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Examples of Simply Connected Compact Complex 3-folds

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In this note, we shall construct a series of compact complex manifolds $\{M_n\}_{n=1,2,3,\dots}$ of dimension 3 which are non-algebraic and non-kähler with the numerical characters $\pi_1(M_n)=0$, $\pi_2(M_n)=\boldsymbol{Z}$, $b_3(M_n)=4n$, $\dim H^1(M_n, \mathcal{O})\geqq n$, and $\dim H^1(M_n, \Omega^1)\geqq n$, where Ω^p is the sheaf of germs of holomorphic p-forms. These examples show, in particular, that, it is impossible to estimate $h^{p,q}(M)=\dim H^q(M, \Omega^p)$ of a compact complex manifold M in terms of its $(p+q)$-th Betti number, contrary to the case of dimension 2 or the case of kähler manifolds. To construct these examples, we employ a method of connecting two manifolds together to obtain a new one (see §§ 3 and 4).

The discussions with Mr. H. Tsuji was very stimulating, to whom the author would like to express his hearty thanks.

§ 1. We shall construct, in this section, a complex manifold X of dimension 3 with a projection

$$p: X \longrightarrow C$$

such that

(i) *$X-p^{-1}(0)$ is biholomorphic to the product of a primary Hopf surface $S_\alpha=\boldsymbol{C}^2-\{0\}/\langle\alpha\rangle$ and $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$ with $\alpha=\exp 2\pi i a$;*

(ii) *$p^{-1}(0)$ is simply connected, and is a union of two primary Hopf surfaces biholomorphic to $S_{\beta_j}=\boldsymbol{C}^2-\{0\}/\langle\beta_j\rangle$ $(j=0, 1)$ with $\beta_j=\exp 2\pi i b_j$ which intersect each other normally in an elliptic curve, where $a\in\boldsymbol{C}$ is a fixed constant satisfying $\operatorname{Im} a>0$, $b_0=a^{-1}$, and $b_1=(1-a)^{-1}$.*

Let $a\in\boldsymbol{C}$ be a fixed number such that $\operatorname{Im} a>0$. Then $\alpha=\exp 2\pi i a$ satisfies $0<|\alpha|<1$. The multiplication $\xi\mapsto\alpha\xi$ for $\xi\in\boldsymbol{C}^*=\{\xi\in\boldsymbol{C}: \xi\neq0\}$ defines a holomorphic automorphism of $\boldsymbol{C}^*$ and the quotient space $C=\boldsymbol{C}^*/\langle\alpha\rangle$ is an elliptic curve. Denote by $[\xi]$ the point on C corresponding to $\xi\in\boldsymbol{C}^*$. Take three copies W_j, $j=1, 2, 3$, of $\boldsymbol{C}^2$, on which we fix standard systems of coordinates (x_j, y_j). Let $X_j=W_j\times C$, and let $(x_j, y_j$:

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$[\xi_j]$) be their coordinates. We form the complex 3-fold X by patching X_j's as follows:

$$X=\bigcup_{j=1}^{3} X_j ,$$

$$(1)\quad \begin{cases} x_2=x_1y_1 \\ y_2=x_1^{-1} \\ [\xi_2]=[\xi_1 x_1^a] \end{cases} \text{on } X_1\cap X_2 , \qquad \begin{cases} x_3=x_2y_2 \\ y_3=x_2^{-1} \\ [\xi_3]=[\xi_2 x_2 x_2^{-a}] \end{cases} \text{on } X_2\cap X_3 ,$$
$$\begin{cases} x_1=x_3^{-1}y_3^{-1} \\ y_1=x_3 \\ [\xi_1]=[\xi_3 x_3^a y_3] \end{cases} \text{on } X_3\cap X_1 .$$

It is easy to check that the patching is well-defined. Let p be the holomorphic mapping of X onto C given by

$$(2)\qquad p=\begin{cases} y_1 & \text{on } X_1 \\ x_2y_2 & \text{on } X_2 \\ x_3 & \text{on } X_3 . \end{cases}$$

We shall show that the fibre space

$$p: X \longrightarrow C$$

has the desired properties (i) and (ii), and see also some additional facts. Consider the following two 2-folds S_0 and S_1 in X:

$$(3)\qquad \begin{aligned} S_0 &: y_1=0 \text{ in } X_1, \text{ and } x_2=0 \text{ in } X_2 \\ S_1 &: y_2=0 \text{ in } X_2, \text{ and } x_3=0 \text{ in } X_3 , \end{aligned}$$

which are biholomorphic, respectively, to the primary Hopf surfaces

$$S_{\beta_j}=C^2-\{0\}/\langle\beta_j\rangle , \qquad j=0, 1 ,$$

where $\langle\beta_j\rangle$ is the infinite cyclic group generated by the holomorphic automorphism

$$\begin{array}{ccc} \beta_j: C^2-\{0\} & \longrightarrow & C^2-\{0\} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (x, y) & \longmapsto & (\beta_j x, \beta_j y) . \end{array}$$

In fact, let

$$\varphi_{01}: S_0\cap X_1 \longrightarrow S_{\beta_0}$$

be given by

$$\begin{cases} x = \xi_1^{a/1} \\ y = x_1 \xi_1^{1/a} \end{cases}$$

and let

$$\varphi_{02}: S_0 \cap X_2 \longrightarrow S_{\beta_0}$$

be given by

$$\begin{cases} x = y_2 \xi_2^{1/a} \\ y = \xi_2^{1/a} . \end{cases}$$

Then

$$\varphi_0 = \begin{cases} \varphi_{01} & \text{on } S_0 \cap X_1 \\ \varphi_{02} & \text{on } S_0 \cap X_2 \end{cases}$$

gives a biholomorphic mapping of S_0 onto S_{β_0}. Similarly, let

$$\psi_{12}: S_1 \cap X_2 \longrightarrow S_{\beta_1}$$

be given by

$$\begin{cases} x = \xi_2^{1/(1-a)} \\ y = x_2 \xi_2^{1/(1-a)} \end{cases}$$

and let

$$\psi_{13}: S_1 \cap X_3 \longrightarrow S_{\beta_1}$$

be given by

$$\begin{cases} x = y_3 \xi_3^{1/(1-a)} \\ y = \xi_3^{1/(1-a)} . \end{cases}$$

Then

$$\psi_1 = \begin{cases} \psi_{12} & \text{on } S_1 \cap X_2 \\ \psi_{13} & \text{on } S_1 \cap X_3 \end{cases}$$

gives a biholomorphic mapping of S_1 onto S_{β_1}. By (2) and (3), we see that $p^{-1}(0) = S_0 \cup S_1$. Since $S_0 \cap S_1$ is in X_2 and $S_0 \cup S_1$ is given in X_2 by $x_2 y_2 = 0$, S_0 and S_1 intersect with each other normally in the elliptic curve

$$\{(x_2, y_2: [\xi_2]): x_2 = y_2 = 0\} ,$$

which is biholomorphic to $C^*/\langle\alpha\rangle$. Note that $[\xi] \mapsto [\xi^{b_j}]$ gives a biholomorphic map of $C^*/\langle\alpha\rangle$ to $C^*/\langle\beta_j\rangle$. Thus we see that X has the property (ii) when we show the following.

