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Non-solvable Multiplicative Subgroups of Simple Algebras of Degree 2

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Let $M_2(\Delta)$ be the full matrix algebra of degree 2 over a division algebra Δ of characteristic 0. In [8] we determined the non-abelian simple groups which are homomorphic images of multiplicative subgroups of $M_2(\Delta)$. In this paper we will study the non-solvable multiplicative subgroups G of $M_2(\Delta)$ such that $V_{\boldsymbol{Q}}(G)=M_2(\Delta)$, where $V_{\boldsymbol{Q}}(G)=\{\sum \alpha_i g_i \mid \alpha_i \in \boldsymbol{Q},\ g_i \in G\}$. Let N be the largest solvable normal subgroup of G. In §1 we will prove that G/N is isomorphic to a subgroup W of $\mathrm{Aut}(T)$ with $W \supset T$, where $T \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5)\times PSL(2, 5)$. Let H be the largest normal subgroup of G such that $[H, H]=H$. We will prove in §2 and §3 $H \cong SL(2, 5)$, $SL(2, 9)$, $SL(2, 5)\times SL(2, 5)$ or E, where E is an extension of $PSL(2, 5)$ by DQ, the central product of the dihedral group D of order 8 and the quaternion group Q of order 8. In §4 first we will characterize G in the case where G has a normal subgroup M such that $V_{\boldsymbol{Q}}(M)\cong \Delta_1 \oplus \Delta_2$ for some division algebras Δ_1 and Δ_2. In the other case we will show the following;

(1) $O(G)$ is a Z-group (i.e. all Sylow subgroups of $O(G)$ are cyclic).

(2) G has a normal subgroup G_1 such that $G_1 \supset O(G)$, G/G_1 is a 2-group of order $\leqq 8$, and $G_1/O(G)\cong SL(2, 5)P$, $SL(2, 9)$ or E, where P is a cyclic 2-group or a dihedral group of order $2^n \geqq 4$, and $SL(2, 5)P$ is the central product of $SL(2, 5)$ and P.

§1. The largest solvable normal subgroup.

All division algebras considered in this paper are of characteristic 0. As usual $\boldsymbol{Q}$ and $\boldsymbol{C}$ denote respectively the rational number field and the complex number field. By a subgroup of $M_2(\Delta)$ we mean a finite multiplicative subgroup of $M_2(\Delta)$. Let Δ be a division algebra and let K be a field contained in the center of Δ. Let G be a subgroup of $M_2(\Delta)$. We define $V_K(G)=\{\sum \alpha_i g_i \mid \alpha_i \in K,\ g_i \in G\}$ as a K-subalgebra of $M_2(\Delta)$.

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Let $\mathscr{C}$ be the class of finite groups G which satisfies the following conditions (a) and (b):

(a) A Sylow 3-subgroup of G is an abelian group generated by at most 2-elements.

(b) A non-abelian simple group which occurs as a composition factor of G is isomorphic to $PSL(2, 5)$ or $PSL(2, 9)$.

If G is a subgroup of $M_2(\Delta)$, then by [6] and [8] $G \in \mathscr{C}$. Let N be the largest solvable normal subgroup of G. As is easily seen, $G/N \in \mathscr{C}$ and the largest solvable normal subgroup of G/N is trivial.

LEMMA 1.1. *Let G be an element of $\mathscr{C}$. We assume that G is non-solvable and that the largest solvable normal subgroup of G is trivial. Then we have*

(1) *Let H be a normal subgroup of G which is the direct product of non-abelian simple groups S_i, $H = S_1 \times S_2 \times \cdots \times S_n$. Then $n \leqq 2$ and $H \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$.*

(2) *Let M be a minimal normal subgroup of G with $M \neq 1$. Then $M \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$.*

(3) *If $C_G(H) \supset M$, then $H \cong M \cong PSL(2, 5)$.*

(4) *There exists a normal subgroup T of G such that $C_G(T) = 1$ and $T \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$.*

PROOF. (1) By the condition (b) S_i is isomorphic to $PSL(2, 5)$ or $PSL(2, 9)$, $i = 1, 2, \cdots, n$. Since a Sylow 3-subgroup of $PSL(2, 5)$ (resp. $PSL(2, 9)$) is a cyclic group (resp. an elementary abelian group of order 9), (1) follows directly from the condition (a).

(2) It is well known that $M \cong S \times S \times \cdots \times S$ for some simple group S. Since the largest solvable normal subgroup of G is trivial, S is non-abelian. Therefore by (1) $M \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$.

