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Class Field Theory of p-Extensions over a Formal Power Series Field with a p-Quasifinite Coefficient Field

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Introduction

Local class field theory has been generalized by M. Moriya and T. Nakayama [4, 5] for the case where the ground field K is a complete local field (i.e., a complete field under a discrete valuation) with a quasifinite residue field C. Here a field C is called quasifinite if C satisfies the following two conditions;

1. C is a perfect field.
2. $\mathrm{Gal}(C_s/C)\cong\hat{\boldsymbol{Z}}$,

where C_s is the separable algebraic closure of C and $\hat{\boldsymbol{Z}}=\mathrm{proj.\,lim}\,\boldsymbol{Z}/n\boldsymbol{Z}$. Thereafter, G. Whaples [7, 8] proved explicitly the existence theorem over a complete local field K with a quasifinite residue field of characteristic $p>0$, introducing the notion of analytic subgroups of the multiplicative group $K^\times$ of K. J. P. Serre [6] reconstructed class field theory over such a field K by using the class formation theory which was introduced by E. Artin. But the existence theorem was discussed only in the case where the residue field of the ground field is finite.

On the other hand, Y. Kawada and I. Satake [2] applied the residue vectors defined in E. Witt [9] to the class formation theory of p-extensions over a formal power series field in one variable with a finite coefficient field of characteristic $p>0$. Thereafter, K. Kanesaka and K. Sekiguchi [3] carried out explicitly the calculation of the residue vectors of the formal power series field with a perfect coefficient field of characteristic $p>0$.

In this paper we consider the generalized local class field theory by the method of Y. Kawada and I. Satake [2] using the explicit calculation of the residue vectors. Since we consider only p-extensions, the condition of the residue field to be quasifinite can be replaced by a weaker condition to be p-quasifinite (see Definition 1.2.1). Namely, we shall prove

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here the fundamental theorems (including the existence theorem) of class field theory of p-extensions over a formal power series field K with a p-quasifinite coefficient field, using the theory of abelian p-extensions and residue vectors of Witt. For the existence theorem, we define a new topology in $K^{\times}$ which we shall call the weak topology. In order to define the weak topology in $K^{\times}$, we use the explicit calculation of residue vectors.

In § 1 we shall prove a class formation of p-extensions over a complete local field with a p-quasifinite residue field by a similar method as that of J. P. Serre [6] (see Theorem 1.2.1).

In § 2 we shall explain well-known theory of Witt for abelian p-extensions of a field of characteristic $p>0$, following Y. Kawada and I. Satake [2]. In order to give a preparation for class field theory of infinite extensions which will be considered in forthcoming paper, we shall also consider infinite abelian p-extensions here. Especially, Theorem 2.2.1 is a natural generalization of Witt theory of finite extensions and show that the generalized Witt theory is dual to the Galois theory in the sense of Pontrjagin.

In § 3 we define first two group pairings by the use of the residue vectors (see § 3.1). One of these pairings is the same as defined in Y. Kawada and I. Satake [2] § 2, (32). Next we define the weak topology in $K^{\times}$. Using the results in § 1 and § 2 together, we shall prove the orthogonal theorem (Theorem 3.2.1) and the fundamental theorem (Theorem 3.2.2) (see § 3.2). The orthogonal theorem is a natural generalization of [2] § 2, (XII). Most of the results in § 3 can be proved by the results in § 2 and by explicit calculations only. The result of the general theory of class formation in § 1 is used only for a part of the proof of the orthogonal theorem. Finally we consider the relation between the existence theorem of G. Whaples [7, 8] and our result.

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§ 1. Class formation of p-extensions over a complete local field with a p-quasifinite residue field.

1.1. First we shall fix some necessary notations which will be used later. Let K be a field, L/K a finite or an infinite Galois extension with the Galois group $G=\mathrm{Gal}(L/K)$. We always consider Krull's compact topology in G. Let A be a G-group. A is called a topological G-group if, for any $a \in A$, the set of those $\sigma \in G$ such that $\sigma a=a$ is an open subgroup of G; a trivial G-group A, $A=L$ and $A=L^{\times}$ are such examples. For a topological G-group A and a positive integer r, the r-cohomology group of G over A is defined by

$$H^r(G, A)=\text{ind.lim}\, H^r(G/H, A^H) \tag{1}$$

where $A^H=\{a\in A\,|\,\sigma a=a \text{ for any } \sigma\in H\}$ and H runs over the set of open normal subgroups of G. Here the inductive limit is defined with respect to the inductive system with inflation homomorphisms. In particular, we write

$$H^r(L/K)=H^r(G, L^{\times}) . \tag{2}$$

Then, we have

$$H^1(L/K)=0 . \tag{3}$$

Let E/K be a finite or an infinite Galois extension containing the Galois extension L/K. Then we have the following exact sequence:

$$0 \longrightarrow H^2(L/K) \xrightarrow{\text{inf}} H^2(E/K) \xrightarrow{\text{res}} H^2(E/L) \quad \text{(exact)} . \tag{4}$$

For a finite Galois extension L/K of degree n, we have

$$n\cdot H^2(L/K)=0 . \tag{5}$$

By (1) and (4), we have also

$$H^2(E/K)=\cup_L H^2(L/K) , \tag{6}$$

where L runs over the set of finite Galois extensions over K contained in E.

For an additive group A and a prime number p, we put

$$A^{(p)}=\{a\in A\,|\,p^e a=0 \text{ for some } e\geqq 0\} ; \tag{7}$$

and for an additive group homomorphism $f: A\to B$, we put

$$f^{(p)}=f|_{A^{(p)}} \quad (\text{the restriction of } f \text{ on } A^{(p)}) . \tag{7'}$$

By (7) and (7′), we have a functor of the category of additive groups into itself. If $0\to A\xrightarrow{f} B\xrightarrow{g} C$ (exact), then $0\to A^{(p)}\xrightarrow{f^{(p)}} B^{(p)}\xrightarrow{g^{(p)}} C^{(p)}$ (exact). If $0\to A\xrightarrow{f} B\xrightarrow{g} C\to 0$ (split, exact), then $0\to A^{(p)}\xrightarrow{f^{(p)}} B^{(p)}\xrightarrow{g^{(p)}} C^{(p)}\to 0$ (split, exact).

Let K be a field, K_{cl} an algebraic closure of K, K_s the separable algebraic closure of K contained in K_{cl} and $K_s^{[p]}$ the maximal separable

p-extension of K contained in K_s. Then $K_s^{[p]}/K$ is a Galois extension and $K \subset L \subset K_s^{[p]}$ implies $L_s^{[p]} = K_s^{[p]}$.

LEMMA 1.1.1. *Let L/K be a finite or an infinite Galois extension contained in K_s, then*

$$\text{Cont. Hom}\,(\text{Gal}\,(L/K),\ \boldsymbol{R}/\boldsymbol{Z})^{(p)} \cong \text{Cont. Hom}\,(\text{Gal}\,(L \cap K_s^{[p]}/K),\ \boldsymbol{R}/\boldsymbol{Z})$$

where we denote by $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{Z}$ the field of real numbers, rational numbers, and the ring of integers respectively. Here we denote by Cont. Hom (,) *the set of all continuous homomorphisms.*

PROOF. For $\chi \in \text{Cont. Hom}\,(\text{Gal}\,(L/K), \boldsymbol{R}/\boldsymbol{Z})$, the order of χ is a power of p if and only if $\text{Gal}\,(L/L \cap K_s^{[p]}) \subset \text{Ker}\,\chi$. Hence if $\chi \in \text{Cont. Hom}\,(\text{Gal}\,(L/K), \boldsymbol{R}/\boldsymbol{Z})^{(p)}$, then χ induces the homomorphism $\bar{\chi}$ of $\text{Gal}\,(L \cap K_s^{[p]}/K)$ into $\boldsymbol{R}/\boldsymbol{Z}$ such that

$$\begin{array}{ccc} \text{Gal}\,(L/K) & \xrightarrow{\ \chi\ } & \boldsymbol{R}/\boldsymbol{Z} \\ \downarrow & \nearrow_{\bar{\chi}} & \\ \text{Gal}\,(L \cap K_s^{[p]}/K) & & \end{array}$$

is a commutative diagram, where the vertical arrow is defined by the canonical homomorphism. Then the mapping $\chi \mapsto \bar{\chi}$ is an isomorphism of $\text{Cont.Hom}\,(\text{Gal}\,(L/K), \boldsymbol{R}/\boldsymbol{Z})^{(p)}$ onto $\text{Cont.Hom}\,(\text{Gal}\,(L \cap K_s^{[p]}/K), \boldsymbol{R}/\boldsymbol{Z})$. Q.E.D.

Let K be a field. Then the Brauer group of the field K is defined by

$$(8) \qquad B_K = H^2(K_s/K)\ ,$$

and we have

$$(9) \qquad B_K^{(p)} = H^2(K_s^{[p]}/K)\ .$$

LEMMA 1.1.2. *Let C be a perfect field of characteristic p, then $B_C^{(p)} = 0$.*

PROOF. By (6) and (9), it is enough to prove that $H^2(C'/C) = H^2(\text{Gal}\,(C'/C), C'^{\times}) = 0$ for a finite Galois p-extension C' over C. This relation is clear by the fact that $C'^{\times}$ is uniquely p-divisible and $\text{Gal}\,(C'/C)$ is a finite p-group. Q.E.D.

Let K be a complete field under a discrete valuation. If the residue field C of K is perfect, then there is a split exact sequence:

$$(10) \qquad 0 \longrightarrow B_C \longrightarrow B_K \xrightarrow[\alpha]{} \text{Cont. Hom}\,(\text{Gal}\,(C_s/C), \boldsymbol{R}/\boldsymbol{Z}) \longrightarrow 0\ .$$

Moreover, let L/K be a finite extension with the ramification index e, and C' the residue field of L, then there is a commutative diagram:

$$
(11)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & B_C & \longrightarrow & B_K & \xrightarrow[\alpha]{} & \text{Cont. Hom}\,(\text{Gal}\,(C_s/C),\, \boldsymbol{R}/\boldsymbol{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle e\cdot\text{res}} & & \\
0 & \longrightarrow & B_{C'} & \longrightarrow & B_L & \xrightarrow[\alpha']{} & \text{Cont. Hom}\,(\text{Gal}\,(C'_s/C'),\, \boldsymbol{R}/\boldsymbol{Z}) & \longrightarrow & 0\,.
\end{array}
$$

Hence we have the following proposition:

PROPOSITION 1.1.1. *Let K be a complete local field with a perfect residue field C of characteristic $p>0$, L/K a finite p-extension with the ramification index e, and C' the residue field of L. Then, we have the following commutative diagram*:

$$
(12)\qquad
\begin{array}{ccc}
B_K^{(p)} & \xrightarrow[\alpha^{(p)}]{\sim} & \text{Cont. Hom}\,(\text{Gal}\,(C_s^{[p]}/C),\, \boldsymbol{R}/\boldsymbol{Z}) \\
\downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle e\cdot\text{res}} \\
B_L^{(p)} & \xrightarrow[\alpha'^{(p)}]{\sim} & \text{Cont. Hom}\,(\text{Gal}\,(C_s^{[p]}/C'),\, \boldsymbol{R}/\boldsymbol{Z})\,.
\end{array}
$$

1.2. We denote by $\boldsymbol{Z}_p$ the ring of all p-adic integers. $\boldsymbol{Z}_p$ is a topological ring by the usual topology.