PROPOSITION 1. *$p^{-1}(0)$ is simply connected.*

PROOF. Since $\pi_1(p^{-1}(0))$ is generated by the elements of $\pi_1(S_0\cap S_1)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$, it is enough to show that each generator of $\pi_1(S_0\cap S_1)$ is null-homotopic in $\pi_1(p^{-1}(0))$. Let s and t be real numbers such that $0\leqq s\leqq 1$, and $0\leqq t\leqq 1$. Put

$$\gamma_s\colon [0, 1]\longrightarrow X_2\,, \qquad \theta_1\in[0, 1]\,,$$
$$\begin{cases} x_2=0 \\ y_2=se^{-2\pi i\theta_1} \\ [\xi_2]=[e^{2\pi i a\theta_1}] \end{cases}$$

and

$$\delta_t\colon [0, 1]\longrightarrow X_2\,, \qquad \theta_2\in[0, 1]\,,$$
$$\begin{cases} x_2=te^{-2\pi i\theta_2} \\ y_2=0 \\ [\xi_2]=[e^{2\pi i(1-a)\theta_2}]\,. \end{cases}$$

Then we see easily the following:

$$\gamma_s([0, 1])\subset S_0 \text{ for all } s, \text{ and } \gamma_0([0, 1])\subset S_0\cap S_1\,,$$
$$\delta_t([0, 1])\subset S_1 \text{ for all } t, \text{ and } \delta_0([0, 1])\subset S_0\cap S_1\,.$$

Moreover γ_0 and δ_0 generate $\pi_1(S_0\cap S_1)$. To prove the proposition, it is enough to show that γ_1 is null-homotopic in S_0, and that δ_1 is null-homotopic in S_1. By (1), γ_1 is given in X_1 by

$$\begin{cases} x_1=e^{2\pi i\theta_1} \\ y_1=0 \\ [\xi_1]=[1]\,. \end{cases}$$

Hence γ_1 is null-homotopic in $S_0\cap X_1\subset S_0$. Similarly, δ_1 is given in X_3 by

$$\begin{cases} x_3=0 \\ y_3=e^{2\pi i\theta_2} \\ [\xi_3]=[1]\,. \end{cases}$$

Hence δ_1 is null-homotopic in $S_1\cap X_3\subset S_1$. Q.E.D.

Let

$$W=\bigcup_{j=1}^{3} W_j$$

be the complex 2-fold defined by patching W_j's as follows:

$$\begin{cases} x_2 = x_1 y_1 \\ y_2 = x_1^{-1} \end{cases} \text{on } W_1 \cap W_2 , \quad \begin{cases} x_3 = x_2 y_2 \\ y_3 = x_2^{-1} \end{cases} \text{on } W_2 \cap W_3 ,$$

$$\begin{cases} x_1 = x_3^{-1} y_3^{-1} \\ y_1 = x_3 \end{cases} \text{on } W_3 \cap W_1 .$$

Then the projections

$$(x_j, y_j : [\xi_j]) \longmapsto (x_j, y_j)$$

define a projection

$$\pi_X : X \longrightarrow W .$$

Note that X becomes a complex analytic fibre bundle over W with the fibre $C^*/\langle\alpha\rangle$ by means of this projection. Let t be the holomorphic mapping of W onto C given by

$$t = \begin{cases} y_1 & \text{on } W_1 \\ x_2 y_2 & \text{on } W_2 \\ x_3 & \text{on } W_3 . \end{cases}$$

Then we have the commutative diagram of projections:

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & W \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle t} & \\ & C . & \end{array}$$

Take the primary Hopf surface

$$S_\alpha = C^2 - \{0\}/\langle\alpha\rangle ,$$

which is defined by identifying $(x, y) \in C^2 - \{0\}$ with $(\alpha x, \alpha y) \in C^2 - \{0\}$, where $\alpha = \exp 2\pi i a$. Let $[x, y] \in S$ denote the point corresponding to $(x, y) \in C^2 - \{0\}$. Put

$$Y = S_\alpha \times C$$

and consider the set

$$E = \{([x, y], s) \in Y : y = s = 0\} ,$$

which is biholomorphic to $C^*/\langle\alpha\rangle$. Let

$$q : Y \longrightarrow C$$

be the projection to the 2nd component. Take two copies Z_j, $j = 1, 2$, of C^2, and we form a complex 2-fold

$$Z=\bigcup_{j=1}^{2} Z_j$$

as follows. Letting (u_j, v_j) be a standard system of coordinates on Z_j, we identify (u_1, v_1) with (u_2, v_2), if and only if

$$\begin{cases} u_1=v_2 \\ v_1=u_2^{-1} . \end{cases}$$

There is a holomorphic projection

$$\pi_Y\colon Y \longrightarrow Z$$

defined as follows. Let

$$Y_1=\{([x, y], s) \in Y\colon x \neq 0\}$$
$$Y_2=\{([x, y], s) \in Y\colon y \neq 0\} .$$

Then π_Y is given by

$$\pi_Y|Y_1\colon u_1=s, \qquad v_1=x^{-1}y ,$$
$$\pi_Y|Y_2\colon u_2=xy^{-1}, \ v_2=s .$$

Note that Y becomes a complex analytic fibre bundle over Z with the fibre $C^*/\langle\alpha\rangle$ by means of this projection. There is also a holomorphic mapping

$$\mu'\colon W \longrightarrow Z$$

given by

$$\begin{cases} u_1=y_1 \\ v_1=x_1y_1 \end{cases} \text{on } W_1 , \quad \begin{cases} u_1=x_2y_2 \\ v_1=x_2 \end{cases} \text{on } W_2 , \quad \begin{cases} u_2=y_3 \\ v_2=x_3 \end{cases} \text{on } W_3 .$$

Then μ is the blowing-down of W which contracts

$$l=\{(x_1, y_1) \in W_1\colon y_1=0\} \cup \{(x_2, y_2) \in W_2\colon x_2=0\}$$

to the point

$$P=\{(u_1, v_1) \in Z_1\colon u_1=v_1=0\} .$$

Now we shall prove the following proposition, from which the property (i) of $p\colon X \to C$ follows easily.

PROPOSITION 2. *There is a biholomorphic mapping*

$$\Psi'\colon X-S_0 \longrightarrow Y-E$$

which makes the diagram

$$\begin{array}{ccc} X-S_0 & \xrightarrow{\Psi'} & Y-E \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X-l & \xrightarrow{\mu'} & Z-P \end{array}$$

commutative.

PROOF. Define Ψ' as follows:

$$\Psi'|X_1: \begin{cases} x=\xi_1 y_1^{-a} \\ y=\xi_1 x_1 y_1 y_1^{-a} \\ s=y_1 , \end{cases}$$

$$\Psi'|X_2: \begin{cases} x=\xi_2 x_2^{-a} \\ y=\xi_2 x_2 x_2^{-a} \\ s=x_2 y_2 , \end{cases}$$

$$\Psi'|X_3: \begin{cases} x=\xi_3 y_3 \\ y=\xi_3 \\ s=x_3 . \end{cases}$$

It is easy to see that Ψ' is well-defined and gives the desired biholomorphic mapping. Q.E.D

§2. We shall construct a compact complex 3-fold M_1 with $\pi_1(M_1)=0$, $\pi_2(M_1)=\boldsymbol{Z}$, and $b_3(M_1)=4$. Let $\tilde{V}$ be the vector bundle of rank 2 defined by the Whitney sum $\mathcal{O}_{P^1}(1)\oplus\mathcal{O}_{P^1}(1)$ of two line bundles of degree 1 on $\boldsymbol{P}^1$. Take two copies $\tilde{V}_1$, $\tilde{V}_2$ of $\boldsymbol{C}^3$. Let (ξ_j, ζ_j, s_j) be a standard system of coordinates on $\tilde{V}_j$. Then $\tilde{V}$ is obtained by taking the union $\tilde{V}_1\cup\tilde{V}_2$ identifying (ξ_1, ζ_1, s_1) with (ξ_2, ζ_2, s_2), if and only if

$$\begin{cases} \xi_1=\xi_2 s_2^{-1} \\ \zeta_1=\zeta_2 s_2^{-1} \\ s_1=s_2^{-1} . \end{cases}$$