(3) The condition $C_G(H) \supset M$ means $MH \cong M \times H$, because $M \cap H \subset C_G(H) \cap H = 1$. Since $MH \lhd G$, it follows from (1) and (2) that $M \cong H \cong PSL(2, 5)$.

(4) Let L be a non-trivial minimal normal subgroup of G. By (2) $L \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$. If $C_G(L)$ is solvable, then by the assumption $C_G(L) = 1$. Thus we may assume that $C_G(L)$ is non-solvable. Let M be a minimal normal subgroup of G such that $1 \neq M \subset C_G(L)$. By (3) $M \cong L \cong PSL(2, 5)$. Suppose that $C_G(LM)$ is not solvable. Let K be a minimal normal subgroup of G such that $1 \neq K \subset C_G(LM)$. Then by (3) $LM \cong K \cong PSL(2, 5)$, which contradicts the fact $LM \cong PSL(2, 5) \times PSL(2, 5)$. Hence $C_G(LM)$ is solvable, and $C_G(LM) = 1$. In this case, if we put $T = LM$, then we get the assertion (4).

Using this lemma we have

PROPOSITION 1.2. *Let Δ be a division algebra. Let G be a non-solvable subgroup of $M_2(\Delta)$. Then we have*

(1) *The largest solvable normal subgroup N of G is non-trivial.*

(2) *G/N is isomorphic to a subgroup W of $\mathrm{Aut}(T)$ with $W \supset T$, where $T \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$.*

PROOF. By (1.1) (4) there exists a normal subgroup T of G/N such that $C_{G/N}(T)=1$ and $T \cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5) \times PSL(2, 5)$. Hence G/N is isomorphic to a subgroup of $\mathrm{Aut}(T)$. If $N=1$, then either $PSL(2, 5)$ or $PSL(2, 9)$ is a subgroup of $M_2(\Delta)$. But it contradicts the main result in [8]. Therefore $N \neq 1$.

As is well known, $\mathrm{Aut}(PSL(2, 5))/PSL(2, 5)$ and $\mathrm{Aut}(PSL(2, 9))/PSL(2, 9)$ are 2-groups.

LEMMA 1.3. $\mathrm{Aut}(PSL(2, 5) \times PSL(2, 5))/(PSL(2, 5) \times PSL(2, 5))$ *is a 2-group.*

PROOF. Let τ_1 (resp. τ_2) be the morphism from $PSL(2, 5)$ to $PSL(2, 5) \times PSL(2, 5)$ determined by $\tau_1(a)=(a, 1)$ (resp. $\tau_2(a)=(1, a)$). Let μ_i be the projection of $PSL(2, 5) \times PSL(2, 5)$ on the i-th component. Let σ be an automorphism of $PSL(2, 5) \times PSL(2, 5)$. We denote by σ_{ij} the morphism $\mu_i \sigma \tau_j$ from $PSL(2, 5)$ to $PSL(2, 5)$. Since $PSL(2, 5)$ is simple, $\mathrm{Ker}\, \sigma_{ij}=1$ or $PSL(2, 5)$.

Now we will prove that one of the following holds:

(1) $\mathrm{Ker}\, \sigma_{11}=\mathrm{Ker}\, \sigma_{22}=1$, $\mathrm{Ker}\, \sigma_{12}=\mathrm{Ker}\, \sigma_{21}=PSL(2, 5)$; or

(2) $\mathrm{Ker}\, \sigma_{11}=\mathrm{Ker}\, \sigma_{22}=PSL(2, 5)$, $\mathrm{Ker}\, \sigma_{12}=\mathrm{Ker}\, \sigma_{21}=1$.

Since $\mu_i \sigma$ is a surjection, $\mathrm{Ker}\, \sigma_{i1}=1$ or $\mathrm{Ker}\, \sigma_{i2}=1$. We assume that $\mathrm{Ker}\, \sigma_{i1}=\mathrm{Ker}\, \sigma_{i2}=1$. Let a, b be a pair of elements of $PSL(2, 5)$ satisfying $[a, b] \neq 1$. We put $a'=\sigma_{i1}^{-1}(a)$, $b'=\sigma_{i2}^{-1}(b)$. Then $\tau_1(a')=(a', 1)$ and $\tau_2(b')=(1, b')$, which implies $[\tau_1(a'), \tau_2(b')]=1$ and $[\mu_i \sigma \tau_1(a'), \mu_i \sigma \tau_2(b')]=1$. It is impossible because $a=\mu_i \sigma \tau_1(a')$ and $b=\mu_i \sigma \tau_2(b')$. Next we assume that $\mathrm{Ker}\, \sigma_{1j}=\mathrm{Ker}\, \sigma_{2j}=1$. Then $\sigma \tau_1(PSL(2, 5))=\sigma \tau_2(PSL(2, 5))=PSL(2, 5) \times 1$ if $j=1$, $=1 \times PSL(2, 5)$ if $j=2$. It is a contradiction.