LEMMA 1.2.1. *Let G be a compact topological group with the fundamental system of neighborhoods of unity consisting of all open normal subgroups of G of finite indices p^n $(n\geqq 1)$, and let $\sigma\in G$. Then there is a unique continuous homomorphism f_σ of $\boldsymbol{Z}_p$ to G such that $f_\sigma(1)=\sigma$.*

PROOF. We define a mapping of $\boldsymbol{Z}$ to G by $n\mapsto\sigma^n$ for $n\in\boldsymbol{Z}$. This mapping is continuous homomorphism with respect to the relative topology of $\boldsymbol{Z}\subset\boldsymbol{Z}_p$. Since $\boldsymbol{Z}$ is dense in $\boldsymbol{Z}_p$ and G is compact, this continuous homomorphism is uniquely extended to the continuous homomorphism f_σ of $\boldsymbol{Z}_p$ to G. Q.E.D.

We denote σ^ν instead of $f_\sigma(\nu)$, for $\nu\in\boldsymbol{Z}_p$; and $\sigma^{\boldsymbol{Z}_p}$ instead of $f_\sigma(\boldsymbol{Z}_p)$.

COROLLARY. *Let G be as in Lemma 1.2.1.*

(i) *If $\sigma\in G$ is of finite order, then the order is a power of p, and*

$$
\sigma^{\boldsymbol{Z}_p}\cong\begin{cases}\boldsymbol{Z}_p\colon \textit{ if } \sigma \textit{ is of infinite order}\\ \boldsymbol{Z}/p^n\boldsymbol{Z}\colon \textit{ if } \sigma \textit{ is of finite order } p^n\ (n\geqq 0)\,.\end{cases}
$$

(ii) *If G is commutative, then G is a $\boldsymbol{Z}_p$-group by the following action:*

$$\begin{array}{ccc} Z_p \times G & \longrightarrow & G \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ (\nu, \sigma) & \longmapsto & \sigma^\nu . \end{array}$$

DEFINITION 1.2.1. Let C be a perfect field of characteristic p. We say that C is p-quasifinite if the following condition holds:

$$\text{(13)} \qquad \mathrm{Gal}\,(C_s^{[p]}/C) \cong \boldsymbol{Z}_p .$$

REMARK. If C is quasifinite of characteristic p, then C is p-quasifinite.

Let $F \in \mathrm{Gal}\,(C_s^{[p]}/C)$. We say that F provides C with a structure of p-quasifinite field or F is a free generator of $\mathrm{Gal}\,(C_s^{[p]}/C)$ if $\mathrm{Gal}\,(C_s^{[p]}/C) = F^{\boldsymbol{Z}_p}$.

PROPOSITION 1.2.1. *Let C be a p-quasifinite field, F a free generator of* $\mathrm{Gal}\,(C_s^{[p]}/C)$, *$C'/C$ a finite extension of degree p^n ($n \geqq 0$), and $F' = F^{p^n}$. Then $F' \in \mathrm{Gal}\,(C_s^{[p]}/C')$ and F' provides C' with a structure of p-quasifinite field. Moreover, C'/C is a cyclic extension.*

The proof is clear.

Let C be a p-quasifinite field, F a free generator of $\mathrm{Gal}\,(C_s^{[p]}/C)$, then, we define a mapping β of Cont. Hom $(\mathrm{Gal}\,(C_s^{[p]}/C), \boldsymbol{R}/\boldsymbol{Z})$ into $\boldsymbol{R}/\boldsymbol{Z}$ by

$$\text{(14)} \qquad \beta(\chi) = \chi(F) \quad \text{for} \quad \chi \in \text{Cont. Hom}\,(\mathrm{Gal}\,(C_s^{[p]}/C), \boldsymbol{R}/\boldsymbol{Z}) .$$

Let C'/C be a finite extension of degree p^n ($n \geqq 0$), then similarly we define a mapping β' by

$$\text{(14')} \qquad \beta'(\chi') = \chi'(F^{p^n}) \quad \text{for} \quad \chi' \in \text{Cont. Hom}\,(\mathrm{Gal}\,(C_s^{[p]}/C'), \boldsymbol{R}/\boldsymbol{Z}) .$$

PROPOSITION 1.2.2. *We have the following commutative diagram:*

$$\text{(15)} \qquad \begin{array}{ccc} \text{Cont. Hom}\,(\mathrm{Gal}\,(C_s^{[p]}/C), \boldsymbol{R}/\boldsymbol{Z}) & \xrightarrow[\beta]{\sim} & (\boldsymbol{Q}/\boldsymbol{Z})^{(p)} \\ \Big\downarrow \text{res} & & \Big\downarrow \times p^n \\ \text{Cont. Hom}\,(\mathrm{Gal}\,(C_s^{[p]}/C'), \boldsymbol{R}/\boldsymbol{Z}) & \xrightarrow[\beta']{\sim} & (\boldsymbol{Q}/\boldsymbol{Z})^{(p)} . \end{array}$$

PROOF. $\beta' \,\mathrm{res}\,(\chi) = \mathrm{res}\,\chi(F^{p^n}) = \chi(F^{p^n}) = p^n \chi(F) = p^n \beta(\chi)$ for any $\chi \in$ Cont. Hom $(\mathrm{Gal}\,(C_s^{[p]}/C), \boldsymbol{R}/\boldsymbol{Z})$. Q.E.D.

By (12) and (15), we have

PROPOSITION 1.2.3. *Let K be a complete local field with a p-quasifinite residue field, L/K a finite separable p-extension of degree $[L: K]$. We put $\mathrm{inv}_K^{[p]} = \beta \circ \alpha^{(p)}$, $\mathrm{inv}_L^{[p]} = \beta' \circ \alpha'^{(p)}$. Then, we have the following commutative diagram:*

$$\begin{array}{ccc} B_K^{(p)} & \xrightarrow[\mathrm{inv}_K^{[p]}]{\sim} & (\boldsymbol{Q}/\boldsymbol{Z})^{(p)} \\ \mathrm{res}\Big\downarrow & & \Big\downarrow \times[L:K] \\ B_L^{(p)} & \xrightarrow[\mathrm{inv}_L^{[p]}]{\sim} & (\boldsymbol{Q}/\boldsymbol{Z})^{(p)} . \end{array} \tag{16}$$

COROLLARY. *Let K be as in Proposition* 1.2.3, *L/K a finite Galois p-extension. Then, we have*

$$H^2(L/K) \cong \boldsymbol{Z}/[L:K]\boldsymbol{Z} . \tag{17}$$

Finally, we have the following theorem:

THEOREM 1.2.1. *Let K be a complete local field with a p-quasifinite residue field C. Put*

$$\begin{cases} \mathfrak{K} = \{L \mid K \subset L \subset K_s^{[p]}, [L:K] < \infty\} , & G = \mathrm{Gal}(K_s^{[p]}/K) , \\ G_L = \mathrm{Gal}(K_s^{[p]}/L) \quad \text{for} \quad L \in \mathfrak{K} . \end{cases} \tag{18}$$

Then we have a class formation $(G, \{G_L \mid L \in \mathfrak{K}\}, K_s^{[p]\times}, \mathrm{inv}_L^{[p]})$. (See J. P. Serre [6].)

In other words, we define a functor $E^\times$ of $\mathfrak{K}$ to the category of abelian groups by

$$\begin{cases} E^\times(L) = L^\times \quad \text{for} \quad L \in \mathfrak{K} \\ E^\times(i_{L/L'}) = i_{L/L'}|_{L^\times} \ (\text{restriction}) \\ \qquad\qquad \text{for} \quad L, L' \in \mathfrak{K}, \ i_{L/L'}: L \hookrightarrow L' \ (\text{natural injection}) . \end{cases} \tag{19}$$

Then $E^\times$ satisfies all the conditions of a class formation. (See Y. Kawada [1].)

By the general theory of class formation, we have

$$K^\times/N_{L/K}L^\times \cong \mathrm{Gal}(L \cap K_{ab}/K) \cong G/[G:G] \tag{20}$$

for $L \in \mathfrak{K}$, L/K: Galois extension, where $G = \mathrm{Gal}(L/K)$, $N_{L/K}$ is the norm map of L to K, K_{ab} is the maximal abelian extension of K contained in K_s and $[G:G]$ means the commutator subgroup of G. Especially, if $L \in \mathfrak{K}$ is a finite abelian p-extension over K, then

$$K^\times/N_{L/K}L^\times \cong \mathrm{Gal}(L/K) . \tag{20'}$$

§ 2. Abelian p-extensions over a field of characteristic p.

2.0. First we shall reproduce here well-known Witt theory for abelian p-extension in characteristic p for our further development. Let

K be a field of characteristic p, $\boldsymbol{F}_p$ the prime field of K. We denote by $W_\infty(K)$, $W_n(K)$ ($n\geqq 1$) the ring of Witt vectors over K of infinite length, of length n, respectively. For a Witt vector $x=(x_0, x_1, \cdots)$ ($x_n\in K$, $n=0, 1, 2, \cdots$) of infinite length, we define the mappings V, π_n ($n\geqq 1$) by $Vx=(0, x_0, x_1, \cdots)\in W_\infty(K)$ and by $\pi_n x=(x_0, x_1, \cdots, x_{n-1})\in W_n(K)$. Then V is an injective additive homomorphism, π_n is a surjective ring homomorphism, and $\operatorname{Ker}\pi_n=V^n W_\infty(K)$ holds. Hence we have the exact sequence:

$$0 \longrightarrow V^n W_\infty(K) \longrightarrow W_\infty(K) \xrightarrow{\pi_n} W_n(K) \longrightarrow 0 .$$

We define a topology in $W_\infty(K)$ by taking $\{V^n W_\infty(K)\,|\,n\geqq 1\}$ as the fundamental system of neighborhoods of 0. Then $W_\infty(K)$ is a topological ring and $W_n(K)$ is a discrete ring with the quotient topology. We define the ring homomorphism A_n of $W_n(K)$ onto $W_{n-1}(K)$ by $(x_0, x_1, \cdots, x_{n-2}, x_{n-1}) \overset{A_n}{\mapsto} (x_0, x_1, \cdots, x_{n-2})$. Then we have $\pi_n=A_{n+1}\cdot\pi_{n+1}$. Hence we have $W_\infty(K)=\text{proj. lim}\, W_n(K)$ with respect to the projective system $(W_n(K), A_n)$. For a Witt vector $x=(x_0, x_1, \cdots)$ of infinite length or of finite length n, we define the mappings P, $\wp$ as usual by $Px=(x_0^p, x_1^p, \cdots)$ and by $\wp x=Px-x$. Then P is an injective ring homomorphism, $\wp$ is an additive homomorphism and $\operatorname{Ker}\wp=W_*(\boldsymbol{F}_p)$ ($*=\infty$ or n) holds. Moreover, we have $px=P\circ Vx=V\circ Px$ and $p^n x=V^n x+\wp\circ V^n(x+Px+P^2x+\cdots+P^{n-1}x)$, for $x\in W_\infty(K)$, $n\geqq 1$. Hence, we have

$$(1)\qquad \pi_n^{-1}\wp W_n(K)=\wp W_\infty(K)+V^n W_\infty(K)=\wp W_\infty(K)+p^n W_\infty(K) .$$

In particular, we have $W_n(\boldsymbol{F}_p)\cong \boldsymbol{Z}/p^n\boldsymbol{Z}$ and $W_\infty(\boldsymbol{F}_p)\cong \boldsymbol{Z}_p$. We define the injective additive homomorphism η_n of $W_n(\boldsymbol{F}_p)$ into $\boldsymbol{R}/\boldsymbol{Z}$ by

$$\eta_n(1)=\frac{1}{p^n}+\boldsymbol{Z}\quad (\in \boldsymbol{R}/\boldsymbol{Z}) .$$

Then we have $p\cdot\eta_{n+1}=\eta_n\circ A_{n+1}$, and $p\eta_{n+1}\circ\pi_{n+1}=\eta_n\circ\pi_n$.