Put $l_0=\{\xi_1=\zeta_1=0\}\cup\{\xi_2=\zeta_2=0\}$ and $\tilde{V}^*=\tilde{V}-l_0$. Let α be a holomorphic automorphism of $\tilde{V}^*$ defined by

$$(\xi_j, \zeta_j, s_j)\longmapsto(\alpha\xi_j, \alpha\zeta_j, s_j) \qquad \text{on } \tilde{V}^*\cap\tilde{V}_j ,$$

$j=1, 2$. Put

$$M=\tilde{V}^*/\langle\alpha\rangle .$$

Then the canonical projection $\tilde{\pi}$: $\tilde{V}\to\boldsymbol{P}^1$ induces a projection

$$\pi: M \longrightarrow P^1$$

and define a structure on M of a complex analytic fibre bundle over P^1 with the fibre S_α. Now we shall modify M to obtain M_1. Put $V_j = (\tilde{V}_j \cap \tilde{V})/\langle\alpha\rangle$ $(j=1, 2)$. Obviously, V_j, $j=1, 2$, are subdomains in M, and $M = V_1 \cup V_2$. We replace V_1 by X constructed in § 1 as follows. Let

$$\Phi_1: V_1 \longrightarrow Y = S_\alpha \times C$$

be the natural isomorphism induced by

$$(\xi_1, \zeta_1, s_1) \longmapsto ([\xi_1, \zeta_1], s_1) .$$

We have another isomorphism

$$\Phi: X - p^{-1}(0) \longrightarrow S_\alpha \times C^* \subset Y ,$$

which is given by

$$\Phi = \Psi' | (X - p^{-1}(0)) .$$

Therefore we can define a compact complex 3-fold

$$M_1 = X \cup V_2$$

by identifying $x \in V_1 - \pi^{-1}(0) = V_1 \cap V_2$ with $\Phi^{-1} \circ \Phi_1(x) \in X - p^{-1}(0)$, where $0 \in P^1$ indicates the point $s_1 = 0$. Then M_1 is a complex analytic fibre space over P^1 with the projection

$$p_1 = \begin{cases} p & \text{on } X \\ \pi & \text{on } V_2 . \end{cases}$$

Note that, for $s \in P^1$, $s \neq 0$, $p_1^{-1}(s)$ is biholomorphic to S_α and $p_1^{-1}(0)$ is biholomorphic to $S_0 \cup S_1$.

PROPOSITION 3.
(i) $\pi_1(M_1) = 0$,
(ii) $\pi_2(M_1) = Z$,
(iii) $b_3(M_1) = 4$, *in particular, the Euler number* $e(M_1) = 0$.

PROOF. (i) It is clear that

$$i_*: \pi_1(M - \pi^{-1}(0)) \longrightarrow \pi_1(M_1)$$

is surjective, where i_* is induced by the natural inclusion. Note that $\pi_1(M - \pi^{-1}(0)) \cong Z$ is generated by a closed path contained in a fibre of π. Since $p_1^{-1}(0)$ is simply connected by Proposition 1, we infer that i_*

is a zero mapping. Hence $\pi_1(M_1)=0$. (ii) Since $\pi_1(M_1)=0$, it is enough to show that $H_2(M_1, \boldsymbol{Z})=\boldsymbol{Z}$ by the Hurewicz isomorphism theorem. Let $\varDelta$ be a small disc around $0\in \boldsymbol{P}^1$. Then we have the Mayer-Vietoris sequence with $\boldsymbol{Z}$-coefficients:

$$\begin{aligned}(4)\qquad \cdots &\longrightarrow H_2(M_1-p_1^{-1}(0))\oplus H_2(p_1^{-1}(\varDelta)) \xrightarrow{q_2} H_2(M_1) \longrightarrow H_1(p_1^{-1}(\partial\varDelta)) \\ &\xrightarrow{i_1\oplus j_1} H_1(M_1-p_1^{-1}(0))\oplus H_1(p_1^{-1}(\varDelta)) \longrightarrow \cdots .\end{aligned}$$

First we claim that $H_2(p_1^{-1}(\varDelta))=0$. Since $p_1^{-1}(0)=p^{-1}(0)$ is a deformation retract of $p_1^{-1}(\varDelta)$, it is enough to show that $H_2(p^{-1}(0))=0$. Recall that $p^{-1}(0)=S_0\cup S_1$. We consider the Mayer-Vietoris sequence with $\boldsymbol{Z}$-coefficients

$$\begin{aligned}\cdots &\longrightarrow H_2(S_0)\oplus H_2(S_1) \longrightarrow H_2(p^{-1}(0)) \longrightarrow H_1(S_0\cap S_1) \\ &\longrightarrow H_1(S_0)\oplus H_2(S_1) \longrightarrow \cdots .\end{aligned}$$

By the argument in the proof of Proposition 1, we see that

$$H_1(S_0\cap S_1) \longrightarrow H_1(S_0)\oplus H_1(S_1)$$

is bijective. Moreover, it is clear that $H_2(S_j)=0$, $j=0, 1$. Therefore we have $H_2(p^{-1}(0))=0$. Next we claim that the kernel of

$$i_1\colon H_1(p_1^{-1}(\partial\varDelta)) \longrightarrow H_1(M_1-p_1^{-1}(0))$$

is isomorphic to $\boldsymbol{Z}$. Note that, by Proposition 2,

$$p_1^{-1}(\partial\varDelta)\cong S^1\times S_\alpha \cong S^1\times S^1\times S^3$$

and

$$(5)\qquad M_1-p_1^{-1}(0)\cong M-\pi^{-1}(0)\cong \boldsymbol{C}\times S_\alpha\cong \boldsymbol{R}^2\times S^1\times S^3 .$$

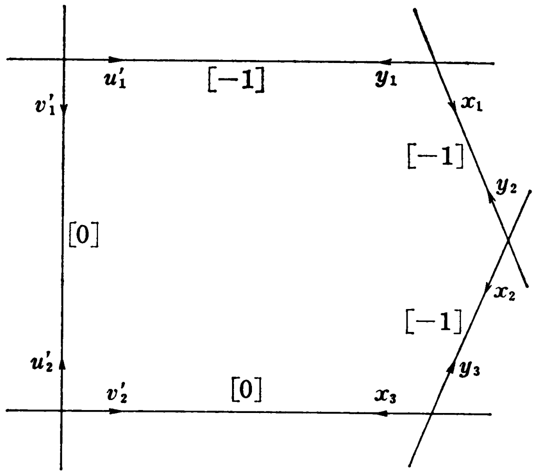
Therefore the 1-cycle γ_b in $p_1^{-1}(\partial\varDelta)$ defined by $S^1\times\{q\}$, $q\in S_\alpha$, is a free basis of the kernel of i_1. Hence $H_2(M_1)=\boldsymbol{Z}$ follows from (5) and (4). (iii) Since the Euler number $e(M_1)$ is equal to that of M, we have $b_3(M_1)=4$. Q.E.D.

