |x_2|) r_1 / \rho)^{k-1} .$$

Thus the formal sum (1.19) converges uniformly with respect to x in a neighborhood of the origin. This ends the proof of Theorem 1.2.

PROOF OF THEOREM 1.3. By the same arguments as in the proof of Theorem 1.2 we get (2.24). On the other hand it follows from the assumption $t=q_0/p_0$, (1.17), (2.14) and (2.15) that the l-th diagonal element of D_k vanishes if and only if l and k satisfy

$$q_0/p_0=l/(k+1)\ . \tag{2.29}$$

Hence equation (2.24) has a formal solution if and only if the condition $\tilde{h}_{k,l}=0$ is fulfilled for all positive integers l and k satisfying (2.29), that is, if and only if $h(x)$ satisfies (1.20). Moreover, by (2.24) the formal solution has the expression (1.22).

By using (2.17), (2.26) and (2.27) we can prove that the first term of (1.22) converges. Thus we have proved the former half of Theorem 1.3. The latter half is the direct consequence of the expression (1.22). This completes the proof of Theorem 1.3.

PROOF OF THEOREM 1.4. Since we have $\varepsilon\neq\varepsilon_l^k(1\leqq l\leqq k,\ k\geqq 1)$ by the definition of the case II_b, we see that the formal solution of (1.8) exists and has the representation (1.19).

First we shall prove (1.24). For this we have to prove the convergence of (1.19). Suppose that $t\in J_\gamma$ and $h\in B_\eta$. Then, by Lemma 2.3 and (2.16) there exist a constant $r_3>0$ and an integer k_0 such that

$$|\varepsilon-\varepsilon_l^k|>r_3^{k+1}k^{-\gamma(k+1)},\ 1\leqq l\leqq k,\ k\geqq k_0\ . \tag{2.30}$$

On the other hand the definition of B_η implies,

$$|h_{\nu,\mu}|\leqq M_0r_1^{k-1}((k-1)!)^{1-\eta},\ \nu+\mu=k-1,\ k\geqq 1$$

for some $M_0>0$, $r_1>0$. Hence it follows from (2.22) that

$$|\tilde{h}_{k,l}|\leqq M_0r_1^{k-1}k((k-1)!)^{1-\eta},\ 1\leqq l\leqq k,\ k=1,2,\cdots \tag{2.31}$$

By (2.27), (2.30) and (2.31) we have

$$\sum_{k\geqq k_2}\sum_{l=1}^{k}|\varepsilon-\varepsilon_l^k|^{-1}|\tilde{h}_{k,l}|\,|f_l^k(x_1,x_2)|\leqq\sum\sum r_3^{-k-1}k^{\gamma(k+1)+1}M_0r_1^{k+1}((k-1)!)^{-\eta}(|x_1|+|x_2|)^{k-1}\ .$$

It follows from the condition $\eta\geqq\gamma$ and the Stirling's formula that the last term converges if the condition $|x_1|+|x_2|<e^{-\gamma}r_3/r_1$ is satisfied. This ends the proof of (1.24).

Next we shall prove (1.23). By the similar arguments as in the proof of Theorem 1.2 we can prove that the image PH_0 of H_0 by P contains $H_m (m=1, 2, \cdots)$. By (1.24) and that $B_\eta \not\subset H_m (m=1, 2, \cdots)$ we see that $PH_0 \neq H_m$ for any positive integer m.

To complete the proof of (1.23) we shall construct a function $h \in H_0$ such that $h \notin PH_0$. It follows from Remark 2.1 and the condition $\rho(t)=0$ that

$$|I_k| < r_m^{k(m)+1}, \quad m=1, 2, \cdots, \tag{2.32}$$

for some monotone increasing sequence $\{k(m)\}_{m=1}^{\infty}$ of integers. We set

$$\operatorname{sgn} a = \begin{cases} 1 & (a>0) \\ -1 & (a<0) \end{cases}$$

and define the desired function $h(x)=\sum h_{ij} x_1^i x_2^j/(i!j!)$ by

$$h_{k-n,n-1} = \begin{cases} (k-1)! \operatorname{sgn}(I_{k-n}/I_k) & \text{when } k=k(m) \\ 0 & \text{when } k \neq k(m) \end{cases}$$

where $n=1, \cdots, k$, $m=1, 2, \cdots$, and $I_p (p=1, 2, \cdots)$ are given by (2.8). Then, by using (2.21), (2.32) and the result of the proof of Lemma 2.1, we see that equation (1.8) has no analytic solution for this h. Thus we have proved (1.23) and this ends the proof of Theorem 1.4.

§ 3. An application of the eigenfunction expansion.