Let K_{cl} be the algebraic closure of K. For $B\in W_\infty(K_{cl})$ or $B\in W_n(K_{cl})$ ($B=(B_0, B_1, \cdots)$), we define the field $K(B)$ generated by B over K by $K(B)=K(B_0, B_1, \cdots)$. Similarly, for a set $Z\subset W_\infty(K_{cl})$ or $Z\subset W_n(K_{cl})$, we define the field $K(Z)$ generated by Z over K by $K(Z)=K(B\,|\,B\in Z)$, and for a set $M\subset W_\infty(K)$ or $M\subset W_n(K)$, we define the field $K(\wp^{-1}M)$ by $K(\wp^{-1}M)=K(B\,|\,\wp B\in M)$.

Let L/K be a finite Galois extension with the Galois group $G=\operatorname{Gal}(L/K)$, then $W_\infty(L)$ and $W_n(L)$ are G-groups by the action: $\sigma B=(\sigma B_0, \sigma B_1, \cdots)$ for $\sigma\in G$ and $B\in W_\infty(L)$ or $B\in W_n(L)$. Then we have

$W_*(K)=\{B\in W_*(L)\,|\,\sigma B=B$ for any $\sigma\in G\}$ ($*=\infty$ or n). Hence we can define the trace mapping of $W_*(L)$ onto $W_*(K)$ by $T_{W_*(L/K)}B=\sum_{\sigma\in G}\sigma B$ ($*=\infty$ or n). Then 1-cohomology group of G over $W_n(L)$ is trivial:

$$H^1(G,\ W_n(L))=0\ . \tag{2}$$

From (2) follows, as usual, the following theorem of Witt. Let L/K be a cyclic extension of degree p^n, then there exists a vector $B\in W_n(L)$ such that $\wp B=b\in W_n(K)$ and $L=K(B)=K(\wp^{-1}b)$. Conversely, for any vector $b\in W_n(K)$, there exists a vector $B\in W_n(K_s)$ such that $\wp B=b$ and $L=K(B)=K(\wp^{-1}b)$ is a cyclic extension of K with the degree p^r $(0\leqq r\leqq n)$.

2.1. Let $\Omega=K_{ab}^{[p]}=K_s^{[p]}\cap K_{ab}$ be the maximal abelian p-extension of K contained in K_s $(\subset K_{cl})$, Ω_n the composite field of all cyclic extensions of degree p^n of K contained in Ω. Then we have

$$\Omega=\bigcup_{n\geqq 1}\Omega_n\ , \tag{3}$$

and

$$\Omega=K(\wp^{-1}W_\infty(K))\ ,\qquad \Omega_n=K(\wp^{-1}W_n(K))\ . \tag{4}$$

We put

$$\Gamma(K)=\mathrm{Gal}\,(\Omega/K)\ ,\qquad \Gamma^n(K)=\mathrm{Gal}\,(\Omega_n/K)\ . \tag{5}$$

Then $\Gamma(K)$ and $\Gamma^n(K)$ are compact abelian groups. By (3), we have

$$\Gamma(K)=\text{proj.}\lim\Gamma^n(K)\ . \tag{6}$$

We define the discrete abelian groups $\mathfrak{W}(K)$ and $\mathfrak{W}_n(K)$ $(n\geqq 1)$ by

$$\begin{cases}\mathfrak{W}(K)=(W_\infty(K)/\wp W_\infty(K))\otimes(\boldsymbol{Q}/\boldsymbol{Z})^{(p)}\ ,\\ \mathfrak{W}_n(K)=(W_\infty(K)/\wp W_\infty(K))\otimes(p^{-n}\boldsymbol{Z}/\boldsymbol{Z})\ ,\end{cases} \tag{7}$$

where $\otimes$ means the tensor product over $\boldsymbol{Z}$. Since $\mathfrak{W}_n(K)\subset\mathfrak{W}_{n+1}(K)$ for all $n\geqq 1$ and $\mathfrak{W}(K)=\bigcup_{n\geqq 1}\mathfrak{W}_n(K)$, we have

$$\mathfrak{W}(K)=\text{ind.}\lim\mathfrak{W}_n(K)\ , \tag{8}$$

with respect to the inductive system of the injection mappings. Let L/K be any extension, then $\mathfrak{W}(L)$ is defined similarly by (7). We define the additive homomorphism $\mathfrak{W}_{K/L}$ of $\mathfrak{W}(K)$ to $\mathfrak{W}(L)$ by

$$\mathfrak{W}_{K/L}\colon (b+\wp W_\infty(K))\otimes(1/p^n+\boldsymbol{Z})\longmapsto(b+\wp W_\infty(L))\otimes(1/p^n+\boldsymbol{Z})\ . \tag{9}$$

By (7) and (9), we have a functor $\mathfrak{W}$ of the category of fields of characteristic $p>0$ to the category of discrete additive groups.

PROPOSITION 2.1.1. (i) *$\mathfrak{W}(K)$ is a divisible torsion p-group.*

(ii) *Let Q be any subgroup of $\mathfrak{W}(K)$, then Q is divisible if and only if $Q=pQ$.*

The proof is clear.

PROPOSITION 2.1.2.

$$\mathfrak{W}_n(K)\cong W_n(K)/\wp W_n(K) \quad (n\geqq 1)\,. \tag{10}$$

PROOF. We define a bilinear mapping of $(W_\infty(K)/\wp W_\infty(K))\times(p^{-n}Z/Z)$ onto $W_n(K)/\wp W_n(K)$ by $(b+\wp W_\infty(K),\ i/p^n+Z)\mapsto i\pi_n b+\wp W_n(K)$. By the property of tensor product, we have the homomorphism $(b+\wp W_\infty(K))\otimes(1/p^n+Z)\mapsto\pi_n b+\wp W_n(K)$ of $\mathfrak{W}_n(K)$ onto $W_n(K)/\wp W_n(K)$. From (1) follows that this homomorphism is injective. Q.E.D.

Hence we can define the additive homomorphism ϕ_n of $W_n(K)$ into $\mathfrak{W}(K)$ by

$$\begin{aligned}\phi_n\colon (b_0, b_1, \cdots, b_{n-1}) \\ \longmapsto ((b_0, b_1, \cdots, b_{n-1}, 0, 0, \cdots)+\wp W_\infty(K))\otimes(1/p^n+Z)\,,\end{aligned} \tag{11}$$

and we have $\operatorname{Ker}\phi_n=\wp W_n(K)$, $\operatorname{Im}\phi_n=\mathfrak{W}_n(K)$,

$$(b+\mathfrak{p}W_\infty(K))\otimes(1/p^n+Z)=\phi_n\cdot\pi_n(b) \quad \text{for} \quad b\in W_\infty(K)\,, \tag{12}$$

$$p\phi_{n+1}\circ\pi_{n+1}=\phi_n\circ\pi_n\,, \tag{13}$$

$$\mathfrak{W}_n(K)=\{\beta\in\mathfrak{W}(K)\,|\,p^n\beta=0\} \quad (n\geqq 1)\,, \tag{14}$$

$$\beta \text{ is of order } p^n \Longleftrightarrow b_0\notin\wp K \quad \text{for} \quad \beta=\phi_n\circ\pi_n(b)\,. \tag{15}$$

LEMMA 2.1.1. *If $\phi_n\circ\pi_n(b)=\phi_m\circ\pi_m(b')$ holds for $m, n\geqq 1$, $b, b'\in W_\infty(K)$, then, we have $K(\wp^{-1}\pi_n b)=K(\wp^{-1}\pi_m b')$.*

PROOF. We may assume $n\leqq m$. By (13), we have $\phi_n\circ\pi_n(b)=\phi_m\circ\pi_m(p^{m-n}b)$ and so $\pi_m b'-\pi_m(p^{m-n}b)\in\wp W_m(K)$. Hence we have $K(\wp^{-1}\pi_m b')=K(\wp^{-1}\pi_m(p^{m-n}b))$. On the other hand, we have $\pi_m(p^{m-n}b)\equiv\pi_m(V^{m-n}b)=(0, \cdots, 0, b_0, \cdots, b_{n-1}) \bmod \wp W_m(K)$. This implies that $K(\wp^{-1}\pi_m(p^{m-n}b))=K(\wp^{-1}\pi_n b)$. Q.E.D.

By this lemma, we can define the field $K(\wp^{-1}\beta)$ generated by $\wp^{-1}\beta$ over K as

$$K(\wp^{-1}\beta)=K(\wp^{-1}\pi_n b) \quad \text{for} \quad \beta=\phi_n\circ\pi_n(b)\in\mathfrak{W}(K),\ b\in W_\infty(K)\,. \tag{16}$$

Moreover, for any subset Q of $\mathfrak{W}(K)$, we define the field $K(\wp^{-1}Q)$ generated by $\wp^{-1}Q$ over K as $K(\wp^{-1}Q)=K(\wp^{-1}\beta \mid \beta\in Q)$.

Let K be any field of characteristic p, then we define an invariant $\lambda(K)$ (which is a cardinal number) by

$$\lambda(K)=\dim_{F_p} K/\wp K\,. \tag{17}$$

PROPOSITION 2.1.3.

$$\mathfrak{W}(K)\cong\bigoplus_{\lambda(K)}(\boldsymbol{Q}/\boldsymbol{Z})^{(p)}\quad(\textit{direct sum})\,.$$

The proof is clear by (10), (14) (put $n=1$) and Proposition 2.1.1. (i).