Take two copies Z', Z'' of Z. Let (u_j', v_j') (resp. (u_j'', v_j'')) be the local coordinates on Z' (resp. Z'') corresponding to (u_j, v_j) on Z. We form the union

$$R_1=Z'\cup W$$

by the identifications:

$$(u'_1, v'_1)=(x_1, y_1) \text{ iff } x_1=u'_1v'_1 , \quad y_1=u'^{-1}_1 ,$$
$$(u'_1, v'_1)=(x_2, y_2) \text{ iff } x_2=v'_1 , \quad y_2=u'^{-1}_1v'^{-1}_1 ,$$
$$(u'_1, v'_1)=(x_3, y_3) \text{ iff } x_3=u'^{-1}_1 , \quad y_3=v'^{-1}_1 ,$$
$$(u'_2, v'_2)=(x_1, y_1) \text{ iff } x_1=u'^{-1}_2v'_2 , \quad y_1=v'^{-1}_2 ,$$
$$(u'_2, v'_2)=(x_2, y_2) \text{ iff } x_2=u'^{-1}_2 , \quad y_2=u'_2v'^{-1}_2 ,$$
$$(u'_2, v'_2)=(x_3, y_3) \text{ iff } x_3=v'^{-1}_2 , \quad y_3=u'_2 .$$



[ ] indicates the self-intersection number of a curve.
FIGURE R_1

Let

$$\pi_{V_2}: V_2 \longrightarrow Z'$$

be the holomorphic mapping given by

$$\begin{cases} u'_1=s_2 \\ v'_1=\zeta_2\xi_2^{-1} \end{cases} \text{ if } \xi_2\neq 0 , \quad \text{and}$$

$$\begin{cases} v'_2=s_2 \\ u'_2=\zeta_2^{-1}\xi_2 \end{cases} \text{ if } \zeta_2\neq 0 .$$

Define

$$\pi_{M_1}: M_1 \longrightarrow R_1$$

by

$$\pi_{M_1}=\begin{cases} \pi_X & \text{on } X \\ \pi_{V_2} & \text{on } V_2 . \end{cases}$$

Then M_1 is a complex analytic fibre bundle over R_1 with the fibre

$C^*/\langle\alpha\rangle$ and with the projection π_{M_1}. Similarly we form the union

$$R=Z''\cup Z$$

by the identifications:

$$\begin{aligned}
(u_1'', v_1'')&=(u_1, v_1) \text{ iff } u_1''u_1=1\,, \quad v_1''=v_1\,,\\
(u_1'', v_1'')&=(u_2, v_2) \text{ iff } u_1''u_2=1\,, \quad v_1''u_2=1\,,\\
(u_2'', v_2'')&=(u_1, v_1) \text{ iff } u_2''v_1=1\,, \quad v_2''u_1=1\,,\\
(u_2'', v_2'')&=(u_2, v_2) \text{ iff } u_2''=u_2\,, \quad v_2''v_2=1\,.
\end{aligned}$$

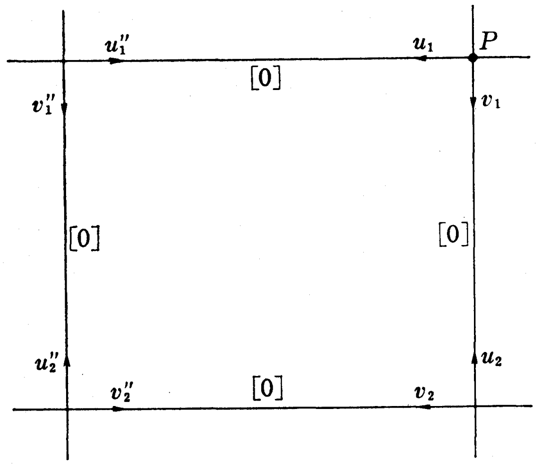


FIGURE R

Clearly, R is biholomorphic to $P^1\times P^1$ and R_1 is the blowing-up of R at $P=\{u_1=v_1=0\}$. Let μ: $R_1\to R$ be the blowing-up. There is a projection

$$\pi_M\colon M\longrightarrow R$$

given by

$$\pi_M|V_1\colon \begin{cases} u_1=s_1\,, \quad v_1=\zeta_1\xi_1^{-1}\,, & \text{if } \xi_1\neq 0\,,\\ v_2=s_1\,, \quad u_2=\zeta_1^{-1}\xi_1\,, & \text{if } \zeta_1\neq 0\,,\end{cases}$$

$$\pi_M|V_2\colon \begin{cases} u_1''=s_2\,, \quad v_1''=\zeta_2\xi_2^{-1}\,, & \text{if } \xi_2\neq 0\,,\\ v_2''=s_2\,, \quad u_2''=\zeta_2^{-1}\xi_2\,, & \text{if } \zeta_2\neq 0\,.\end{cases}$$

The following proposition is clear from the above construction.

PROPOSITION 4. *The biholomorphic mapping*

$$\Psi'\colon X-S_0\longrightarrow Y-E$$

of Proposition 2 *extends naturally to a biholomorphic mapping*

$$\Psi: M_1 - S_0 \longrightarrow M - E ,$$

which makes the diagram

$$\begin{array}{ccc} M_1 - S_0 & \xrightarrow{\Psi} & M - E \\ \downarrow{\scriptstyle \pi_{M_1}} & & \downarrow{\scriptstyle \pi_M} \\ R_1 - l & \xrightarrow{\mu} & R - P \end{array}$$

commutative.

PROPOSITION 5. *There are non-singular rational curves l_q in M_1, parametrized by $q=\begin{pmatrix} q_1 & r_1 \\ q_2 & r_2 \end{pmatrix} \in GL(2, C)$ with $q_2 \neq 0$, such that each l_q is a section of p_1: $M_1 \rightarrow P^1$, and has a neighborhood isomorphic to that of a section of $\tilde{V}=\mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{P^1}(1)$.*

PROOF. For each $q=\begin{pmatrix} q_1 & r_1 \\ q_2 & r_2 \end{pmatrix} \in GL(2, C)$ with $q_2 \neq 0$, we define the section $\tilde{l}_q$ of $\tilde{V}^* = \tilde{V} - l_0$ by

$$\begin{cases} \xi_1 = q_1 + r_1 s_1 \\ \zeta_1 = q_2 + r_2 s_1 \end{cases} \text{ on } \tilde{V}_1 , \quad \text{and} \quad \begin{cases} \xi_2 = q_1 s_2 + r_1 \\ \zeta_2 = q_2 s_2 + r_2 \end{cases} \text{ on } \tilde{V}_2 .$$

Then the image l'_q of $\tilde{l}_q$ in M does not intersect with E, and has a neighborhood in $M-E$ which is biholomorphic to a section of $\tilde{V}$. Put $l_q = \Psi^{-1}(l'_q)$. Then the proposition follows from Proposition 4. Q.E.D.

§ 3. In this section we shall describe a method of connecting two compact complex 3-folds to obtain a new compact complex 3-fold. Let P^3 be a complex projective space of dimension 3 and $[z_0: z_1: z_2: z_3]$ be a system of homogeneous coordinates. We define a holomorphic involution

$$\sigma: P^3 \longrightarrow P^3$$

by

$$\sigma([z_0: z_1: z_2: z_3]) = [z_2: z_3: z_0: z_1] .$$

Let l and l_∞ be skew lines in P^3 given by

$$l: z_0 = z_1 = 0 ,$$

and

$$l_\infty: z_2 = z_3 = 0 .$$

It is easy to check that $\sigma(l) = l_\infty$. For any $r>0$, and $\varepsilon>1$, we define the following subsets in P^3:

$$U_r = \{[z_0: z_1: z_2: z_3] \in P^3: |z_0|^2 + |z_1|^2 < r(|z_2|^2 + |z_3|^2)\} ,$$
$$U = U_1 ,$$
$$N(\varepsilon) = U_\varepsilon - [U_{1/\varepsilon}] ,$$

and

$$\begin{aligned}\textstyle\sum &= \partial U \\ &= \{[z_0: z_1: z_2: z_3] \in P^3: |z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^2\} .\end{aligned}$$

Then U_r and $N(\varepsilon)$ are connected and open, and $\sum$ is a non-singular real hypersurface in P^3. It is easy to show the following two lemmas.