It follows from (1.19) that the solution $u(x, \varepsilon)$ of (1.8) is analytic with respect to ε in $C \backslash [-2, 2]$. On the contrary it is not analytic when $\varepsilon \in [-2, 2]$. By making use of the eigenfunction expansions (1.19) and (1.22) we shall study the regularity of $u(x, \varepsilon)$ when $\varepsilon \in [-2, 2]$ in this section.

Let ε_0 and θ satisfy

$$\theta > 0, \quad -2 \leqq \varepsilon_0 \leqq 2 . \tag{3.1}$$

Then we define the sectors $S_\theta(\varepsilon_0)$ by

$$S_\theta(\varepsilon_0) = \{\varepsilon \in C;\ \varepsilon \neq \varepsilon_0,\ \theta < |\arg(\varepsilon-\varepsilon_0)| < \pi-\theta\} \text{ for } \varepsilon_0 \neq \pm 2 .$$
$$S_\theta(2) = \{\varepsilon \in C;\ \varepsilon \neq 2,\ |\arg(\varepsilon-2)| < \pi-\theta\} ,$$
$$S_\theta(-2) = \{\varepsilon \in C;\ \varepsilon \neq -2,\ |\arg(\varepsilon+2)| > \theta\} .$$

We set

$$\varepsilon_0 = 2\cos(\pi t_0), \quad -1 \leqq t_0 \leqq 1 . \tag{3.2}$$

First we describe the asymptotic estimates of ε in these sectors.

LEMMA 3.1. *Let* $\varepsilon_l^k = 2\cos(l\pi/(k+1))$ *and suppose that the numbers* θ *and* ε_0 *satisfy* (3.1). *Then*

$$|\varepsilon_0 - \varepsilon_l^k| \leqq K|\varepsilon - \varepsilon_l^k| , \tag{3.3}$$

$$|\varepsilon - \varepsilon_0| \leqq K|\varepsilon - \varepsilon_l^k| , \tag{3.4}$$

for $1 \leqq l \leqq k$, $k=1, 2, \cdots$, $\forall \varepsilon \in S_\theta(\varepsilon_0)$. *Here* $K = 1 + (\tan\theta)^{-1}$.

PROOF. We shall prove (3.3) when $-2<\varepsilon_0<2$ and $\operatorname{Im}\varepsilon \geqq 0$. For this we consider following six cases:

(1) $\varepsilon_0 \leqq \varepsilon_l^k \leqq \operatorname{Re}\varepsilon$. (2) $\varepsilon_0 \leqq \operatorname{Re}\varepsilon \leqq \varepsilon_l^k$.
(3) $\varepsilon_l^k \leqq \varepsilon_0 \leqq \operatorname{Re}\varepsilon$. (4) $\operatorname{Re}\varepsilon \leqq \varepsilon_l^k \leqq \varepsilon_0$.
(5) $\varepsilon_l^k \leqq \operatorname{Re}\varepsilon \leqq \varepsilon_0$. (6) $\operatorname{Re}\varepsilon \leqq \varepsilon_0 \leqq \varepsilon_l^k$.

Case (1). Let ε' and ε'' be the points on the boundary of $S_\theta(\varepsilon_0)$ such that

$$\operatorname{Re}\varepsilon = \operatorname{Re}\varepsilon',\ \operatorname{Re}\varepsilon'' = \varepsilon_l^k,\ \operatorname{Im}\varepsilon' \geqq 0,\ \operatorname{Im}\varepsilon'' \geqq 0 .$$

Then we have

$$\begin{aligned} &|\varepsilon - \varepsilon_l^k| \geqq |\varepsilon' - \varepsilon_l^k| . \\ &|\varepsilon' - \varepsilon_l^k| \geqq |\varepsilon'' - \varepsilon_l^k| . \end{aligned} \tag{3.5}$$

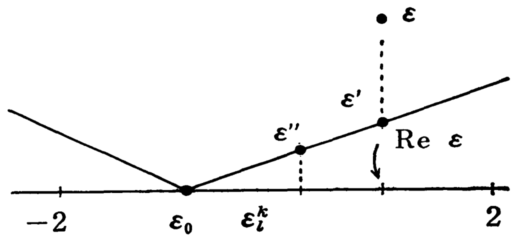

FIGURE 1

By (3.5) we have

$$|\varepsilon_0 - \varepsilon_l^k| = (\tan\theta)^{-1}|\varepsilon'' - \varepsilon_l^k| \leqq (\tan\theta)^{-1}|\varepsilon - \varepsilon_l^k| .$$

Case (2). Let ε' be the point on the boundary of $S_\theta(\varepsilon_0)$ such that $\operatorname{Re}\varepsilon = \operatorname{Re}\varepsilon'$ and $\operatorname{Im}\varepsilon' \geqq 0$. Then we have that $|\varepsilon - \varepsilon_l^k| \geqq |\varepsilon' - \varepsilon_l^k|$. Therefore we get

$$|\varepsilon_0-\varepsilon_l^k|=|\varepsilon_0-\text{Re}\,\varepsilon|+|\text{Re}\,\varepsilon-\varepsilon_l^k|=(\tan\theta)^{-1}|\varepsilon'-\text{Re}\,\varepsilon|+|\text{Re}\,\varepsilon-\varepsilon_l^k|$$
$$\leqq(\tan\theta)^{-1}|\varepsilon-\varepsilon_l^k|+|\varepsilon-\varepsilon_l^k|=((\tan\theta)^{-1}+1)|\varepsilon-\varepsilon_l^k|\ .$$

Case (3). We can easily see that $|\varepsilon_0-\varepsilon_l^k|\leqq|\varepsilon-\varepsilon_l^k|$.