COROLLARY. *Let K_1, K_2 be two fields of characteristic p, then we have $\lambda(K_1)=\lambda(K_2)$ if and only if $\mathfrak{W}(K_1)\cong\mathfrak{W}(K_2)$.*

2.2. For any vector $b\in W_\infty(K)$, there exists a vector $B\in W_\infty(\Omega)$ such that $\wp B=b$. For any $\sigma\in\Gamma(K)$, we have $\sigma B-B\in W_\infty(\boldsymbol{F}_p)$ and $\sigma B-B=\sigma B'-B'$ holds for different B' with $\wp B'=b$. Hence we can write $\sigma B-B=\sigma(\wp^{-1}b)-\wp^{-1}b$. We define a mapping $\langle\ ,\ \rangle_\infty^\Gamma$ of $\Gamma(K)\times W_\infty(K)$ onto $W_\infty(\boldsymbol{F}_p)$ by

$$\langle\sigma, b\rangle_\infty^\Gamma=\sigma(\wp^{-1}b)-\wp^{-1}b \tag{18}$$

for $\sigma\in\Gamma(K)$, $b\in W_\infty(K)$.

PROPOSITION 2.2.1. (i) *The mapping $\langle\ ,\ \rangle_\infty^\Gamma$ is a group pairing.*
(ii) *We denote by $\boldsymbol{B}_\infty^\Gamma$ the annihilator (operation) of the pairing $\langle\ ,\ \rangle_\infty^\Gamma$. Then, we have*

$$\boldsymbol{B}_\infty^\Gamma(W_\infty(K))=1\,,\qquad \boldsymbol{B}_\infty^\Gamma(\Gamma(K))=\wp W_\infty(K)\,. \tag{19}$$

Hence, we have an orthogonal pairing:

$$\langle\ ,\ \rangle_\infty^\Gamma\colon \Gamma(K)\times W_\infty(K)/\wp W_\infty(K)\longrightarrow W_\infty(\boldsymbol{F}_p)\,. \tag{20}$$

PROOF. (i) The bilinearity of $\langle\ ,\ \rangle_\infty^\Gamma$ is evident. Let $(\sigma, b)\in\Gamma(K)\times W_\infty(K)$, $n\geqq 1$, then we put $L=K(\wp^{-1}\pi_n b)$. Since L/K is a finite cyclic p-extension, $H=\mathrm{Gal}(\Omega/L)$ is an open subgroup of $\Gamma(K)$. Moreover, we have $\langle\sigma H, b+V^n W_\infty(K)\rangle_\infty^\Gamma\subset\langle\sigma, b\rangle_\infty^\Gamma+p^n W_\infty(\boldsymbol{F}_p)$. This proves the continuity of $\langle\ ,\ \rangle_\infty^\Gamma$.

(ii) If $\sigma\in\boldsymbol{B}_\infty^\Gamma(W_\infty(K))$, then $\sigma(\wp^{-1}b)=\wp^{-1}b$ holds for any $b\in W_\infty(K)$, and hence $\sigma=1$. It is clear that $\wp W_\infty(K)\subset\boldsymbol{B}_\infty^\Gamma(\Gamma(K))$. Conversely, let $b\in\boldsymbol{B}_\infty^\Gamma(\Gamma(K))$. Then $\sigma(\wp^{-1}b)=\wp^{-1}b$ holds for any $\sigma\in\Gamma(K)$ and so $\wp^{-1}b\in W_\infty(K)$. Hence, we have $b\in\wp W_\infty(K)$. Q.E.D.

By Corollary (ii) of Lemma 1.2.1 and by the isomorphism $W_\infty(\boldsymbol{F}_p)\cong \boldsymbol{Z}_p$, $\Gamma(K)$ is a $W_\infty(\boldsymbol{F}_p)$-group. Then we have

$$\langle\sigma^\nu, b\rangle_\infty^\Gamma=\nu\langle\sigma, b\rangle_\infty^\Gamma=\langle\sigma, \nu b\rangle_\infty^\Gamma \tag{21}$$

for $\sigma\in\Gamma(K)$, $b\in W_\infty(K)$ and $\nu\in W_\infty(\boldsymbol{F}_p)$.

LEMMA 2.2.1. *If $\phi_n\circ\pi_n(b)=\phi_m\circ\pi_m(b')$ holds for $m, n\geqq 1$, $b, b'\in W_\infty(K)$, then $\eta_n\circ\pi_n\langle\sigma, b\rangle_\infty^\Gamma=\eta_m\circ\pi_m\langle\sigma, b'\rangle_\infty^\Gamma$ for any $\sigma\in\Gamma(K)$.*

PROOF. We may assume $n\leqq m$. By (13), we have $\pi_m(b'-p^{m-n}b)\in \wp W_m(K)$. By (19), we have $\pi_m\langle\sigma, b'\rangle_\infty^\Gamma=\pi_m\langle\sigma, p^{m-n}b\rangle_\infty^\Gamma$. Hence we have $\eta_m\circ\pi_m\langle\sigma, b'\rangle_\infty^\Gamma=\eta_m\circ\pi_m p^{m-n}\langle\sigma, b\rangle_\infty^\Gamma=\eta_n\circ\pi_n\langle\sigma, b\rangle_\infty^\Gamma$. Q.E.D.

By this lemma, we can define the mapping $\langle\ ,\ \rangle^\Gamma$ of $\Gamma(K)\times\mathfrak{W}(K)$ into $\boldsymbol{R}/\boldsymbol{Z}$ by

$$\langle\sigma, \beta\rangle^\Gamma=\eta_n\circ\pi_n\langle\sigma, b\rangle_\infty^\Gamma \tag{22}$$

for $\sigma\in\Gamma(K)$, $b\in W_\infty(K)$, $\beta=\phi_n\circ\pi_n(b)\in\mathfrak{W}(K)$.

PROPOSITION 2.2.2. (i) *The mapping $\langle\ ,\ \rangle^\Gamma$ is a group pairing.*

(ii) *We denote by $\boldsymbol{B}^\Gamma$ the annihilator (operation) of the pairing $\langle\ ,\ \rangle^\Gamma$. Then, we have*

$$\boldsymbol{B}^\Gamma(\mathfrak{W}(K))=1\ ,\qquad \boldsymbol{B}^\Gamma(\Gamma(K))=0\ . \tag{23}$$

Hence we have an orthogonal pairing:

$$\langle\ ,\ \rangle^\Gamma\colon \Gamma(K)\times\mathfrak{W}(K)\longrightarrow \boldsymbol{R}/\boldsymbol{Z}\ . \tag{24}$$

By the duality theorem of Pontrjagin, the compact group $\Gamma(K)$ is dual to the discrete group $\mathfrak{W}(K)$.

PROOF. (i) is clear by Proposition 2.2.1. (i).

(ii) If $\sigma\in\boldsymbol{B}^\Gamma(\mathfrak{W}(K))$, then $\pi_n\langle\sigma, b\rangle_\infty^\Gamma=0$ holds for any $b\in W_\infty(K)$, $n\geqq 1$. Since $\langle\sigma, b\rangle_\infty^\Gamma=0$ for any $b\in W_\infty(K)$, we have $\sigma=1$. If $\beta=\phi_n\circ\pi_n(b)\in \boldsymbol{B}^\Gamma(\Gamma(K))$, then $\sigma(\wp^{-1}\pi_n b)=\wp^{-1}\pi_n b$ holds any $\sigma\in\Gamma(K)$. Since $\pi_n b\in\wp W_n(K)$, we have $\beta=\phi_n\circ\pi_n(b)=0$. Q.E.D.

We put

$$\begin{cases}\mathscr{L}=\{\text{the set of all subfields of } \Omega \text{ containing } K\}\ ,\\ \mathscr{L}^{\text{fin}}=\{L\in\mathscr{L}\mid [L\colon K]<\infty\}\ ,\end{cases} \tag{25}$$

$$\begin{cases}\mathscr{H}=\{\text{the set of all closed subgroups of } \Gamma(K)\}\ ,\\ \mathscr{H}_{\text{open}}=\{H\in\mathscr{H}\mid H\colon \text{open in } \Gamma(K)\}\ ,\end{cases} \tag{26}$$

$$(27)\qquad \begin{cases} \mathscr{Q}=\{\text{the set of all subgroups of } \mathfrak{W}(K)\}\,, \\ \mathscr{Q}^{\text{fin}}=\{Q\in\mathscr{Q}\mid Q\text{: finite}\}\,. \end{cases}$$

We define the mappings between $\mathscr{L}$ and $\mathscr{H}$, between $\mathscr{H}$ and $\mathscr{Q}$, between $\mathscr{Q}$ and $\mathscr{L}$, by

$$(28)\qquad H\longmapsto L=\Omega^H\,,\qquad L\longmapsto H=\mathrm{Gal}\,(\Omega/L)\quad \text{(Galois correspondences)},$$

$$(29)\qquad Q\longmapsto H=\boldsymbol{B}^\Gamma(Q)\,,\qquad H\longmapsto Q=\boldsymbol{B}^\Gamma(H)\,,$$

$$(30)\qquad L\longmapsto Q=\mathrm{Ker}\,\mathfrak{W}_{K/L}\,,\qquad Q\longmapsto L=K(\wp^{-1}Q)\,,$$

for $L\in\mathscr{L}$, $H\in\mathscr{H}$, $Q\in\mathscr{Q}$, respectively.

LEMMA 2.2.2. (i) $\boldsymbol{B}^\Gamma(Q)=\mathrm{Gal}\,(\Omega/K(\wp^{-1}Q))$ *holds for any* $Q\in\mathscr{Q}$.
(ii) $\boldsymbol{B}^\Gamma(\mathrm{Gal}\,(\Omega/L))=\mathrm{Ker}\,\mathfrak{W}_{K/L}$ *holds for any* $L\in\mathscr{L}$.

The proof is clear.

THEOREM 2.2.1. (i) *The mappings* (28), (29), (30) *between* $\mathscr{L}$, $\mathscr{H}$ *and* $\mathscr{Q}$ *are all bijective.*

(ii) *If* L_i, H_i, Q_i $(i=1, 2)$ *correspond to one another by* (28), (29), (30), *then* $L_1\subset L_2\Leftrightarrow H_1\supset H_2\Leftrightarrow Q_1\subset Q_2$.

Since $\mathscr{L}$, $\mathscr{H}$, $\mathscr{Q}$ *are all complete lattices,* (28) *and* (29) *are dual latticeisomorphisms,* (30) *is a lattice-isomorphism.*

(iii) *If L and Q correspond to each other by* (30), *then we have an orthogonal pairing*:

$$(31)\qquad \langle\ ,\ \rangle^\Gamma\colon \mathrm{Gal}\,(L/K)\times Q\longrightarrow \boldsymbol{R}/\boldsymbol{Z}\,.$$

Hence, $\mathrm{Gal}\,(L/K)$ *is dual to* Q.

(iv) *If* L, H, Q *correspond to one another by* (28), (29), (30), *then we have* $L\in\mathscr{L}^{\text{fin}}\Leftrightarrow H\in\mathscr{H}_{\text{open}}\Leftrightarrow Q\in\mathscr{Q}^{\text{fin}}\Leftrightarrow \mathrm{Gal}\,(L/K)\cong Q$.