Let ν be an automorphism of $PSL(2, 5) \times PSL(2, 5)$ determined by $\nu(a, b)=(b, a)$. In the case (1) $\sigma=(\sigma_{11}, \sigma_{22}) \in \mathrm{Aut}(PSL(2, 5)) \times \mathrm{Aut}(PSL(2, 5))$. In the case (2) $\nu\sigma \in \mathrm{Aut}(PSL(2, 5)) \times \mathrm{Aut}(PSL(2, 5))$. Thus $\mathrm{Aut}(PSL(2, 5) \times PSL(2, 5))/(PSL(2, 5) \times PSL(2, 5))$ is a 2-group.

In [7] we proved that a solvable subgroup of $M_2(\Delta)$ has a normal

Hall $\{2, 3, 5, 7\}'$-subgroup. This result can be generalized to any subgroup of $M_2(\Delta)$.

COROLLARY 1.4. *Let Δ be a division algebra. Let G be a subgroup of $M_2(\Delta)$. Then G has a normal Hall $\{2, 3, 5, 7\}'$-subgroup.*

PROOF. We may assume that G is non-solvable. Let N be the largest solvable normal subgroup of G. Let $\pi=\{2, 3, 5, 7\}$. Let H be a normal Hall π'-subgroup of N. Since $PSL(2, 5)$ and $PSL(2, 9)$ are π-groups, $\mathrm{Aut}(PSL(2, 5))$, $\mathrm{Aut}(PSL(2, 9))$ and $\mathrm{Aut}(PSL(2, 5)\times PSL(2, 5))$ are π-groups. By (1.2) H is a normal Hall π'-subgroup of G.

§ 2. Perfect groups.

A group G is perfect if $[G, G]=G$. In this paper we denote by D, Q, DQ and DD respectively the dihedral group of order 8, the quaternion group of order 8, the central product of D and Q and the central product of D and D. In this section we will determine all perfect subgroups of $M_2(\Delta)$ such that no normal subgroup of G is isomorphic to DQ. Let m, r be relatively prime integers, and put $s=(r-1, m)$, $t=m/s$; $n=$the minimal positive integer satisfying $r^n\equiv 1 \bmod m$. Denote by $G_{m,r}$ the group generated by two elements a, b with the relations: $a^m=1$, $b^n=a^t$ and $bab^{-1}=a^r$. Let ζ_m be a fixed primitive m-th root of unity and let $\sigma=\sigma_r$ be the automorphism of $\boldsymbol{Q}(\zeta_m)$ determined by the mapping $\zeta_m\rightarrow\zeta_m^r$. We denote by $\Lambda_{m,r}$ the cyclic algebra $(\boldsymbol{Q}(\zeta_m), \sigma_r, \zeta_s)$.

First we recall the results in Amitsur [1].

(2.1) ([1]). Let G be a finite group and let Δ be a divison algebra. Assume that $G\subset\Delta$. Then we have

(1) The odd Sylow subgroups of G are cyclic and the even Sylow subgroups of G are cyclic or generalized quaternion.

(2) If all Sylow subgroups of G are cyclic, then $G\cong G_{m,r}$ for some relatively prime integers m, r with $(n, t)=1$.

(3) A group $G_{m,r}$ can be embedded in a division algebra if and only if $\Lambda_{m,r}$ is a division algebra; then we have $V_Q(G_{m,r})\cong\Lambda_{m,r}$ and the isomorphism is obtained by the correspondence $a\leftrightarrow\zeta_m$, $b\leftrightarrow\sigma_r$.

(4) If G is not solvable, then $G\cong SL(2, 5)\times G_{m,r}$ and $V_Q(G)\cong\Lambda_{10,-1}\otimes_Q\Lambda_{m,r}\cong(\Lambda_{4,-1}\otimes_Q\boldsymbol{Q}(\sqrt{5}))\otimes_Q\Lambda_{m,r}$.