The proofs of the cases (4), (5), (6) are quite similar to those of the cases (1), (2), (3) respectively. Therefore we have proved (3.3) when $\varepsilon\in S_\theta(\varepsilon_0)$, $\text{Im}\,\varepsilon\geqq 0$, $-2<\varepsilon_0<2$. Similarly we have (3.3) when $\varepsilon\in S_\theta(\varepsilon_0)$ $(-2<\varepsilon_0<2)$, $\text{Im}\,\varepsilon\leqq 0$. Also we can prove (3.3) when ε is in $S_\theta(2)$ or $S_\theta(-2)$. Hence we have proved (3.3).

The proof of (3.4) is quite similar to that of (3.3). So we omit the proof. This ends the proof of Lemma 3.1.

THEOREM 3.1. *Suppose that either the condition* $t_0=\pm 1$ *or the condition* $\rho(t_0)>0$ *is satisfied. Then the analytic solution* $u(x,\varepsilon)$ *of* (1.8) *is* C^∞ *at* $\varepsilon=\varepsilon_0$ *with respect to* ε *in the following sense:*

For any $\theta>0$ *and for each non-negative integer* n *we have*

$$\lim_{\substack{\varepsilon\to\varepsilon_0\\ \varepsilon\in S_\theta(\varepsilon_0)}}((\partial/\partial\varepsilon)^n u)(x,\varepsilon)=(\partial/\partial\varepsilon)^n u(x,\varepsilon)|_{\varepsilon=\varepsilon_0}\ . \tag{3.6}$$

Here the right-hand side means that we differentiate $u(x,\varepsilon)$ *n-times in* $C\backslash[-2,2]$ *and substitute* ε *with* ε_0.

PROOF. We make use of the expression (1.19). By (2.16) with $\gamma=0$ and the definition of $\rho(t_0)$ we obtain (2.28) with $\varepsilon=\varepsilon_0$. On the other hand the holomorphy of $h(x)$ implies (2.26). Hence, by (2.26), (2.27) and (2.28) with $\varepsilon=\varepsilon_0$, the right-hand side of (3.6) exists. Then, by (1.19) we have

$$\begin{aligned}&|\textstyle\sum\sum(-1)^n(n!)(\varepsilon-\varepsilon_l^k)^{-n-1}\tilde{h}_{k,l}f_l^k(x_1,x_2)\\ &\quad-\textstyle\sum\sum(-1)^n(n!)(\varepsilon_0-\varepsilon_l^k)^{-n-1}\tilde{h}_{k,l}f_l^k(x_1,x_2)|\\ &\quad\leqq\textstyle\sum\sum(n!)|(\varepsilon-\varepsilon_l^k)^{-n-1}-(\varepsilon_0-\varepsilon_l^k)^{-n-1}|\,|\tilde{h}_{k,l}|\,|f_l^k(x_1,x_2)|\ .\end{aligned} \tag{3.7}$$

By making use of Lemma 3.1 we can prove, for any $\sigma(0\leqq\sigma\leqq 1)$ and every $\varepsilon\in S_\theta(\varepsilon_0)$,

$$|(\varepsilon-\varepsilon_l^k)^{-n-1}-(\varepsilon_0-\varepsilon_l^k)^{-n-1}|\leqq(n+1)K^{n+1}|\varepsilon-\varepsilon_0|^\sigma|\varepsilon_0-\varepsilon_l^k|^{-n-1-\sigma} \tag{3.8}$$

where K is given in Lemma 3.1. It follows from (3.8) that the right-hand side of (3.7) is smaller than

$$|\varepsilon-\varepsilon_0|^\sigma(n!)(n+1)K^{n+1}\textstyle\sum\sum|\varepsilon_0-\varepsilon_l^k|^{-n-1-\sigma}|\tilde{h}_{k,l}|\,|f_l^k(x_1,x_2)|\ .$$

On the other hand, by (2.26), (2.27) and (2.28) with $\varepsilon=\varepsilon_0$ we get

$$\sum_{k=1}^{\infty}\sum_{l=1}^{k}|\varepsilon_0-\varepsilon_l^k|^{-n-1-\sigma}|\tilde{h}_{k,l}||f_l^k(x_1, x_2)|<\infty$$

uniformly with respect to x in a neighborhood of the origin. Thus we get (3.6) from (3.7). This completes the proof.