(v) Ω_n, $\Gamma(K)^{p^n}$, $\mathfrak{W}_n(K)$ $(n\geqq 1)$ *correspond to one another by* (28), (29), (30).

Hence we have

$$(32)\qquad \Gamma^n(K)=\Gamma(K)/\Gamma(K)^{p^n}\,,$$

and we have an orthogonal pairing:

$$(33)\qquad \langle\ ,\ \rangle^\Gamma\colon \Gamma^n(K)\times\mathfrak{W}_n(K)\longrightarrow \boldsymbol{R}/\boldsymbol{Z}\,.$$

Hence $\Gamma^n(K)$ *is dual to* $\mathfrak{W}_n(K)$.

PROOF. (i) The mapping (28) is bijective by the fundamental theorem of Galois theory. The mapping (29) is bijective by the duality theorem of Pontrjagin. By Lemma 2.2.2, the mapping (30) is the composite of (28) and (29), so (30) is also bijective.

(ii), (iii), (iv) hold also by the fundamental theorem of Galois theory and the duality theorem of Pontrjagin.

(v) By (4), (16), we have $\Omega_n=K(\wp^{-1}\mathfrak{W}_n(K))$. On the other hand, by (14) we have $B^{\Gamma}(\Gamma(K)^{p^n})=\mathfrak{W}_n(K)$. Q.E.D.

EXAMPLES. 1. $\lambda(K)=0$. This is equivalent to $K=\wp K$, $\mathfrak{W}(K)=0$, $\Gamma(K)=1$ and $K_{ab}^{[p]}=K$. Especially, if K is an algebraically closed field of characteristic p, then $\lambda(K)=0$.

2. $\lambda(K)=1$. This is equivalent to $K\cong\wp K\oplus \boldsymbol{F}_p$, $\mathfrak{W}(K)\cong(\boldsymbol{Q}/\boldsymbol{Z})^{(p)}$ and $\Gamma(K)\cong \boldsymbol{Z}_p$. Especially, if K is a finite field of characteristic p, a quasifinite field of characteristic p or a p-quasifinite field, then $\lambda(K)=1$.

3. $\lambda(K)=\aleph_0$. If K is a formal power series field with the finite coefficient field of characteristic p, then we have $\lambda(K)=\aleph_0$.

Now we consider a field C of characteristic p such that $\lambda(C)=1$. Let $\sigma\in\Gamma(C)$. We define a mapping S_σ of $W_\infty(C)$ to $W_\infty(\boldsymbol{F}_p)$ by

$$S_\sigma\colon b\longmapsto\langle\sigma, b\rangle_\infty^{\Gamma}\quad\text{for}\quad b\in W_\infty(C)\,. \tag{34}$$

By (21), S_σ is a $W_\infty(\boldsymbol{F}_p)$-homomorphism of $W_\infty(C)$ to $W_\infty(\boldsymbol{F}_p)$ and satisfies

$$S_{\sigma^\nu}=\nu\cdot S_\sigma\quad(\nu\in W_\infty(\boldsymbol{F}_p))\,. \tag{35}$$

If $F\in\Gamma(C)$ satisfies $\Gamma(C)=F^{W_\infty(\boldsymbol{F}_p)}$, then we have

LEMMA 2.2.3. (i) $S_F\in\text{Cont. Hom}\,(W_\infty(C), W_\infty(\boldsymbol{F}_p))$.

(ii) $\text{Ker}\,S_F=\wp W_\infty(C)$, $\text{Im}\,S_F=W_\infty(\boldsymbol{F}_p)$.

(iii) $S_F^{-1}(p^nW_\infty(\boldsymbol{F}_p))=\wp W_\infty(C)+p^nW_\infty(C)$ $(n\geqq 1)$.

(iv) *The exact sequence* $0\to\wp W_\infty(C)\to W_\infty(C)\xrightarrow{S_F}W_\infty(\boldsymbol{F}_p)\to 0$ *splits. In other words, if we put* $b(F)\in W_\infty(C)$ *such that* $S_F(b(F))=1$, *then we have*

$$W_\infty(C)=\wp W_\infty(C)\oplus W_\infty(\boldsymbol{F}_p)b(F)\,. \tag{36}$$

PROOF. (i) and (ii) are clear.

(iii) We have $b\in S_F^{-1}(p^nW_\infty(\boldsymbol{F}_p))\Leftrightarrow\langle F, b\rangle_\infty^{\Gamma}\in p^nW_\infty(\boldsymbol{F}_p)\Leftrightarrow\pi_n\langle F, b\rangle_\infty^{\Gamma}=0\Leftrightarrow\pi_nb\in\wp W_n(C)\Leftrightarrow b\in\wp W_\infty(C)+p^nW_\infty(C)$.

(iv) is also clear. Q.E.D.

Moreover, we assume $C_s^{[p]}=C_{ab}^{[p]}$. Let C'/C be a separable extension of degree p^n, then we have $\Gamma(C')=\Gamma(C)^{p^n}=F^{p^n\boldsymbol{Z}_p}\cong\boldsymbol{Z}_p$. Hence we have $\lambda(C')=1$ and so $S_{F^{p^n}}\colon W_\infty(C')\to W_\infty(\boldsymbol{F}_p)$ is defined similarly by (34).

LEMMA 2.2.4.

(37) $$S_{Fp^n} = S_F \cdot T_{W_\infty(C'/C)} .$$

PROOF. Since $\Gamma(C') = \Gamma(C)^{p^n}$, we have Gal $(C'/C) = \{1, F, \cdots, F^{p^n-1}\}$, and so $T_{W_\infty(C'/C)} b' = \sum_{i=0}^{p^n-1} F^i(b')$ (for $b' \in W_\infty(C')$). Then we have $S_F \cdot T_{W_\infty(C'/C)} b' = \sum_{i=0}^{p^n-1} S_F \cdot F^i(b') = \sum_{i=0}^{p^n-1} (F-1)\wp^{-1}(F^i b') = \sum_{i=0}^{p^n-1} (F^{i+1} - F^i)\wp^{-1} b' = (F^{p^n} - 1)\wp^{-1} b' = S_{Fp^n} b'$. Q.E.D.

This lemma means that S_F is a generalization of the trace mapping $T_{W_\infty(C/F_p)}$. Especially, if $C = \boldsymbol{F}_q$ ($q = p^f$, $f \geqq 1$) is the finite field of characteristic p with q elements, and $F = P^f$ is the Frobenius automorphism, then $S_F = T_{W_\infty(C/F_p)}$.

§ 3. Class field theory of p-extensions over a formal power series field with a p-quasifinite coefficient field.

3.0. Let C be a perfect field of characteristic p, K a formal power series field in one variable t over the field C: $K = C((t))$. For $a \in K^\times$ and $b \in W_\infty(K)$, the residue vector $\mathrm{Res}_\infty^K(a, b) \in W_\infty(C)$ is defined in E. Witt [9], and satisfies the following properties: let $a, a' \in K^\times$, $b, b' \in W_\infty(K)$, $c \in W_\infty(C)$, then

R-1. $\mathrm{Res}_\infty^K(a \cdot a', b) = \mathrm{Res}_\infty^K(a, b) + \mathrm{Res}_\infty^K(a', b)$.

R-2. $\mathrm{Res}_\infty^K(a, b + b') = \mathrm{Res}_\infty^K(a, b) + \mathrm{Res}_\infty^K(a, b')$.

R-3. $\mathrm{Res}_\infty^K(a, cb) = c \cdot \mathrm{Res}_\infty^K(a, b)$.

R-4. $\mathrm{Res}_\infty^K(a, Vb) = V(\mathrm{Res}_\infty^K(a, b))$.

R-5. $\mathrm{Res}_\infty^K(a, Pb) = P(\mathrm{Res}_\infty^K(a, b))$.

R-6. $\mathrm{Res}_\infty^K : K^\times \times W_\infty(K) \to W_\infty(C)$ is continuous, with respect to the usual topology in $K^\times$, $W_\infty(K)$, $W_\infty(C)$ defined by the discrete valuation.

R-7. $\mathrm{Res}_\infty^{K'}(\sigma a, \sigma b) = \sigma(\mathrm{Res}_\infty^K(a, b))$, for another formal power series field K' and a ring isomorphism $\sigma: K \to K'$ such that $\mathrm{ord}_K = \mathrm{ord}_{K'} \circ \sigma$, where we denote by ord_K, $\mathrm{ord}_{K'}$ the discrete normal exponential valuation of K, K', respectively.

Let L/K be a finite extension. Then, L is also a formal power series field. Let C' be the coefficient field of L. Then we have

R-8. $T_{W_\infty(C'/C)} \circ \mathrm{Res}_\infty^L(a, \tilde{b}) = \mathrm{Res}_\infty^K(a, T_{W_\infty(L/K)} \tilde{b})$, for $a \in K^\times$, $\tilde{b} \in W_\infty(L)$.

We can calculate the residue vectors explicitly as follows. We denote by $\mathfrak{o}_K$, m_K, U_K, $U_K^{(1)} = 1 + m_K$ the valuation ring, the valuation ideal, the unit group, the 1-unit group of K, respectively. We shall use here the mapping $\mathfrak{f}_u \in \mathrm{Cont.\,Hom}(W_\infty(C), U_K^{(1)})$ for $u \in m_K$, which is defined in K. Kanesaka and K. Sekiguchi [3]. Then we have the following decomposition of $K^\times$:

$$K^{\times}=C^{\times}\times t^{Z}\times U_K^{(1)}\,,\quad U_K^{(1)}=\prod_{\substack{j\geq 1\\ (j,p)=1}} \mathfrak{f}_{t^j}(W_\infty(C))\,. \tag{1}$$

On the other hand, we have the following decomposition of $W_\infty(K)$:

$$\begin{cases} W_\infty(K)=W_\infty(m_K)\oplus W_\infty(C)\oplus W_\infty(t^{-1}C[t^{-1}])\,,\\ W_\infty(t^{-1}C[t^{-1}])=\overline{\bigoplus_{e\geq 0}\bigoplus_{\substack{m\geq 1\\ (m,p)=1}} W_\infty(C)P^e\{t^{-m}\}}\oplus\prod_{i\geq 1} V^i(\overline{\bigoplus_{\substack{m\geq 1\\ (m,p)=1}} W_\infty(C)\{t^{-m}\}})\,, \end{cases} \tag{2}$$

where $\{a\}=(a,0,0,\cdots)\in W_\infty(K)$ for $a\in K$, and $\overline{\bigoplus_{e\geq 0}\bigoplus_{m\geq 1;\,(m,p)=1} W_\infty(C)\times P^e\{t^{-m}\}}=\{\sum_{e\geq 0}\sum_{m\geq 1;\,(m,p)=1} b(e,m)P^e\{t^{-m}\}\,|\,b(e,m)\in W_\infty(C),\ \lim_{e\to\infty} b(e,m)=0$ (the convergence is uniform with respect to $m\geq 1$, $(m,p)=1$), $\lim_{m\to\infty;\,(m,p)=1} b(e,m)=0$ (the convergence is uniform with respect to $e\geq 0$)$\}$,

$$\overline{\bigoplus_{\substack{m\geq 1\\ (m,p)=1}} W_\infty(C)\{t^{-m}\}}=\{\sum_{\substack{m\geq 1\\ (m,p)=1}} b(m)\{t^{-m}\}\,|\,b(m)\in W_\infty(C),\ \lim_{\substack{m\to\infty\\ (m,p)=1}} b(m)=0\}\,.$$