COROLLARY 2.2. *Let G be a non-trivial perfect subgroup of $M_2(\Delta)$.*

(1) *If $V_Q(G)\cong\Delta_1$ for some division algebra Δ_1, then $G\cong SL(2, 5)$ and $V_Q(G)\cong\Lambda_{10,-1}$.*

(2) *If* $V_Q(G)\cong\Delta_1\oplus\Delta_2$ *for some division algebras* Δ_1, Δ_2, *then one of the following holds*:

(i) $G\cong SL(2, 5)$ *and* $V_Q(G)\cong \boldsymbol{Q}\oplus\Lambda_{10,-1}$; *or*

(ii) $G\cong SL(2, 5)\times SL(2, 5)$ *and* $V_Q(G)\cong\Lambda_{10,-1}\oplus\Lambda_{10,-1}$.

PROOF. Since $[G, G]=G$, the assertion (1) follows directly from (2.1)(4).

We now assume that $V_Q(G)\cong\Delta_1\oplus\Delta_2$ for some division algebras Δ_1, Δ_2. Let ρ_i be the projection of $V_Q(G)$ on the i-th component of $\Delta_1\oplus\Delta_2$. Since $G\subset V_Q(G)$, the morphism $\rho: G\to\rho_1(G)\times\rho_2(G)$ determined by the mapping $g\to(\rho_1(g), \rho_2(g))$ is injective. Because $[G, G]=G$, $[\rho_i(G), \rho_i(G)]=\rho_i(G)$ and $V_Q(\rho_i(G))=\Delta_i$. By (1), $\rho_i(G)\cong 1$ and $\Delta_i\cong\boldsymbol{Q}$, or $\rho_i(G)\cong SL(2, 5)$ and $\Delta_i\cong\Lambda_{10,-1}$. Therefore $V_Q(G)\cong\Delta_1\oplus\Delta_2\cong\boldsymbol{Q}\oplus\Lambda_{10,-1}$ or $\Lambda_{10,-1}\oplus\Lambda_{10,-1}$, because $G\neq 1$. In the case where $V_Q(G)\cong\boldsymbol{Q}\oplus\Lambda_{10,-1}$, we may assume that $\rho_1(G)=1$ and $\rho_2(G)\cong SL(2, 5)$. Then since $|\rho_2(G)|\leqq|G|\leqq|\rho_1(G)\times\rho_2(G)|=|\rho_2(G)|$, $G\cong\rho_2(G)\cong SL(2, 5)$.

Next we assume that $\Delta_1\cong\Delta_2\cong\Lambda_{10,-1}$. Put $K_i=\operatorname{Ker}\rho_i$, $i=1, 2$. Since ρ is injective, $K_1\cap K_2=1$. Since $K_1K_2/K_i\lhd SL(2, 5)$, $K_1K_2/K_i\cong 1$, $Z(SL(2, 5))$ or $SL(2, 5)$. The fact $|G: K_i|=|SL(2, 5)|$ implies $|K_1|=|K_2|$. If $|K_1K_2/K_1|=|K_1K_2/K_2|=1$, then $K_1=K_2=1$ and $G\cong SL(2, 5)$. By [10] $\boldsymbol{Q}[SL(2, 5)]\cong\boldsymbol{Q}\oplus M_5(\boldsymbol{Q})\oplus M_3(\Lambda_{4,-1})\oplus M_2(\Delta_3)\oplus M_4(\boldsymbol{Q})\oplus M_3(\boldsymbol{Q}(\sqrt{5}))\oplus\Lambda_{10,-1}$, where $\Delta_3\cong(\boldsymbol{Q}(\zeta_3), \tau, -1)$. Hence $V_Q(SL(2,5))\not\cong\Lambda_{10,-1}\oplus\Lambda_{10,-1}$. Thus $|K_1K_2/K_1|=|K_1K_2/K_2|\neq 1$. Suppose that $K_1K_2/K_1\cong K_1K_2/K_2\cong Z(SL(2, 5))$. Since $\rho(K_1K_2)\subset\rho_1(K_1K_2)\times\rho_2(K_1K_2)\subset Z(SL(2, 5))\times Z(SL(2, 5))$, we get $K_1K_2\subset Z(G)$. Therefore G is a central extension of $PSL(2, 5)$ with $[G, G]=G$. Since the Schur multiplier of $PSL(2, 5)$ is 2, we have that $|K_1K_2|\leqq 2$. But it is impossible. In fact, by the assumption, $|K_1K_2|=|K_1\times K_2|=|K_1|^2=|Z(SL(2, 5))|^2=4$. Thus $K_1K_2/K_i\cong SL(2, 5)$. Since $K_1\cong K_1K_2/K_2\cong SL(2, 5)$, $|SL(2, 5)\times SL(2, 5)|=|K_1K_2|\leqq|G|\leqq|\rho_1(G)\times\rho_2(G)|=|SL(2, 5)\times SL(2, 5)|$. Hence we conclude that $G\cong SL(2, 5)\times SL(2, 5)$ if $V_Q(G)\cong\Lambda_{10,-1}\oplus\Lambda_{10,-1}$.