LEMMA 1. *For any $r>0$, U_r is biholomorphic to U, and* $\lim_{r\to 0} U_r = l$.

LEMMA 2. *For any $\varepsilon>1$, we have*
(i) $\sum \subset N(\varepsilon)$,
(ii) $\sigma(\sum) = \sum$,
(iii) $\sigma(N(\varepsilon)) = N(\varepsilon)$, *and*
(iv) $\sigma(U) = P^3 - [U]$.

A compact complex 3-fold M is said to be *of type Class L if and only if M contains a subdomain which is biholomorphic to $N(\varepsilon)$ for some $\varepsilon>1$.*

Let

$$F: \tilde{V} \longrightarrow P^3 - l_\infty$$

be the biholomorphic mapping defined by

$$F|V_1: (\xi_1, \zeta_1, s_1) \longmapsto [\xi_1: \zeta_1: s_1: 1]$$

and

$$F|V_2: (\xi_2, \zeta_2, s_2) \longmapsto [\xi_2: \zeta_2: 1: s_2] .$$

From this we have

LEMMA 3. *Each section of $\tilde{V}$ is mapped by F to a projective line in P^3 outside l_∞.*

For any $\varepsilon>1$, $N(\varepsilon)$ contains (infinitely many) projective lines in P^3. Therefore, by Lemma 1, we have

LEMMA 4. *Suppose that M is of Class L. Then there is a non-singular rational curve C and its neighborhood in M which is biholomorphic to U_ε for some $\varepsilon>1$.*

Suppose that M_1 and M_2 are of Class L. For some $\varepsilon>1$, there are open embeddings

$$i_\nu\colon U_\varepsilon \longrightarrow M_\nu \quad (\nu=1, 2)\ .$$

Let

$$M_\nu^* = M_\nu - [i_\nu(U_{1/\varepsilon})]$$

and form the union

$$M_1^* \cup M_2^*$$

by identifying a point $x_1 \in i_1(N(\varepsilon)) \subset M_1^*$ with the point $x_2 = i_2 \circ \sigma \circ i_1^{-1}(x_1) \in M_2^*$.

LEMMA 5. *$M_1^* \cup M_2^*$ is a compact complex 3-fold.*

Proof is easy.

REMARK 1. If $M_1 = M_2 = P^3$ and i_ν are the natural inclusions, then $M_1^* \cup M_2^* = P^3$.

We denote $M_1^* \cup M_2^*$ by $M(M_1, M_2, i_1, i_2)$. It is clear that $M(M_1, M_2, i_1, i_2)$ is defined independently of the choice of ε, but may depend on the choice of i_ν's. The process to construct $M(M_1, M_2, i_1, i_2)$ out of M_ν's and i_ν's is called a *connecting operation*. Note that $M(M_1, M_2, i_1, i_2)$ is also of Class L.

§4. By means of connecting operations, we shall construct inductively a series of compact complex 3-folds $\{M_n\}_{n=1,2,3,\dots}$ stated in the beginning of this note. Let M_1 be the manifold constructed in §2, which is of Class L by Proposition 5. To construct M_2, we take two copies of M_1, say M_1 and M_1'. In the following, A' indicates a subset in M_1' corresponding to A in M_1. Let l_{q_1} (resp. l'_{q_1}) be one of the non-singular rational curves in M_1 (resp. M_1') described in Proposition 5. Let L_1 (resp. L_1') be a neighborhood of l_{q_1} (resp. l'_{q_1}) in $M_1 - S_0$ (resp. $M_1' - S_0'$) which is biholomorphic to U_{ε_1} for some $\varepsilon_1 > 1$. This is possible by Lemma 1. Let $i_1\colon U_{\varepsilon_1} \to L_1 \subset M_1$ (resp. $i_1'\colon U_{\varepsilon_1} \to L_1' \subset M_1'$) be an isomorphism. By the connecting operation, we obtain a compact complex 3-fold

$$M_2 = M(M_1, M_1', i_1, i_1')\ .$$

Note that M_2 contains at least two Hopf surfaces H_1 and H_2, corresponding to S_0 and S_0' in M_1 and M_1', respectively. Now we regard $i_1(N(\varepsilon_1))$ as a subdomain in M_2. In $i_1(N(\varepsilon_1))$, there are a non-singular rational curve l_{q_2} and its neighborhood L_2 which is biholomorphic to that of a section of $\tilde{V}$. Let $i_2\colon U_{\varepsilon_2} \to L_2$ $(\subset i_1(N(\varepsilon_1)) \subset M_2)$ be an isomorphism, where we can assume that $1 < \varepsilon_2 \leq \varepsilon_1$. By using $i_1 | U_{\varepsilon_2}$ and i_2, we can connect M_1 and M_2, and obtain

$$M_3 = M(M_1, M_2, i_1 | U_{\varepsilon_2}, i_2) .$$

Since $i_1(N(\varepsilon_1)) \subset M_2 - (H_1 \cup H_2)$, M_3 contains at least 3 Hopf surfaces H_1, H_2, and H_3 which correspond, respectively, to H_1 and H_2 in M_2, and S_0 in M_1. Now again, regarding $i_1(N(\varepsilon_2))$ as a subdomain in M_3, we can repeat the above step, and we have inductively a series $\{M_n\}_{n=1,2,\cdots}$

$$M_n = M(M_1, M_{n-1}, i_1 | U_{\varepsilon_{n-1}}, i_{n-1})$$

of compact complex 3-folds. M_n contains at least n Hopf surfaces, one of which is from M_1 and the others are from M_{n-1}.

THEOREM. *For all* $n \geqq 1$,
(i) M_n *is non-algebraic and non-kähler,*
(ii) $\pi_1(M_n) = 0$, $\pi_2(M_n) = \boldsymbol{Z}$, *and* $b_3(M_n) = 4n$,
(iii) $\dim H^1(M_n, \mathcal{O}) \geqq n$,
(iv) $\dim H^1(M_n, \Omega^1) \geqq n$.