THEOREM 3.2. *Let $u(x, \varepsilon)$ be an analytic solution of* (1.8) *and suppose that t_0 is rational. Then, for every $\varepsilon \in C\backslash[-2, 2]$ we can decompose $u(x, \varepsilon)$ into*

$$u(x, \varepsilon)=u_0(x, \varepsilon)+(\varepsilon-\varepsilon_0)^{-1}u_1(x) \tag{3.9}$$

and $u_0(x, \varepsilon)$ is C^∞ at $\varepsilon=\varepsilon_0$ in the sense of Theorem 3.1.

PROOF. We write $t_0=q_0/p_0$ with relatively prime integers p_0, q_0 and define $u_0(x, \varepsilon)$ by

$$u_0(x, \varepsilon)=\sum(-1)^n(n!)(\varepsilon-\varepsilon_l^k)^{-n-1}\tilde{h}_{k,l}f_l^k(x_1, x_2)$$

where the summation is taken for all positive integers k and l such that $1\leqq l\leqq k$, $l/(k+1)\neq q_0/p_0$. Then, by (1.19) and Remark 1.4 we have the expression (3.9). Moreover, by the same reasoning as in the proof of Theorem 3.1 we have, for $n=0, 1, \cdots$,

$$\lim_{\substack{\varepsilon\to\varepsilon_0 \\ \varepsilon\in S_\theta(\varepsilon_0)}} (\partial/\partial\varepsilon)^n u_0(x, \varepsilon)=(\partial/\partial\varepsilon)^n u_0(x, \varepsilon)|_{\varepsilon=\varepsilon_0}$$

where the right-hand side means that we differentiate $u_0(x, \varepsilon)$ n-times in $C\backslash[-2, 2]$ and substitute ε with ε_0. This implies that $u_0(x, \varepsilon)$ is C^∞ at $\varepsilon=\varepsilon_0$ in the sense of Theorem 3.1. Thus we have proved Theorem 3.2.

Next we shall study the case where t_0 is irrational and satisfies $\rho(t_0)=0$. Let J_γ and B_η be defined by (1.15) and (1.16) respectively. Then

THEOREM 3.3. *Suppose that $t_0\in J_\gamma$ and that the function $h(x)$ in* (1.8) *is contained in B_η for some η with $\gamma<\eta$. Then, for any non-negative integer n satisfying the condition $(n+1)\gamma<\eta$ we have* (3.6).

REMARK 3.1. Note that the regularity of the solution $u(x, \varepsilon)$ of (1.8) with respect to ε at $\varepsilon=\varepsilon_0$ is connected with the algebraic-transcendental property of t_0 given by (3.2).

PROOF OF THEOREM 3.3. We make use of the expression (1.19). Since, $t_0\in J_\gamma$, it follows from (2.16) and Lemma 2.3 that

$$|\varepsilon_0-\varepsilon_l^k|\geqq k^{-\gamma(k+1)}\rho^{k+1},\ 1\leqq l\leqq k,\ k\geqq k_1$$

for some number $\rho>0$ and integer k_1. In view of the definition of B_η and by using (2.22) for h instead of g, we have the estimate

$$|\tilde{h}_{k,l}|\leqq M_0 r_1^{k-1}k((k-1)!)^{1-\eta},\ 1\leqq l\leqq k,\ k=1,2,\cdots.$$

Hence, by using the estimate of $f_l^k(x_1, x_2)$ in (2.27) we have

$$\sum_{k\geqq k_1}\sum_{l=1}^{k}|\varepsilon_0-\varepsilon_l^k|^{-n-1-\sigma}|\tilde{h}_{k,l}|\,|f_l^k(x_1,x_2)|$$
$$\leqq M_0\sum_{k\geqq k_1}\sum_{l=1}^{k}k^{(k+1)\gamma(n+1+\sigma)}\rho^{-(n+1+\sigma)(k+1)}r_1^{k-1}k((k-1)!)^{-\eta}(|x_1|+|x_2|)^{k-1}.$$

By the assumption $(n+1)\gamma<\eta$ we can take $\sigma>0$ so small that $\gamma(n+1+\sigma)-\eta<0$. Then, by using Stirling's formula we see that the last term converges for sufficiently small $|x_1|+|x_2|$. Thus, by using the same arguments as in the proof of Theorem 3.1 we get (3.6). This ends the proof of Theorem 3.3.

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[7] M. YOSHINO, Remarks on the Goursat Problems, Tokyo J. Math., **3** (1980), 115–130.

Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
TOKYO METROPOLITAN UNIVERSITY
FUKAZAWA, SETAGAYA-KU, TOKYO 158