LEMMA 2.3. *Let* Δ *be a division algebra. Let* G_1 *and* G_2 *be subgroups of* $M_2(\Delta)$. *Let* 1 *be the unit element of* $M_2(\Delta)$. *Assume that* $V_Q(G_i)$ *contains the simple algebra* A_i *with* $A_i\ni 1$, $i=1, 2$. *If* A_1 *and* A_2 *satisfy one of the following conditions* (1)-(4), *then we have* $[G_1, G_2]\neq 1$.

(1) $A_1\cong A_2\cong M_2(\boldsymbol{Q})$.

(2) $A_1\cong A_2\cong\Lambda_{4,-1}$.

(3) $A_1\cong\Lambda_{4,-1}$ *and* $A_2\cong M_2(\boldsymbol{Q}(\zeta_3))$.

(4) $A_1\cong\Lambda_{4,-1}$ *and* $A_2\cong M_2(\boldsymbol{Q}(i))$.

PROOF. Suppose that $[G_1, G_2]=1$. In any case the center of $A_1=\boldsymbol{Q}$. Since $a_1a_2=a_2a_1$ for any element $a_i\in A_i$, $i=1, 2$, $A_1\otimes_Q A_2$ is isomorphic to a Q-subalgebra A_1A_2 of $M_2(\Delta)$. On the other hand $M_2(Q)\otimes_Q M_2(Q)\cong M_4(Q)$, $\Lambda_{4,-1}\otimes_Q \Lambda_{4,-1}\cong M_4(Q)$, $\Lambda_{4,-1}\otimes_Q M_2(Q(\zeta_8))\cong M_4(Q(\zeta_8))$ and $\Lambda_{4,-1}\otimes_Q M_2(Q(i))\cong M_4(Q(i))$. Hence in any case $M_2(\Delta)$ contains a Q-subalgebra which is isomorphic to $M_4(Q)$. It is a contradiction. Thus we obtain $[G_1, G_2]\neq 1$.

LEMMA 2.4. *Let G be a perfect subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. Then $O(G)$ (the largest normal $2'$-subgroup of G) is trivial.*

PROOF. We assume that $O(G)\neq 1$. If $V_Q(O(G))$ is not a division algebra, by [7] (2.3) G has a normal subgroup of index 2, contradicting the assumption $[G, G]=G$. Therefore $V_Q(O(G))$ is a division algebra. By (2.1) all Sylow subgroups of $O(G)$ are cyclic. Let p be the maximal prime number which divides the order of $O(G)$. Let P be a Sylow p-subgroup of $O(G)$. Then it is well known that P is a normal subgroup of $O(G)$ (see [5]). Thus P is a normal subgroup of G. Since $G/C_G(P)$ is abelian, we have $G=C_G(P)$. Let S_p be a Sylow p-subgroup of G. Set $R=S_p\cap Z(N_G(S_p))$. Then $R\supset P$. Since S_p is abelian (See [6] Proposition 2.), by [5] (20.12) there exists a normal subgroup G_0 of G such that $G/G_0\cong R$. Since $[G, G]=G$, we have $G=G_0$, and $R=1$. Hence $P=1$. It is a contradiction. Therefore $O(G)=1$.

LEMMA 2.5. *Let G be a perfect subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. We assume that no normal subgroup of G is isomorphic to DQ. Let N be a normal subgroup of G. If N is a 2-group, then $N\subset Z(G)$ and N is cyclic.*

PROOF. The proof is by induction on $|N|$. Let $\Phi(N)$ be the Frattini subgroup of N. By induction $\Phi(N)\subset Z(G)$ and $\Phi(N)$ is cyclic.