Using these decompositions, let $a\in K^{\times}$, $b\in W_\infty(K)$ be

$$\begin{cases} a=c\cdot t^n\cdot \prod_{\substack{j\geq 1\\ (j,p)=1}} \mathfrak{f}_{t^j}(a(j))\,,\\ b=b''+b'+\sum_{e\geq 0}\sum_{\substack{m\geq 1\\ (m,p)=1}} b(e,m)P^e\{t^{-m}\}+\sum_{i\geq 1} V^i(\sum_{\substack{m\geq 1\\ (m,p)=1}} b'(i,m)\{t^{-m}\})\,, \end{cases} \tag{3}$$

where $c\in C^{\times}$, $n\in Z$, $a(j)$, b', $b(e,m)$, $b'(i,m)\in W_\infty(C)$, $b''\in W_\infty(m_K)$. Then we have

$$\begin{aligned} \mathrm{Res}_\infty^K(a,b)=n\cdot b'+\sum_{e\geq 0}\sum_{\substack{m\geq 1\\ (m,p)=1}} m\cdot b(e,m)\cdot P^e a(m)\\ +\sum_{i\geq 1} V^i(\sum_{\substack{m\geq 1\\ (m,p)=1}} m\cdot b'(i,m)\cdot a(m))\,. \end{aligned} \tag{4}$$

(See K. Kanesaka and K. Sekiguchi [3].)

3.1. From now until the end of this paper, we denote by K a formal power series field in one variable t over a p-quasifinite field C. If F provides C with a structure of p-quasifinite field, then we have $\lambda(C)=1$, $\Gamma(C)=F^{Z_p}$ and so S_F is defined by (34) in § 2 and satisfies Lemma 2.2.3.

LEMMA 3.1.1. *The additive group* $W_\infty(K)$ *is decomposed as*

$$W^\infty(K)=\wp W_\infty(K)\oplus W_\infty(F_p)b(F)\oplus\overline{\bigoplus_{\substack{m\geq 1\\ (m,p)=1}} W_\infty(C)\{t^{-m}\}}\,, \tag{5}$$

where $b(F)\in W_\infty(C)$ *such that* $S_F(b(F))=1$. (See Lemma 2.2.3.)

PROOF. It is clear that $\wp W_\infty(m_K)=W_\infty(m_K)$, $W_\infty(C)=\wp W_\infty(C)\oplus$

$W_\infty(F_p)b(F)$. Hence, it is enough to prove that $W_\infty(t^{-1}C[t^{-1}])=\wp W_\infty(t^{-1}C[t^{-1}])\oplus \overline{\bigoplus_{m\geqq 1;\,(m,p)=1} W_\infty(C)\{t^{-m}\}}$. Obviously, $W_\infty(t^{-1}C[t^{-1}])$ contains both $\wp W_\infty(t^{-1}C[t^{-1}])$, $\overline{\bigoplus_{m\geqq 1;\,(m,p)=1} W_\infty(C)\{t^{-m}\}}$ and so their direct sum. Conversely, let $b=c'P^e\{t^{-m}\}\in W_\infty(C)P^e\{t^{-m}\}$. If we put $c\in W_\infty(C)$ such that $P^e c=c'$, then we have $b=P^e(c\{t^{-m}\})$. Then $b=\wp(\sum_{i=0}^{e-1} P^i(c\{t^{-m}\}))+c\{t^{-m}\}$. Hence we have $W_\infty(C)P^e\{t^{-m}\}\subset\wp(W_\infty(t^{-1}C[t^{-1}]))\oplus W_\infty(C)\{t^{-m}\}$. Let $b=\sum_{i\geqq 1} V^i(\sum_{m\geqq 1;\,(m,p)=1} b(i,m)\{t^{-m}\})\in \prod_{i\geqq 1} V^i(\overline{\bigoplus_{m\geqq 1;\,(m,p)=1} W_\infty(C)\{t^{-m}\}})$, where $\lim_{m\to\infty;\,(m,p)=1} b(i,m)=0$. If we put $b(i)=\sum_{m\geqq 1;\,(m,p)=1} b(i,m)\{t^{-m}\}$, $b'(i)=b(i)+pb(i)+\cdots+p^{i-1}b(i)$, then we have $b=\sum_{i\geqq 1} V^i(b(i))=\sum_{i\geqq 1} p^i b(i)-\wp(\sum_{i\geqq 1} V^i b'(i))\in\wp W_\infty(t^{-1}C[t^{-1}])\oplus \overline{\bigoplus_{m\geqq 1;\,(m,p)=1} W_\infty(C)\{t^{-m}\}}$. Hence we have $W_\infty(t^{-1}C[t^{-1}])\subset\wp W_\infty(t^{-1}C[t^{-1}])\oplus \overline{\bigoplus_{m\geqq 1;\,(m,p)=1} W_\infty(C)\{t^{-m}\}}$. Q.E.D.

We define a mapping $\langle\ ,\ \rangle_\infty^K$ of $K^\times\times W_\infty(K)$ onto $W_\infty(F_p)$ by

$$\langle a, b\rangle_\infty^K=S_F(\mathrm{Res}_\infty^K(a, b)) \tag{6}$$

for $a\in K^\times$, $b\in W_\infty(K)$.

PROPOSITION 3.1.1. (i) *The mapping $\langle\ ,\ \rangle_\infty^K$ is a group pairing.*
(ii) *We denote by B_∞^K the annihilator (operation) of the pairing $\langle\ ,\ \rangle_\infty^K$. Then, we have*

$$B_\infty^K(W_\infty(K))=C^\times\ ,\qquad B_\infty^K(K^\times)=\wp W_\infty(K)\ . \tag{7}$$

Hence we have an orthogonal pairing:

$$\langle\ ,\ \rangle_\infty^K\colon K^\times/C^\times\times W_\infty(K)/\wp W_\infty(K)\longrightarrow W_\infty(F_p)\ . \tag{8}$$

(iii) *Using the decomposition* (1) *and* (5), *let $a\in K^\times$, $b\in W_\infty(K)$ be*

$$\begin{cases} a=c\cdot t^n\cdot \prod\limits_{\substack{j\geqq 1\\ (j,p)=1}} \mathfrak{f}_{tj}(a(j)) \\ b=\wp(b')+\nu\cdot b(F)+\sum\limits_{\substack{m\geqq 1\\ (m,p)=1}} b(m)\{t^{-m}\} \end{cases} \tag{9}$$

where $c\in C^\times$, $n\in \boldsymbol{Z}$, $a(j), b(m)\in W_\infty(C)$, $b'\in W_\infty(K)$, $\nu\in W_\infty(F_p)$, $\lim_{m\to\infty;\,(m,p)=1} b(m)=0$. *Then we have*

$$\langle a, b\rangle_\infty^K=n\cdot\nu+\sum_{\substack{m\geqq 1\\ (m,p)=1}} m\cdot S_F(a(m)b(m))\ . \tag{10}$$

PROOF. (i) is clear by R-1, R-2, R-6 and Lemma 2.2.3. (i).
(ii) and (iii) are also clear by (1), (4), (5). Q.E.D.

LEMMA 3.1.2. *If $\phi_n\circ\pi_n(b)=\phi_m\circ\pi_m(b')$ holds for $m, n\geqq 1$, $b, b'\in W_\infty(K)$, then $\eta_n\circ\pi_n\langle a, b\rangle_\infty^K=\eta_m\circ\pi_m\langle a, b'\rangle_\infty^K$ for any $a\in K^\times$.*

PROOF. We may assume $n \leqq m$. By (13) in § 2 and (7), we have $\pi_m\langle a, b'\rangle_\infty^K = \pi_m\langle a, p^{m-n}b\rangle_\infty^K$. Hence, we have $\eta_m \circ \pi_m\langle a, b'\rangle_\infty^K = \eta_m \circ \pi_m p^{m-n}\langle a, b\rangle_\infty^K = \eta_n \circ \pi_n\langle a, b\rangle_\infty^K$. Q.E.D.

By this lemma, we can define the mapping $\langle\ ,\ \rangle^K$ of $K^\times \times \mathfrak{W}(K)$ into $\boldsymbol{R}/\boldsymbol{Z}$ by

$$\langle a, \beta\rangle^K = \eta_n \circ \pi_n\langle a, b\rangle_\infty^K \tag{11}$$

for $a \in K^\times$, $b \in W_\infty(K)$, $\beta = \phi_n \circ \pi_n(b) \in \mathfrak{W}(K)$.

PROPOSITION 3.1.2. (i) *The mapping $\langle\ ,\ \rangle^K$ is a group pairing.*

(ii) *We denote by $\boldsymbol{B}^K$ the annihilator (operation) of the pairing $\langle\ ,\ \rangle^K$. Then we have*

$$\boldsymbol{B}^K(\mathfrak{W}(K)) = C^\times\ , \qquad \boldsymbol{B}^K(K^\times) = 0\ . \tag{12}$$

Hence we have an orthogonal pairing:

$$\langle\ ,\ \rangle^K\colon K^\times/C^\times \times \mathfrak{W}(K) \longrightarrow \boldsymbol{R}/\boldsymbol{Z}\ . \tag{13}$$

Here the duality theorem of Pontrjagin can not be applied for this pairing. Hence $K^\times/C^\times$ is not dual to $\mathfrak{W}(K)$.

PROOF. (i) is clear by Proposition 3.1.1. (i).

(ii) It is clear that $C^\times \subset \boldsymbol{B}^K(\mathfrak{W}(K))$. Conversely, let $a \in \boldsymbol{B}^K(\mathfrak{W}(K))$, then $\pi_n\langle a, b\rangle_\infty^K = 0$ for all $b \in W_\infty(K)$, $n \geqq 1$. Since $\langle a, b\rangle_\infty^K = 0$ for all $b \in W_\infty(K)$, we have $a \in C^\times$. If $\beta = \phi_n \circ \pi_n(b) \in \boldsymbol{B}^K(K^\times)$, then $\pi_n\langle a, b\rangle_\infty^K = 0$ for all $a \in K^\times$. Since $\pi_n b \in \wp W_n(K)$, we have $\beta = \phi_n \circ \pi_n(b) = 0$. Q.E.D.