PROOF. (i) is clear, since M_n contains Hopf surfaces. (ii) By the Mayer-Vietoris sequence with $\boldsymbol{Z}$-coefficients

$$(6) \qquad \cdots \longrightarrow H_2(M_1^{n-1} \cap M_{n-1}^{\#}) \xrightarrow{i_2 \oplus j_2} H_2(M_1^{n-1}) \oplus H_2(M_{n-1}^{\#}) \longrightarrow H_2(M_n) \longrightarrow H_1(M_1^{n-1} \cap M_{n-1}^{\#}) \longrightarrow \cdots ,$$

where

$$M_1^{n-1} = M_1 - [i_1(U_{1/\varepsilon_{n-1}})] ,$$

and

$$M_{n-1}^{\#} = M_{n-1} - [i_{n-1}(U_{1/\varepsilon_{n-1}})] ,$$

we have

$$H_1(M_1^{n-1} \cap M_{n-1}^{\#}) = 0 ,$$

and

$$H_2(M_1^{n-1} \cap M_{n-1}^{\#}) = \boldsymbol{Z} ,$$

since $M_1^{n-1} \cap M_{n-1}^{\#}$ is homotopy equivalent to $S^2 \times S^3$. Note that l_{q_n} generates both $H_2(M_1^{n-1} \cap M_{n-1}^{\#})$ and $H_2(M_1^{n-1})$. Hence

$$i_2 \colon H_2(M_1^{n-1} \cap M_{n-1}^{\#}) \longrightarrow H_2(M_1^{n-1})$$

is bijective. Therefore, from (6), we have

$$(7) \qquad H_2(M_n) = H_2(M_{n-1}^{\#}) .$$

By the exact sequence

$$\cdots \longrightarrow H_3(M_{n-1}, M^{\sharp}_{n-1}) \longrightarrow H_2(M^{\sharp}_{n-1}) \longrightarrow H_2(M_{n-1}) \longrightarrow H_2(M_{n-1}, M^{\sharp}_{n-1}) \longrightarrow \cdots ,$$

and the duality

$$H_3(M_{n-1}, M^{\sharp}_{n-1}) = H^3(l_{q_{n-1}}) = 0 ,$$

and

$$H_2(M_{n-1}, M^{\sharp}_{n-1}) = H^4(l_{q_{n-1}}) = 0 ,$$

we have

$$H_2(M^{\sharp}_{n-1}) = H_2(M_{n-1}) .$$

Hence, by (7) and the induction assumption, we obtain

$$H_2(M_n) = H_2(M_{n-1}) = \boldsymbol{Z} .$$

Since $\pi_1(M_n)=0$ is clear, $\pi_2(M_n)=\boldsymbol{Z}$ follows from the Hurewicz isomorphism theorem. Since $e(M_n)=e(M_{n-1})+e(M_1)-4=e(M_{n-1})-4=-4(n-1)$ by the induction assumption and Proposition 3, we have $b_3(M_n)=2+2b_2(M_n)-e(M_n)=4n$.

To prove (iii) of the theorem, we shall make some preparations. Recall that

$$M^{\sharp}_n = M_n - [i_n(U_{1/\varepsilon_n})] ,$$
$$M^n_1 = M_1 - [i_1(U_{1/\varepsilon_n})] ,$$

and that

$$M_{n+1} = M^{\sharp}_n \cup M^n_1 .$$

Let

$$f^1_n \colon M^{\sharp}_n \longrightarrow M_{n+1} , \quad \text{and}$$
$$f^2_n \colon M^n_1 \longrightarrow M_{n+1}$$

be the natural inclusions. Then we have

$$s_n \colon = (f^2_n \circ i_1)|N(\varepsilon_n) = (f^1_n \circ i_n \circ \sigma)|N(\varepsilon_n)$$

which defines an embedding

$$N(\varepsilon_n) \longrightarrow M_{n+1} .$$

Let

$$\rho_n \colon N(\varepsilon_n) \longrightarrow M^{\sharp}_n ,$$
$$\sigma_n \colon M^{\sharp}_n \longrightarrow M_n , \quad \text{and}$$
$$\tau_n \colon N(\varepsilon_n) \longrightarrow M_n$$

be the open embeddings defined, respectively, by

$$\rho_n=(i_n\circ\sigma)|N(\varepsilon_n)\ ,$$
$$\sigma_n=\text{the natural inclusion, and}$$
$$\tau_n=\sigma_n\circ\rho_n\ .$$

Let

$$t_n\colon N(\varepsilon_{n+1})\longrightarrow N(\varepsilon_n)$$

be the open embedding defined by

$$t_n=s_n^{-1}\circ\tau_{n+1}=s_n^{-1}\circ(i_{n+1}\circ\sigma|N(\varepsilon_{n+1}))$$

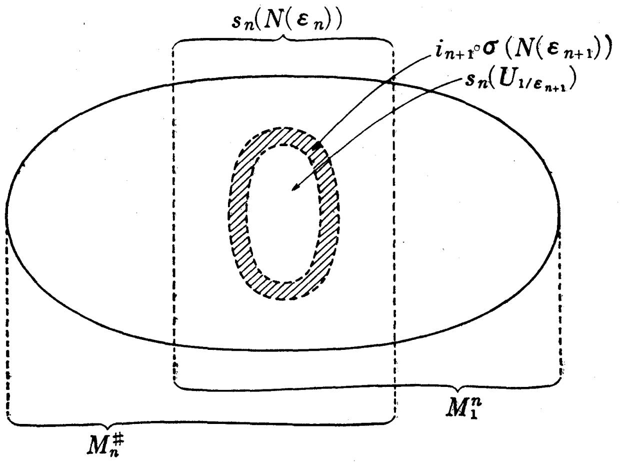


FIGURE M_{n+1}

LEMMA 6. *$\sigma_n^*\colon H^1(M_n, \mathcal{O})\to H^1(M_n^\#, \mathcal{O})$ is injective for all $n\geqq 1$.*

PROOF. Since the homomorphism

$$r_1\colon H^1(M_n-l_{q_n}, \mathcal{O})\longrightarrow H^1(M_n^\#, \mathcal{O})$$

induced by the natural inclusion is injective by Andreotti-Siu [1, Proposition 1.2], it is enough to show that

$$H^1_{l_{q_n}}(M_n, \mathcal{O})=0\ . \tag{8}$$

Since l_{q_n} has a neighborhood in M_n which is biholomorphic to that of a projective line $\boldsymbol{P}^1$ in $\boldsymbol{P}^3$, we have the exact sequence

$$\cdots\longrightarrow H^0(\boldsymbol{P}^3-\boldsymbol{P}^1, \mathcal{O})\longrightarrow H^0(\boldsymbol{P}^3, \mathcal{O})\longrightarrow H^1_{l_{q_n}}(M_n, \mathcal{O})\longrightarrow H^1(\boldsymbol{P}^3, \mathcal{O})\longrightarrow\cdots\ .$$

From this sequence, (8) follows easily. Q.E.D.

Let

$$L_1 = R^1(\pi_{M_1})_* \mathcal{O}_{M_1}, \quad \text{and}$$
$$L = R^1(\pi_M)_* \mathcal{O}_M .$$

Then we have

LEMMA 7. *$L_1 = \mathcal{O}_{R_1}$, and $L = \mathcal{O}_R$.*

PROOF. First we consider L_1. By a theorem of Grauert, L_1 is a vector bundle of rank $1 = \dim H^1(C^*/\langle\alpha\rangle, \mathcal{O})$. Recall that R_1 is the blowing-up of R, μ: $R_1 \to R$, and that $R \cong P^1 \times P^1$. Let E_1 be the proper inverse image of $P^1 \times \{0\} \subset R$, and E_2 the proper inverse image of $\{0\} \times P^1 \subset R$. Then $H^2(R_1, Z)$ is generated by E_1, E_2, and the exceptional curve $l = \mu^{-1}(P)$. Note that $H^1(R_1, \mathcal{O}) = 0$. Hence, to prove the lemma, it is enough to show that the restrictions of L_1 to E_1, E_2, and l are trivial. But these are consequences of the fact that $\pi_{M_1}^{-1}(E_1)$, $\pi_{M_1}^{-1}(E_2)$, and $\pi_{M_1}^{-1}(l)$ are all elliptic bundles with vanishing Chern numbers, by virtue of a result of Kodaira [3, Theorem 12]. By a similar argument, $L = \mathcal{O}_R$ can be proved easily. Q.E.D.