First we will prove that N is generated by at most 3 elements.

By [7] $V_Q(N)\cong\Delta_1$, $\Delta_1\oplus\Delta_2$ or $M_2(\Delta_1)$ for some division algebras Δ_1 and Δ_2. If $V_Q(N)\cong\Delta_1\oplus\Delta_2$, then it follows from [7] (2.3) that G has a normal subgroup of index 2. It contradicts the assumption $[G, G]=G$. Therefore $V_Q(N)\cong\Delta_1$ or $M_2(\Delta_1)$. In the case where $V_Q(N)\cong\Delta_1$ N is cyclic or generalized quaternion. It follows that N is generated by at most 2 elements. Hence we may assume that $V_Q(N)\cong M_2(\Delta_1)$. Suppose that Δ_1 is a commutative field. Then $V_C(N)\cong M_2(\Delta_1)\otimes_{\Delta_1}C\cong M_2(C)$. By [6] Lemma 3 N has a normal subgroup N_0 of index 2 such that $V_C(N_0)\cong C\oplus C$. It is easy to see that N_0 is an abelian group generated by at most 2 elements. Therefore N is generated by at most 3 elements. So it may be assumed that Δ_1 is not commutative. If $|\Phi(N)|=1$, then N is abelian, which contradicts the assumption $V_Q(N)\cong M_2(\Delta_1)$. Therefore $|\Phi(N)|\geqq 2$. Suppose $|Z(N)|>2$. Since $Z(N)\subset$the center of $M_2(\Delta_1)$, $Z(N)$ is cyclic. Put $K=$the center of

$M_2(\Delta_1)$. By $|Z(N)|>2$ K has an element of order 4, which implies $K \ni i$. Since K is a splitting field for N, it follows that $\Delta_1=K$. However Δ_1 is not commutative. Thus $|Z(N)|\leqq 2$. Because $\Phi(N)\subset Z(G)$, $2\leqq|\Phi(N)|\leqq |Z(N)|\leqq 2$. Therefore $\Phi(N)=Z(N)$ and $|\Phi(N)|=2$. On the other hand $N/\Phi(N)$ is an elementary abelian group of order $\leqq 2^4$ by [7]. Suppose that $|N/\Phi(N)|=2^4$. Since N is not abelian, $[N, N]=\Phi(N)$. Thus N is an extra-special 2-group of order 32. It is well known that $N\cong DD$ or DQ (see [3]). And by the assumption $N\cong DD$. Since $\boldsymbol{Q}$ is a splitting field for DD (See [3].), it follows that Δ_1 is commutative. It is a contradiction. Thus $|N/\Phi(N)|\leqq 2^3$ and N is generated by at most 3 elements.

Assume that $G/C_G(N)$ is non-solvable. By (1.2) $G/C_G(N)$ has an element of order 5. Let g be an element of G such that the order of $gC_G(N)$ is 5. Since $N/\Phi(N)$ is an elementary abelian group of order $\leqq 2^3$, $|\mathrm{Aut}(N/\Phi(N))|\,|\,|GL(3, 2)|=2^3\cdot 3\cdot 7$. Therefore for any $n\in N$ $g^{-1}ng\Phi(N)=n\Phi(N)$. We put $z=n^{-1}g^{-1}ng$ and $a=$the order of $\Phi(N)=2^t$. Then $(g^a)^{-1}ng^a=nz^a=n$. And the order of $gC_G(N)$ divides $a=2^t$, which is a contradiction. Thus we obtain that $G/C_G(N)$ is a solvable group. By the assumption $[G, G]=G$ we get $G=C_G(N)$. This means $N\subset Z(G)$. Since $N\subset Z(G)\subset$the center of $M_2(\Delta)$, it follows that N is cyclic. The proof of the lemma is completed.

LEMMA 2.6. *Let G be a perfect subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. We assume that no normal subgroup of G is isomorphic to DQ. Let N be the largest solvable normal subgroup of G. Then we have*

(1) *$G/N\cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5)\times PSL(2, 5)$.*

(2) *N is a cyclic 2-group and $N=Z(G)\neq 1$.*

PROOF. By (1.2) G/N is isomorphic to a subgroup W of $\mathrm{Aut}(T)$ with $W\supset T$, where $T\cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5)\times PSL(2, 5)$. It follows from (1.3) that $\mathrm{Aut}(T)/T$ is a 2-group. Therefore $[G, G]=G$ means that $G/N\cong T$.

