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A Remark on the Duality Mapping on l^∞

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Here we answer a question raised in [1]. In order to formulate this question we are to recall some notations and facts from [1]. We work with the dual $l^{\infty *}$ of l^∞, which can be written as the direct sum $l^1+c_0^\perp$. S denotes the unit sphere in l^∞ and $\mathrm{sm}\, S$ the set of the smooth points of S. The duality mapping $F_0: S \to 2^{l^{\infty *}}$ is defined as follows

$$F_0(v)=\{\lambda \in l^{\infty *}: \lambda(u)=1=\|\lambda\|\}\,, \qquad v \in S\,.$$

$\mathrm{ext}\, F_0(v)$ denotes the set of extremal points of $F_0(v)$. The mentioned question sounds as:

"*Given* $v \in S\backslash \mathrm{sm}\, S$ *and* $\lambda \in \mathrm{ext}\, F_0(v)$, *does there exist a sequence* $\{v_n\} \subset \mathrm{sm}\, S$ *such that* $\|v_n - v\| \to 0$ *and that* λ *is a* w^**-cluster point of the sequence* $\{F_0(v_n)\}$?"

The answer is negative in general as it follows from Propositions 1 and 2. Owing to some reasons from [1] we may and do restrict ourselves to the situation when $v \geqq 0$ and $\lambda \in c_0^\perp$.

We recall that (see [1]) there is a one-to-one correspondence between *ultrafilters* and 0-1-*measures*, namely, given an ultrafilter $\mathscr{U}$ on the set of natural numbers N we can define the measure on N as

$$(*) \qquad \lambda(A)=\begin{cases}1 & \text{iff} \quad A \in \mathscr{U}\\ 0 & \phantom{\text{iff}} \quad A \notin \mathscr{U}\end{cases}$$

and conversely. Also, a 0-1-measure λ is in $c_0^\perp$ if and only if the corresponding $\mathscr{U}$ is free (non-principal), i.e., $\mathscr{U}$ contains no finite sets. It is known [1] that, for $v \in S$, $v \geqq 0$, $\mathrm{ext}\, F_0(v)$ consists only of 0-1-measures.

PROPOSITION 1. *Let* $v \in S\backslash \mathrm{sm}\, S$, $v \geqq 0$, $\lambda \in \mathrm{ext}\, F_0(v) \cap c_0^\perp$ *and* $\mathscr{U}$ *be the ultrafilter associated with* λ *by* $(*)$. *Then the following assertions are equivalent*:

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(i) *There exists* $\{v_n\}\subset \mathrm{sm}\, S$ *such that* $\|v_n-v\|\to 0$ *as* $n\to\infty$ *and that* λ *is a* w^**-cluster point of the sequence* $\{F_0(v_n)\}$.

(ii) *There exists* $\{s_n\}\subset N$ *such that* $v(s_n)\to 1$ *as* $n\to\infty$ *and* $\lambda(\{s_n\})=1$.

(iii) $\mathscr{F}\backslash\mathscr{U}\neq\emptyset$, *where*

$$\mathscr{F}=\{A\in N\colon H_n\backslash A \text{ is finite for each } n\in N\}\ .$$

and

$$H_n=\left\{m\in N\colon \frac{n-1}{n}\leqq v(m)<\frac{n}{n+1}\right\}\ .$$

PROOF. (i)$\Leftrightarrow$(ii) is [1, Proposition 7.6]. Let us show that (ii)$\Leftrightarrow$(iii). If $A\in\mathscr{F}\backslash\mathscr{U}$, then the complement A^c is in $\mathscr{U}$ and so A^c is infinite because $\mathscr{U}$ is free. Hence A^c represents an infinite sequence. Now $\{s_n\}^c\in\mathscr{F}\backslash\mathscr{U}$ if and only if $\lambda(\{s_n\})=1$ and $H_m\cap\{s_n\}$ is finite for each m. But the finiteness of the sets $H_m\cap\{s_n\}$ is equivalent with $v(s_n)\to 1$ as $n\to\infty$.

We can see from Proposition 1 that in order to answer the above question negatively it suffices to find a v in $S\backslash\mathrm{sm}\, S$ and λ in $\mathrm{ext}\, F_0(v)\cap c_0^\perp$ with the corresponding $\mathscr{U}$ in such a way that $\mathscr{F}\subset\mathscr{U}$. This idea leads to the proof of the following proposition.