LEMMA 8. $\dim H^1(M, \mathcal{O}) = \dim H^1(M_1, \mathcal{O}) = 1$.

PROOF. This follows easily from Lemma 7 by using Leray's spectral sequences applied to the fibre bundles π_M: $M \to R$, and π_{M_1}: $M_1 \to R_1$.

LEMMA 9. *The homomorphism*

$$r_2 \colon H^1(M_1, \mathcal{O}) \longrightarrow H^1(M_0 - S_0, \mathcal{O})$$

induced by the natural inclusion is injective.

PROOF. Since $L = \mathcal{O}_R$ by Lemma 7, there is a non-zero section $s \in H^0(R - P, L)$. By Proposition 4, we see that $\mu^* s \in H^0(R_1 - l, L_1)$. Since l is an exceptional curve in R_1, and since L_1 is trivial on R_1 by Lemma 7, $\mu^* s$ extends to a section $\widetilde{\mu^* s}$ of $H^0(R_1, L_1)$. Consider the commutative diagram

$$\begin{array}{ccc} H^1(M_1, \mathcal{O}) & \xrightarrow{r_2} & H^1(M_1 - S_0, \mathcal{O}) \\ \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_2} \\ H^0(R_1, L_1) & \xrightarrow{r_3} & H^0(R_1 - l, L_1) , \end{array} \tag{9}$$

where r_3 is induced by the restrictions, and j_1 and j_2 are the canonical injections of Leray's spectral sequences. Then

$$r_2 \circ j_1(\widetilde{\mu^* s}) = j_2(\mu^* s) .$$

Since j_2 is injective, and since $\mu^* s \neq 0$, we see that

$$r_2 \circ j_1(\widetilde{\mu^* s}) \neq 0 . \tag{10}$$

By Lemmas 7 and 8, j_1 is an isomorphism. Therefore (10) implies that r_2 is injective. Q.E.D.

LEMMA 10. $\dim \operatorname{Ker} \rho_1^* \geqq 1$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} H^1(M_1, \mathcal{O}) & \xrightarrow{\sigma_1^*} & H^1(M_1^\#, \mathcal{O}) \\ & {\scriptstyle \tau_1^*} \searrow \quad \swarrow {\scriptstyle \rho_1^*} & \\ & H^1(N(\varepsilon_1), \mathcal{O}) . & \end{array} \tag{11}$$

Take the element $j_1(\widetilde{\mu^* s}) \in H^1(M_1, \mathcal{O})$ of the proof of Lemma 9. By Lemma 6, $\sigma_1^* \circ j_1(\widetilde{\mu^* s}) \in H^1(M_1^\#, \mathcal{O})$ is not zero. Therefore, to prove the lemma, it suffices to show that

$$\tau_1^* \circ j_1(\widetilde{\mu^* s}) = 0 . \tag{12}$$

The element $s \in H^0(R-P, L)$ extends to an element $\tilde{s} \in H^0(R, L)$. Let j_3: $H^0(R, L) \to H^1(M, \mathcal{O})$ be the inclusion defined by Leray's spectral sequence. Consider the element $j_3(\tilde{s}) \in H^1(M, \mathcal{O})$. Let

$$\psi' : H^1(M, C) \longrightarrow H^1(M_1 - S_0, C) , \quad \text{and}$$
$$\psi'' : H^1(M, \mathcal{O}) \longrightarrow H^1(M_1 - S_0, \mathcal{O})$$

be the homomorphisms defined by the inclusion $M-E \to M$ followed by Ψ^{-1}: $M-E \to M_1 - S_0$ of Proposition 4. Since $S_0 \cap \tau_1(N(\varepsilon_1)) = \varnothing$, we have also the homomorphisms

$$\tau_1' : H^1(M_1 - S_0, C) \longrightarrow H^1(N(\varepsilon_1), C) , \quad \text{and}$$
$$\tau_1'' : H^1(M_1 - S_0, \mathcal{O}) \longrightarrow H^1(N(\varepsilon_1), \mathcal{O})$$

induced by τ_1. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^1(M, C) & \xrightarrow{\psi'} & H^1(M_1 - S_0, C) & \xrightarrow{\tau'} & H^1(N(\varepsilon_1), C) \\ \downarrow {\scriptstyle j_4} & & \downarrow {\scriptstyle j_5} & & \downarrow {\scriptstyle j_6} \\ H^1(M, \mathcal{O}) & \xrightarrow{\psi''} & H^1(M_1 - S_0, \mathcal{O}) & \xrightarrow{\tau_1''} & H^1(N(\varepsilon_1), \mathcal{O}) , \end{array} \tag{13}$$

where j_4, j_5, and j_6 are homomorphisms defined by the natural inclusion $C\to\mathcal{O}$. It is easy to see that $\dim H^0(M, d\mathcal{O})\leqq\dim H^0(M, \Omega^1)=0$, where Ω^1 is the sheaf of germs of holomorphic 1-forms and $d\mathcal{O}$ is the subsheaf of Ω^1 whose elements are d-closed. Moreover $H^1(M, C)=C$. Hence, by Lemma 8 and the exact sequence

$$0\longrightarrow C\longrightarrow\mathcal{O}\longrightarrow d\mathcal{O}\longrightarrow 0\,,$$

we see that j_4 is an isomorphism. Hence, from the diagram (13) and the fact that $H^1(N(\varepsilon_1), C)=0$,

$$\tau_1''\circ\psi''\circ j_3(\tilde{s})=0 \tag{14}$$

follows. Consider the commutative diagram

$$\begin{array}{ccc} H^1(M_1-S_0, \mathcal{O}) & \xleftarrow{\psi''} & H^1(M, \mathcal{O}) \\ \uparrow{\scriptstyle j_2} & & \uparrow{\scriptstyle j_3} \\ H^0(R_1-l, L_1) & \xleftarrow{\mu_1^*} & H^0(R, L)\,, \end{array} \tag{15}$$

where μ_1^* is induced by the inclusion $R-P\to R$ followed by the isomorphism μ: $R_1-l\to R-P$. Note that

$$\mu^*s=\mu_1^*\tilde{s}\,.$$

Then, by the diagrams (9), (11), (13), and (15), we have

$$\begin{aligned}\tau_1^*\circ j_1(\widetilde{\mu^*s})&=\tau_1''\circ r_2\circ j_1(\widetilde{\mu^*s})\\ &=\tau_1''\circ j_2\circ r_3(\widetilde{\mu^*s})\\ &=\tau_1''\circ j_2(\mu^*s)\\ &=\tau_1''\circ j_2(\mu_1^*\tilde{s})\\ &=\tau_1''\circ\psi''\circ j_3(\tilde{s})\,,\end{aligned}$$

which is equal to zero by (14). Thus (12) is obtained. Q.E.D.