Next we will show the assertion (2). Suppose that N is not a 2-group. Since $O(G)=1$ by (2.4), there exist normal subgroups N_0, N_1 of G such that $N\supset N_1\supset N_0\neq 1$, N_0 is a 2-group and N_1/N_0 is an elementary abelian p-group for some odd prime p. By (2.5) N_0 is a cyclic group and $N_0\subset Z(G)$, which implies $N_1\cong N_0\times(N_1/N_0)$. Thus $O(G)\supset N_1/N_0\neq 1$. But it is impossible. Hence we obtain that N is a 2-group. By (2.5) $N\subset Z(G)$, and $N=Z(G)$ because $G/N\cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5)\times PSL(2, 5)$.

Now we determine all perfect subgroups G of $M_2(\Delta)$ such that no normal subgroup of G is isomorphic to DQ.

PROPOSITION 2.7. *Let Δ be a division algebra. Let G be a perfect subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. If no normal subgroup of G is isomorphic to DQ, then $G\cong SL(2, 5)$ or $SL(2, 9)$, and $\Delta\cong(\boldsymbol{Q}(\zeta_8), \tau, -1)$, where $\langle\tau\rangle=\mathrm{Gal}(\boldsymbol{Q}(\zeta_8)/\boldsymbol{Q})$.*

PROOF. Let N be the largest solvable normal subgroup of G. By (2.6) $G/N\cong PSL(2, 5)$, $PSL(2, 9)$ or $PSL(2, 5)\times PSL(2, 5)$, and $Z(G)=N$. This means that G is a central extension of G/N with $[G, G]=G$. The central extensions of $PSL(2, 5)$, $PSL(2, 9)$ and $PSL(2, 5)\times PSL(2, 5)$ are well known (see [9] V § 25).

First we assume that $G/N\cong PSL(2, 5)$. Since $|H^2(PSL(2, 5), \boldsymbol{C}^{\times})|=2$, $|N|=2$ and $G\cong SL(2, 5)$.

In the case where $G/N\cong PSL(2, 9)$, since $|H^2(PSL(2, 9), \boldsymbol{C}^{\times})|=6$ and N is a 2-group, we have that $|N|=2$ and $G\cong SL(2, 9)$.

Suppose that $G/N\cong PSL(2,5)\times PSL(2,5)$. Since $H^2(PSL(2,5)\times PSL(2,5), \boldsymbol{C}^{\times})\cong H^2(PSL(2, 5), \boldsymbol{C}^{\times})\times H^2(PSL(2, 5), \boldsymbol{C}^{\times})$, there exists an epimorphism ρ from $SL(2, 5)\times SL(2, 5)$ to G. Put $G_1=\rho(SL(2, 5)\times 1)$ and $G_2=\rho(1\times SL(2, 5))$. Since N is cyclic and $PSL(2, 5)$ is not a subgroup of $M_2(\Delta)$, $G_i\cong SL(2, 5)$, $|G_1\cap G_2|=2$ and $[G_1, G_2]=1$. If $V_Q(G_i)\cong\Delta_1\oplus\Delta_2$ for some division algebras Δ_1, Δ_2, then G has a normal subgroup of index 2 by [7] (2.3), contradicting the assumption $[G, G]=G$. Thus $V_Q(G_i)\cong\Delta^{(i)}$ or $M_2(\Delta^{(i)})$ for some division algebra $\Delta^{(i)}$, $i=1, 2$. By (2.2) $\Delta^{(i)}\cong A_{10,-1}\cong A_{4,-1}\otimes_Q\boldsymbol{Q}(\sqrt{5})$ if $V_Q(G_i)\cong\Delta^{(i)}$. By [10] $\Delta^{(i)}\cong(\boldsymbol{Q}(\zeta_8), \tau, -1)$ if $V_Q(G_i)\cong M_2(\Delta^{(i)})$. In any case it follows from (2.3) that $[G_1, G_2]\neq 1$. But it is impossible. Thus $G/N\not\cong PSL(2, 5)\times PSL(2, 5)$.

In the case where $G\cong SL(2, 5)$ or $SL(2, 9)$, if $\boldsymbol{Q}G\oplus\!> M_2(\Delta)$, then $\Delta\cong(\boldsymbol{Q}(\zeta_8), \tau, -1)$ (see [10]). The proof of proposition is completed.

§ 3. The extra-special 2-group DQ.