Let L/K be a finite Galois p-extension, C' the coefficient field of L. Then, C'/C is a finite p-extension. If the degree $[C'\colon C] = p^n$, then C' is also a p-quasifinite field and F^{p^n} provides C' with a structure of p-quasifinite field (see Proposition 1.2.1). Hence the pairings $\langle\ ,\ \rangle_\infty^L = S_{F^{p^n}} \cdot \mathrm{Res}_\infty^L(\ ,\ )$, $\langle\ ,\ \rangle^L$ are defined similarly by (6), (11), respectively.

LEMMA 3.1.3. (i) *$\langle \sigma\tilde{a}, \sigma\tilde{b}\rangle_\infty^L = \langle \tilde{a}, \tilde{b}\rangle_\infty^L$ holds for $\tilde{a} \in L^\times$, $\tilde{b} \in W_\infty(L)$, $\sigma \in \mathrm{Gal}(L/K)$.*

(ii) *$\langle a, \tilde{b}\rangle_\infty^L = \langle a, T_{W_\infty(L/K)}\tilde{b}\rangle_\infty^K$ holds for $a \in K^\times$, $\tilde{b} \in W_\infty(L)$.*

PROOF. (i) By R-7, we have $\mathrm{Res}_\infty^L(\sigma\tilde{a}, \sigma\tilde{b}) = \sigma\,\mathrm{Res}_\infty^L(\tilde{a}, \tilde{b})$. Since $\sigma(C') = C'$, we have $\sigma|_{C'} \in \mathrm{Gal}(C'/C)$ and so $S_{F^{p^n}} \cdot \sigma = S_{F^{p^n}}$. Hence we have $S_{F^{p^n}} \cdot \mathrm{Res}_\infty^L(\sigma\tilde{a}, \sigma\tilde{b}) = S_{F^{p^n}} \cdot \mathrm{Res}_\infty^L(\tilde{a}, \tilde{b})$.

(ii) is clear by R-8 and Lemma 2.2.4. Q.E.D.

PROPOSITION 3.1.3. *$\langle N_{L/K}\tilde{a}, b\rangle_\infty^K = \langle \tilde{a}, b\rangle_\infty^L$ holds for $\tilde{a} \in L^\times$, $b \in W_\infty(K)$.*

PROOF. For any $b \in W_\infty(K)$, there exists $\tilde{b} \in W_\infty(L)$ such that $T_{W_\infty(L/K)}\tilde{b} = b$. By Lemma 3.1.3. (i), we have $\langle \sigma\tilde{a}, \tilde{b}\rangle_\infty^L = \langle \tilde{a}, \sigma^{-1}\tilde{b}\rangle_\infty^L$ and so $\langle N_{L/K}\tilde{a}, \tilde{b}\rangle_\infty^L = \langle \tilde{a}, T_{W_\infty(L/K)}\tilde{b}\rangle_\infty^L = \langle \tilde{a}, b\rangle_\infty^L$. On the other hand, by Lemma 3.1.3. (ii), we have $\langle N_{L/K}\tilde{a}, \tilde{b}\rangle_\infty^L = \langle N_{L/K}\tilde{a}, b\rangle_\infty^K$. Q.E.D.

COROLLARY. (i) $\langle N_{L/K}\tilde{a}, \beta\rangle^K = \langle \tilde{a}, \mathfrak{W}_{K/L}\beta\rangle^L$ *holds for* $\tilde{a} \in L^\times$, $\beta \in \mathfrak{W}(K)$.
(ii) $\boldsymbol{B}^K(N_{L/K}L^\times) = \operatorname{Ker} \mathfrak{W}_{K/L}$.

PROOF. (i) is clear by Proposition 3.1.3.
(ii) We have $\beta \in \boldsymbol{B}^K(N_{L/K}L^\times) \Leftrightarrow \langle N_{L/K}L^\times, \beta\rangle^K = 0 \Leftrightarrow \langle L^\times, \mathfrak{W}_{K/L}\beta\rangle^L = 0 \Leftrightarrow \mathfrak{W}_{K/L}\beta = 0 \Leftrightarrow \beta \in \operatorname{Ker} \mathfrak{W}_{K/L}$. Q.E.D.

3.2. We define a new topology in $K^\times$. We denote by $W_\infty(C)^\times$ the unit group of the ring $W_\infty(C)$. We put

$$V(e, m, n, c) = t^{p^e Z} \times \mathfrak{f}_{t^m}(p^n W_\infty(C) + c^{-1}\wp W_\infty(C)) \times \prod_{\substack{j \geq 1 \\ (j,p)=1, j \neq m}} \mathfrak{f}_{t^j}(W_\infty(C)) , \tag{14}$$

where $e \geq 0$, $m \geq 1$, $(m, p) = 1$, $n \geq 0$, $c \in W_\infty(C)^\times$. $V(e, m, n, c)$ is a subgroup of $K^\times$, and the intersection of all $V(e, m, n, c)$ ($e \geq 0$, $m \geq 1$, $(m, p) = 1$, $n \geq 0$, $c \in W_\infty(C)^\times$) consists of only the unity of $K^\times$. Hence there exists a unique topology in $K^\times$ generated by

$$\{V(e, m, n, c) \mid e \geq 0,\ m \geq 1,\ (m, p) = 1,\ n \geq 0,\ c \in W_\infty(C)^\times\} .$$

We call this topology the weak topology in $K^\times$. The set of all finite intersection of $V(e, m, n, c)$ ($e \geq 0$, $m \geq 1$, $(m, p) = 1$, $n \geq 0$, $c \in W_\infty(C)^\times$) is a fundamental system of neighborhoods of unity with respect to the weak topology in $K^\times$. $K^\times$ is a topological group with respect to the weak topology.

LEMMA 3.2.1. *The weak topology in* $K^\times/C^\times$ *is the compact-open topology with respect to the pairing* $\langle\ ,\ \rangle^K$, i.e., $\{\boldsymbol{B}^K(Q)/C^\times \mid Q \in \mathscr{Q}^{\mathrm{fin}}\}$ *is a fundamental system of neighborhoods of unity with respect to the weak topology in* $K^\times/C^\times$.

PROOF. For $e \geq 1$, $m \geq 1$, $(m, p) = 1$, $n \geq 1$, $c \in W_\infty(C)^\times$, we put $Q(e, m, n, c) = \boldsymbol{Z}\cdot\phi_e \circ \pi_e b(F) \oplus \boldsymbol{Z}\cdot\phi_n \circ \pi_n(c\{t^{-m}\}) \in \mathscr{Q}^{\mathrm{fin}}$. By (10) and Lemma 2.2.3, we have $C^\times \times V(e, m, n, c) = \boldsymbol{B}^K(Q(e, m, n, c))$. On the other hand, for any $Q \in \mathscr{Q}^{\mathrm{fin}}$, there exist $e_i \geq 1$, $m_i \geq 1$, $(m_i, p) = 1$, $n_i \geq 1$, $c(i) \in W_\infty(C)^\times$, ($i = 1, 2, \cdots, r$) such that $Q \subset \sum_{i=1}^r Q(e_i, m_i, n_i, c(i))$. Then we have $\bigcap_{i=1}^r C^\times \times V(e_i, m_i, n_i, c(i)) \subset \boldsymbol{B}^K(Q)$. Q.E.D.

Hence $\langle\ ,\ \rangle_\infty^K$ and $\langle\ ,\ \rangle^K$ are also continuous mappings and so are group pairings with respect to the weak topology in $K^\times$. Since we have

Cont. Hom $(W_\infty(K)/\wp W_\infty(K),\ W_\infty(F_p))\cong\Gamma(K)$, we can define the mapping ρ_K of $K^\times$ into $\Gamma(K)$ by

$$\langle\rho_K(a),\ b\rangle_\infty^\Gamma=\langle a,\ b\rangle_\infty^K \tag{15}$$

for $a\in K^\times$, $b\in W_\infty(K)$. For any $L\in\mathscr{L}$, we define the mapping $\rho_{L/K}$ of $K^\times$ into Gal (L/K) by

$$\rho_{L/K}(a)=\rho_K(a)|_L \quad \text{(the restriction of } \rho_K(a) \text{ on } L)\ , \tag{16}$$

for $a\in K^\times$. Obviously, we have $\rho_K=\rho_{\Omega/K}$ and

$$\langle\rho_{L/K}(a),\ \beta\rangle^\Gamma=\langle a,\ \beta\rangle^K \tag{17}$$

for $a\in K^\times$, $\beta\in\operatorname{Ker}\mathfrak{W}_{K/L}$.

PROPOSITION 3.2.1. (i) *Let L, H, Q and L', H', Q' correspond to one another, respectively by* (28), (29), (30) *in* § 2. *If $L'\subset L$, then we have*

$$\rho_{L/K}^{-1}(\mathrm{Gal}\,(L/L'))=\rho_K^{-1}(H')=B^K(Q')\ . \tag{18}$$

(ii) $\rho_{L/K}\in$ Cont. Hom $(K^\times,\ \mathrm{Gal}\,(L/K))$, *with respect to the weak topology in $K^\times$.*

(iii) $\rho_{L/K}(K^\times)$ *is dense in* Gal (L/K).

(iv) $\operatorname{Ker}\rho_K=C^\times$.

(v) *The weak topology in $K^\times/C^\times$ is the relative topology of $\Gamma(K)$ with respect to the injection $\tilde{\rho}_K$: $K^\times/C^\times\hookrightarrow\Gamma(K)$.*

PROOF. (i) For any $a\in K^\times$, we have $a\in\rho_{L/K}^{-1}(\mathrm{Gal}\,(L/L'))\Leftrightarrow\rho_K(a)\in H'\Leftrightarrow\langle\rho_K(a),\ Q'\rangle^\Gamma=0\Leftrightarrow\langle a,\ Q'\rangle^K=0$.

(ii) It is sufficient to prove that $\rho_K\in$ Cont. Hom $(K^\times,\ \Gamma(K))$. For any $a,\ a'\in K^\times$, $b\in W_\infty(K)$, we have $\langle\rho_K(a\cdot a'),\ b\rangle_\infty^\Gamma=\langle a\cdot a',\ b\rangle_\infty^K=\langle a,\ b\rangle_\infty^K+\langle a',\ b\rangle_\infty^K=\langle\rho_K(a),\ b\rangle_\infty^\Gamma+\langle\rho_K(a'),\ b\rangle_\infty^\Gamma=\langle\rho_K(a)\cdot\rho_K(a'),\ b\rangle_\infty^\Gamma$. Hence we have $\rho_K(a\cdot a')=\rho_K(a)\cdot\rho_K(a')$. The continuity of ρ_K is clear by Lemma 3.2.1 and (18).

(iii) We put $\overline{\rho_{L/K}(K^\times)}=\mathrm{Gal}\,(L/L')$, $H'=\mathrm{Gal}\,(\Omega/L')$, and $Q'=\operatorname{Ker}\mathfrak{W}_{K/L'}$. By (18), we have $K^\times=\rho_K^{-1}(H')=B^K(Q')$. Since $K^\times/B^K(Q')\times Q'\to\boldsymbol{R}/\boldsymbol{Z}$ is an orthogonal pairing, $K^\times/B^K(Q')=1$ is compact, and Q' is discrete, we have $Q'=0$ and so $L'=K$.