PROPOSITION 2. *Let* $v\in S\backslash\mathrm{sm}\, S$, $v\geqq 0$. *Then the following assertions are equivalent*:

(i) *For each* 0-1-*measure* $\lambda\in F_0(v)\cap c_0^\perp$, *there exists a sequence* $\{s_n\}\subset N$ *such that* $v(s_n)\to 1$ *and* $\lambda(\{s_n\})=1$.

(ii) *There is* $m\in N$ *such that, for* $n\geqq m$, *the sets* H_n *are finite*.

In this case there exists a sequence $\{v_n\}\subset\mathrm{sm}\, S$ *such that* $\|v_n-v\|\to 0$ *and that each* 0-1-*measure* $\lambda\in F_0(v)\cap c_0^\perp$, *with* $\lambda(v^{-1}(1))=0$, *is a* w^**-cluster point of* $\{F_0(v_n)\}$.

PROOF. Let (ii) hold. Take a 0-1-measure λ in $F_0(v)\cap c_0^\perp$. If $\lambda(v^{-1}(1))=1$, then since $\lambda\in c_0^\perp$, $v^{-1}(1)$ is an infinite set and, writing $v^{-1}(1)=\{s_n\}$, we have the seeking sequence. Further let us assume that $\lambda(v^{-1}(1))=0$. Then we have

$$1=\int v d\lambda=\int v\chi_{\bigcup_1^{m-1}H_n}d\lambda+\int v\chi_{\bigcup_m^\infty H_n}d\lambda\leqq\frac{m-2}{m-1}\lambda\Big(\bigcup_1^{m-1}H_n\Big)+\lambda\Big(\bigcup_m^\infty H_n\Big)$$

and so $\lambda(\bigcup_m^\infty H_n)=1$. Hence the set $\bigcup_m^\infty H_n$ is infinite and, writing $\{s_n\}=\bigcup_m^\infty H_n$, we have that the sets $\{s_n\}\cap H_i$ are finite for $i\geqq m$ by assumption. It follows that $v(s_n)\to 1$. We now define $v_n\in l^\infty$ by

$$v_n(s)=\begin{cases}1 & \text{if} \quad s=s_n\ , \\ v(s) & \text{if} \quad v(s)<v(s_n)\ , \\ v(s_n) & \text{if} \quad v(s)\geqq v(s_n)\ .\end{cases}$$

Then $v_n(s_n)=1$ and $0\leqq v_n(s_n)\leqq v(s_n)<1$ for every $s\neq s_n$ and so, by [1, Corollary 6.5] $v_n\in \mathrm{sm}\, S$, $F_0(v_n)=\delta_{s_n}$. Moreover $\|v_n-v\|\leqq 1-v(s_n)\to 0$ as $n\to\infty$ and it is easily seen that λ is a w^*-cluster point of the sequence $\{F_0(v_n)\}$ so obtained, which proves the last assertion of Proposition 2.

Conversely, let (ii) be violated. Then $\emptyset\notin\mathscr{F}$ since there are infinite H_n. Also, if $A, B\in\mathscr{F}$, then $H_n\backslash(A\cap B)=(H_n\backslash A)\cup(H_n\backslash B)$ is finite and so $A\cap B\in\mathscr{F}$. Further, if $A\in\mathscr{F}$ then $A\cap(\bigcap_1^m H_n^c)\neq\emptyset$ since otherwise A would be in $\bigcup_1^m H_n$ and hence, for $n>m$, $H_n\backslash A=H_n$. But the last set is infinite for some $n>m$, contradicting the definition of $\mathscr{F}$. It follows there is an ultrafilter $\mathscr{U}$ containing $\mathscr{F}$ and all H_n^c. $\mathscr{U}$ is free, since if there would exist a finite set A in $\mathscr{U}$, then $A\subset\bigcup_1^m H_n\cup v^{-1}(1)$ for some m. But $(\bigcup_1^m H_n)^c=\bigcap_1^m H_n^c\in\mathscr{U}$ and $(v^{-1}(1))^c\in\mathscr{F}\subset\mathscr{U}$, which leads to a contradiction. Now let λ be the 0-1-measure associated with $\mathscr{U}$. λ is in $c_0^\perp$ since $\mathscr{U}$ is free. And, as $\mathscr{F}\subset\mathscr{U}$, Proposition 1 says that (i) is violated.

Of course, there exists $v\in S\backslash\mathrm{sm}\, S$ violating (ii) in Proposition 2. So, by Proposition 1, the answer to our question is negative.

Reference

[1] I. HADA, K. HASHIMOTO and S. OHARU, On the duality mapping of l^∞, Tokyo J. Math., **2** (1979), 71-97.

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