PROOF OF (iii) OF THE THEOREM. Consider the following inequalities:

$$\dim H^1(M_n, \mathcal{O})\geqq n\,, \tag{$*$}_n$$

$$\dim \mathrm{Ker}\,\rho_n^*\geqq n\,. \tag{$**$}_n$$

We shall prove, by induction on n, that $(*)_n$ and $(**)_n$ hold for all $n\geqq 1$. By Lemmas 8 and 10, $(*)_1$ and $(**)_1$ hold. Suppose that $(*)_n$ and $(**)_n$ hold for some $n\geqq 1$. Consider the Mayer-Vietoris sequence

$$(16)\qquad \cdots \longrightarrow H^1(M_{n+1}, \mathcal{O}) \xrightarrow{f_n^*} H^1(M_n^\#, \mathcal{O}) \oplus H^1(M_1^n, \mathcal{O}) \xrightarrow{g_n^*} H^1(N(\varepsilon_n), \mathcal{O}) \longrightarrow \cdots ,$$

where

$$f_n^* = f_n^{1*} \oplus f_n^{2*} , \quad \text{and}$$
$$g_n^* = \rho_n^* - (i_1|N(\varepsilon_n))^* .$$

There is the following commutative diagram:

$$\begin{array}{ccc} H^1(M_1^\#, \mathcal{O}) & \xrightarrow{\rho_1^*} & H^1(N(\varepsilon_1), \mathcal{O}) \\ {\scriptstyle j_7}\downarrow & & \downarrow{\scriptstyle j_8'} \\ H^1(M_1^n, \mathcal{O}) & \xrightarrow{(i_1|N(\varepsilon_n))^*} & H^1(N(\varepsilon_n), \mathcal{O}) , \end{array}$$

where j_7 is induced by the inclusion, and j_8' is induced by the inclusion followed by σ. Note that j_7 is injective by Andreotti-Siu [1, Proposition 1.2]. Hence by Lemma 10,

$$(17)\qquad 1 \leqq \dim \operatorname{Ker} \rho_1^* \leqq \dim \operatorname{Ker} (i_1|N(\varepsilon_n))^* .$$

Since the subspace

$$K: = \operatorname{Ker} \rho_n^* \oplus \operatorname{Ker} (i_1|N(\varepsilon_n))^*$$

in $H^1(M_n^\#, \mathcal{O}) \oplus H^1(M_1^n, \mathcal{O})$ is contained in $\operatorname{Ker} g_n^*$, we have

$$\dim \operatorname{Ker} g_n^* \geqq n+1 ,$$

by using (17) and the induction assumptions $(**)_1$ and $(**)_n$. Hence we obtain $(*)_{n+1}$ by the exact sequence (16). Moreover, since

$$f_n^{*-1}(K) \subset \operatorname{Ker} s_n^* ,$$

we have

$$\dim \operatorname{Ker} s_n^* \geqq \dim f_n^{*-1}(K) \geqq n+1 .$$

Then by the commutative diagram

$$\begin{array}{ccc} H^1(M_{n+1}, \mathcal{O}) & \xrightarrow{\tau_{n+1}^*} & H^1(N(\varepsilon_{n+1}), \mathcal{O}) \\ & {\scriptstyle s_n^*}\searrow \quad \nearrow{\scriptstyle t_n^*} & \\ & H^1(N(\varepsilon_n), \mathcal{O}) , & \end{array}$$

we obtain

$$\dim \operatorname{Ker} \tau_{n+1}^* \geqq \dim \operatorname{Ker} s_n^* \geqq n+1 .$$

Therefore, by the commutative diagram

$$\begin{array}{ccc} H^1(M_{n+1}, \mathcal{O}) & \xrightarrow{\sigma^*_{n+1}} & H^1(M^*_{n+1}, \mathcal{O}) \\ & \tau^*_{n+1}\searrow \quad \swarrow\rho^*_{n+1} & \\ & H^1(N(\varepsilon_{n+1}), \mathcal{O}) & \end{array}$$

and Lemma 6, we have

$$\dim \operatorname{Ker} \rho^*_{n+1} \geqq \dim \operatorname{Ker} \tau^*_{n+1} \geqq n+1 ,$$

which proves $(**)_{n+1}$.

PROOF OF (iv) OF THE THEOREM. By the exact sequence

$$0 \longrightarrow C \longrightarrow \mathcal{O} \longrightarrow d\mathcal{O} \longrightarrow 0$$

and $\pi_1(M_n)=0$, we have

$$\dim H^1(M_n, \mathcal{O}) \leqq \dim H^1(M_n, d\mathcal{O}) . \tag{18}$$

Letting $d\Omega^1$ be the subsheaf of Ω^2 whose elements are d-closed, we form the exact sequence

$$0 \longrightarrow d\mathcal{O} \longrightarrow \Omega^1 \longrightarrow d\Omega^1 \longrightarrow 0 . \tag{19}$$

We claim that

$$\dim H^0(M_n, d\Omega^1)=0 . \tag{20}$$

To prove (20), it suffices to show that

$$\dim H^0(M_n, \Omega^2)=0 . \tag{21}$$

Take any $\omega \in H^0(M_n, \Omega^2)$. Then $i^*_n\omega \in H^0(U_{\varepsilon_n}, \Omega^2)$. By Andreotti-Siu [1, Proposition 1.2], we have

$$H^0(U_{\varepsilon_n}, \Omega^2) \cong H^0(P^3, \Omega^2)=0 .$$

Hence $i^*_n\omega=0$. This implies $\omega=0$ and proves (21). Therefore, from (19) and (20),

$$\dim H^1(M_n, d\mathcal{O}) \leqq \dim H^1(M_n, \Omega^1) .$$

Thus combining this with (iii) and the inequality (18), we obtain

$$\dim H^1(M_n, \Omega^1) \geqq n . \qquad \text{Q.E.D.}$$

REMARK 2.*) I don't know whether $\dim H^1(M_n, \mathcal{O})=n$.

*) See the end of the paper.

REMARK 3. For a compact complex manifold X, we put

$$h^{p,q}(X)=\dim H^q(X, \Omega^p) .$$

It is known that, if X is a compact kähler manifold, or, more generally, a compact Fujiki manifold (i.e., of Class $\mathscr{C}$ in Fujiki [2, Definition 1.1]), then the equality

$$h^{p,q}(X)=h^{q,p}(X)$$

holds and the k-th Betti number is given by

$$b_k(X)=\sum_{p+q=k} h^{p,q}(X) .$$

Hence, in particular, we have

$$h^{0,1}(X)=\frac{1}{2}b_1(X) \quad \text{and} \quad h^{1,1}(X)\leqq b_2(X) .$$

By Kodaira [3, Theorem 3], we also see that, if dim $X=2$, then the following equality and inequality hold including the cases where X are non-kähler:

$$h^{0,1}(X)=\begin{cases} \frac{1}{2}b_1(X) , & \text{if } b_1(X)\equiv 0 \pmod 2 \\ \frac{1}{2}(b_1(X)+1) , & \text{if } b_1(X)\equiv 1 \pmod 2 , \end{cases}$$

$$h^{1,1}(X)\leqq b_2(X) .$$

Our example shows, however, that, *for general compact complex manifolds of dimension more than 2, it is impossible to estimate $h^{0,1}(X)$ and $h^{1,1}(X)$ in terms of $b_1(X)$ and $b_2(X)$, respectively.*

REMARK 4. In his recent study of compact complex 3-folds with Hopf surfaces as divisors, H. Tsuji has also found a method of modifying a compact complex manifold as we have used in section §2. Namely, he found that, if a compact complex manifold X, $\dim X\geqq 3$, contains a primary Hopf manifold S of codimension 1 with a certain condition on the normal bundle of S in X, then one can replace S by an elliptic curve E to obtain a new compact complex manifold $Y=(X-S)\cup E$ [4].

Notes added on Dec. 10, 1981. It can be shown that $\dim H^1(M_n, \mathcal{O})=n$, and $\dim H^2(M_n, \mathcal{O})=0$. The differentiable structure of M_n can be described completely by using connected sum operations by virtue of the results of C. T. C. Wall [Invent. Math., 1, 355–374 (1966)]. See the forthcoming paper for these facts.

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