(iv) By (12), (18), we have $\operatorname{Ker}\rho_K=B^K(\mathfrak{W}(K))=C^\times$.

(v) is clear by Lemma 3.2.1 and (18). Q.E.D.

REMARK. There exists $L\in\mathscr{L}$ such that the weak topology in $K^\times/\operatorname{Ker}\rho_{L/K}$ is not the relative topology of Gal (L/K) with respect to the injection $\tilde{\rho}_{L/K}$: $K^\times/\mathrm{Ker}\,\rho_{L/K} \hookrightarrow \mathrm{Gal}\,(L/K)$, in the case where the coefficient field C is infinite.

By (18), we have

$$\mathrm{Ker}\,\rho_{L/K} = \rho_K^{-1}(H) = \boldsymbol{B}^K(Q)\ . \tag{19}$$

We put

$$\begin{cases} \mathscr{A} = \{\text{the set of all closed subgroups of } K^\times \text{ with respect to the weak topology containing } C^\times\}\,, \\ \mathscr{A}_{\mathrm{open}} = \{A \in \mathscr{A} \mid A \text{ is open in } K^\times \text{ with respect to the weak topology}\}\,. \end{cases} \tag{20}$$

We define the mappings between $\mathscr{A}$ and $\mathscr{H}$, between $\mathscr{A}$ and $\mathscr{Q}$ by

$$A \longmapsto H = \overline{\rho_K(A)}\ , \qquad H \longmapsto A = \rho_K^{-1}(H), \tag{21}$$

$$A \longmapsto Q = \boldsymbol{B}^K(A)\ , \qquad Q \longmapsto A = \boldsymbol{B}^K(Q)\ , \tag{22}$$

for $H \in \mathscr{H}$, $Q \in \mathscr{Q}$, $A \in \mathscr{A}$, respectively.

LEMMA 3.2.2. (i) *For any $A \in \mathscr{A}$, we have $\boldsymbol{B}^\Gamma(\rho_K(A)) = \boldsymbol{B}^K(A)$. Hence we have*

$$\overline{\rho_K(A)} = \boldsymbol{B}^\Gamma(\boldsymbol{B}^K(A))\ . \tag{23}$$

(ii) *For any $A \in \mathscr{A}$, we have $A = \rho_K^{-1}(\overline{\rho_K(A)}) = \boldsymbol{B}^K(\boldsymbol{B}^K(A))$.*

PROOF. (i) For any $\beta \in \mathfrak{W}(K)$, we have $\beta \in \boldsymbol{B}({}^\Gamma\rho_K(A)) \Leftrightarrow \langle \rho_K(A), \beta\rangle^\Gamma = 0 \Leftrightarrow \langle A, \beta\rangle^K = 0 \Leftrightarrow \beta \in \boldsymbol{B}^K(A)$.

(ii) It is clear that $A \subset \rho_K^{-1}(\overline{\rho_K(A)})$. Conversely, let $a \in \rho_K^{-1}(\overline{\rho_K(A)})$. Take an open neighborhood V of a in $K^\times$. Then by Proposition 3.2.1. (v), we have an open set W in $\Gamma(K)$ such that $\rho_K(V) = W \cap \rho_K(K^\times)$. Since $\rho_K(a) \in \overline{\rho_K(A)}$, we have $W \cap \rho_K(A) \neq \varnothing$. If $\sigma \in W \cap \rho_K(A)$, then there exists $a' \in A$ such that $\sigma = \rho_K(a')$. Since $\rho_K(a') \in W \cap \rho_K(K^\times) = \rho_K(V)$, there exists $a'' \in V$ such that $\rho_K(a') = \rho_K(a'')$. This implies $a'' \in V \cap A$ and so $V \cap A \neq \varnothing$. Hence we have $a \in A$. Let $Q = \boldsymbol{B}^K(A)$, $H = \overline{\rho_K(A)}$. Then by (23), we have $H = \boldsymbol{B}^\Gamma(Q)$. Hence we have $\boldsymbol{B}^K(\boldsymbol{B}^K(A)) = \boldsymbol{B}^K(Q) = \rho_K^{-1}(H) = \rho_K^{-1}(\overline{\rho_K(A)}) = A$.

Q.E.D.

COROLLARY. *The mapping* (21): $A \mapsto H = \overline{\rho_K(A)}$ *is an injective lattice-homomorphism. Hence we have*

$$\mathscr{A} \hookrightarrow \mathscr{H}\ . \tag{24}$$

THEOREM 3.2.1. *Let L/K be a finite separable p-extension. Then we have*

$$N_{L/K}L^{\times}=B^{K}(\operatorname{Ker}\mathfrak{W}_{K/L}) . \tag{25}$$

PROOF. It is sufficient to prove that

$$N_{L/K}L^{\times}=\operatorname{Ker}\rho_{L/K} , \tag{26}$$

for $L\in\mathscr{L}^{\mathrm{fin}}$. Since $\mathrm{Gal}(L/K)$ is a finite group, by Proposition 3.2.1. (iii), $\rho_{L/K}$ is surjective. On the other hand, by Corollary (ii) of Proposition 3.1.3, we have $N_{L/K}L^{\times}\subset\operatorname{Ker}\rho_{L/K}$. Since $(K^{\times}: N_{L/K}L^{\times})=(K^{\times}:\operatorname{Ker}\rho_{L/K})=[L: K]$, we have $N_{L/K}L^{\times}=\operatorname{Ker}\rho_{L/K}$. Q.E.D.

THEOREM 3.2.2. *The mapping*

$$L\longmapsto A=N_{L/K}L^{\times} \tag{27}$$

is a dual lattice-isomorphism of $\mathscr{L}^{\mathrm{fin}}$ onto $\mathscr{A}_{\mathrm{open}}$. And we have the isomorphism:

$$\tilde{\rho}_{L/K}: K^{\times}/A \xrightarrow{\sim} \mathrm{Gal}(L/K) . \tag{28}$$

PROOF. It is sufficient to prove that the mapping (27) of $\mathscr{L}^{\mathrm{fin}}$ to $\mathscr{A}_{\mathrm{open}}$ is surjective. Let $A\in\mathscr{A}_{\mathrm{open}}$, $H=\overline{\rho_K(A)}$. By Proposition 3.2.1. (v), there exists an open set W in $\Gamma(K)$ such that $\rho_K(A)=W\cap\rho_K(K^{\times})$. For any $\sigma\in W$ and any open neighborhood W' of σ in $\Gamma(K)$, we have $W'\cap\rho_K(A)=W'\cap W\cap\rho_K(K^{\times})\neq\varnothing$. Hence, we have $\sigma\in\overline{\rho_K(A)}$. This implies $W\subset H$, and so $H\in\mathscr{H}_{\mathrm{open}}$. If we put $L=\Omega^{H}\in\mathscr{L}^{\mathrm{fin}}$, then we have $N_{L/K}L^{\times}=\rho_K^{-1}(H)=A$. Q.E.D.

REMARK. We consider the relation between the existence theorem of G. Whaples [7, 8] and our Theorem 3.2.2. First we recall the existence theorem of G. Whaples in the case where the ground field K is a formal power series field. Let C be a quasifinite field of characteristic $p>0$, K the formal power series field over the field C. Consider the subring generated by P over $W_\infty(C)$ of the endomorphism ring of the additive group $W_\infty(C)$. An element $\mathfrak{L}$ in this ring is written by $\mathfrak{L}=\sum_{\nu=0}^{e}c(\nu)P^{\nu}$ for $e\geqq 0$, $c(\nu)\in W_\infty(C)$. Let E be the set of those $\mathfrak{L}=\sum_{\nu=0}^{e}c(\nu)P^{\nu}$ such that $c(\nu)\in W_\infty(C)^{\times}$ for some $\nu=0, 1, \cdots, e$. A subgroup A of $K^{\times}$ is called analytic if the following two conditions hold:

(A-1) A is an open subgroup of $K^{\times}$ with respect to the usual topology defined by the discrete valuation.

(A-2) There exists an element $\mathfrak{L}\in E$ such that

$$\mathfrak{f}_{t^i}(\mathfrak{L}W_\infty(C))\subset A \quad \text{for any} \quad i\geqq 1 .$$

(See Definition 1 and Lemma 8 in G. Whaples [7, 8].)
Then the existence theorem of G. Whaples is described as follows:

THEOREM (Whaples). *Let A be a subgroup of $K^\times$. A is the norm group of a finite abelian extension of K if and only if A is analytic and of finite index* (see Theorem 4 and Theorem 5 in G. Whaples [7, 8]).

Next consider the condition:
(A-2′) For any $j\geqq 1$, $(j, p)=1$, there exists an element $\mathfrak{L}_j\in \boldsymbol{E}$ such that $\mathfrak{f}_{t^j}(\mathfrak{L}_j W_\infty(C))\subset A$.
By Proposition 21 in G. Whaples [8], property (1.7)′: $\mathfrak{f}_{u^p}(a)=\mathfrak{f}_u(Va)$ in K. Kanesaka and K. Sekiguchi [3] and $p=P\circ V=V\circ P$, we can prove that (A-1) and (A-2′) imply (A-2). Hence (A-1) and (A-2′) are equivalent to (A-1) and (A-2). On the other hand, G. Whaples [8] defined a topology in $W_\infty(C)$ by taking $\{\mathfrak{L}W_\infty(C)+p^n W_\infty(C)\mid \mathfrak{L}\in \boldsymbol{E},\ n\geqq 0\}$ as the fundamental system of neighborhoods of 0. We call this topology the W-topology (Whaples-topology) in $W_\infty(C)$. Let $U^{(1)}$ be the 1-unit group of K. We define a topology in $U^{(1)}$ as the direct product topology of W-topology in $W_\infty(C)$ by § 3, (1): $U^{(1)}=\prod_{j\geqq 1;(j,p)=1}\mathfrak{f}_{t^j}(W_\infty(C))$, and call this topology the W-topology in $U^{(1)}$. Then we have the following theorem:

THEOREM (♯). *Let A be a subgroup of $K^\times$. A is analytic if and only if $A\cap U^{(1)}$ is open in $U^{(1)}$ with respect to the W-topology in $U^{(1)}$.*

Since the usual topology in $U^{(1)}$ defined by the discrete valuation is the direct product topology of the usual topology in $W_\infty(C)$ defined by the discrete valuation by § 3, (1), the equivalence of (A-1) and (A-2) to (A-1) and (A-2′) implies Theorem (♯). Hence the condition: $A\in\mathscr{A}_{\text{open}}$ implies that A is analytic and of finite index $p^n (n=0, 1, \cdots)$. Therefore the existence theorem of G. Whaples implies our existence theorem in the case where the coefficient field C is quasifinite of characteristic p. The author can not prove directly the converse, namely the equivalence of the weak topology in $U^{(1)}$ to the W-topology in $U^{(1)}$.

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