In this section we will determine all perfect subgroups of $M_2(\Delta)$. In § 2 we determined these groups G if no normal subgroup of G is isomorphic to DQ. Thus we may assume that G has a normal subgroup which is isomorphic to DQ.

We put $D=\langle a, b\,|\,a^4=1, b^2=1, bab^{-1}=a^{-1}\rangle$ and $Q=\langle c, d\,|\,c^4=1, c^2=d^2, dcd^{-1}=c^{-1}\rangle$. Let set $S=\{x\,|\,x\in DQ, x^2=1\}-\{1\}$. Then S is decomposed into the disjoint conjugate classes of DQ, $S=C_0\cup C_1\cup C_2\cup C_3\cup C_4\cup C_5$, where $C_0=\{a^2\}$, $C_1=\{b, a^2b\}$, $C_2=\{ab, a^3b\}$, $C_3=\{ac, a^3c\}$, $C_4=\{ad, a^3d\}$ and $C_5=\{acd, a^3cd\}$. We set $\Omega=\{C_1, C_2, C_3, C_4, C_5\}$. Let τ be an automorphism of DQ. Since $C_0^{\tau}=C_0$, τ induces a permutation $\tilde{\tau}$ on Ω. Let ϕ be the homomorphism from $\mathrm{Aut}(DQ)$ to S_5 determined by $\phi(\tau)=\tilde{\tau}$. Let $\tau\in\mathrm{Ker}\,\phi$. Then

τ induces the identity map on $DQ/[DQ, DQ]$, and, as is well known, τ is an inner automorphism of DQ. Thus $\operatorname{Ker}\phi = \operatorname{Inn}\operatorname{Aut}(DQ)$. Let $\alpha, \beta, \gamma, \delta$ be the automorphisms of DQ determined by the following;

$a^\alpha = a,\ b^\alpha = ab,\ c^\alpha = c,\ d^\alpha = d,$

$a^\beta = bc^{-1},\ b^\beta = ab,\ c^\beta = c,\ d^\beta = a^{-1}bcd,$

$a^\gamma = bd^{-1},\ b^\gamma = ab,\ c^\gamma = abcd,\ d^\gamma = d,$ and

$a^\delta = a^2bcd,\ b^\delta = ab,\ c^\delta = a^{-1}bd,\ d^\delta = abc.$

Then $\phi(\alpha) = (C_1, C_2)$, $\phi(\beta) = (C_1, C_2, C_3)$, $\phi(\gamma) = (C_1, C_2, C_4)$ and $\phi(\delta) = (C_1, C_2, C_5)$. Since $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$ and $\phi(\delta)$ generate S_5, $\operatorname{Aut}(DQ)/\operatorname{Inn}\operatorname{Aut}(DQ) \cong S_5$ and $\phi(\langle\beta, \gamma, \delta\rangle) \cong A_5 \cong PSL(2, 5)$. It is easy to see that β, γ, δ can be regarded as permutations on $\{b, ab, ac, ad, acd\}$. For any $\sigma \in \operatorname{Aut}(DQ)$, $\sigma = 1$ if σ is the identity permutation on $\{b, ab, ac, ad, acd\}$. Therefore we obtain that $\langle\beta, \gamma, \delta\rangle \cong A_5 \cong PSL(2, 5)$. Let H be a central extension of $\langle\beta, \gamma, \delta\rangle$ by $\langle a^2\rangle$ with $[H, H] = H$. Then $H \cong SL(2, 5)$ (see [9] V § 25). Let $\{u_\sigma \mid \sigma \in \langle\beta, \gamma, \delta\rangle\}$ be a set of representatives of $\langle\beta, \gamma, \delta\rangle$ in H. The set HDQ forms a group if we define $u_\sigma^{-1} x u_\sigma = x^\sigma$, $\sigma \in \langle\beta, \gamma, \delta\rangle$, $x \in DQ$. We denote this group by E. Since $H \cap DQ = \langle a^2\rangle$, E is an extension of $PSL(2, 5)$ by DQ.

LEMMA 3.1. *E is a subgroup of $M_2(\Lambda_{4,-1})$ and $V_Q(DQ) = V_Q(E) = M_2(\Lambda_{4,-1})$.*

PROOF. $\Lambda_{4,-1}$ is the ordinary quaternion algebra over $\boldsymbol{Q}$, i.e. $\Lambda_{4,-1} = \boldsymbol{Q} + \boldsymbol{Q}i + \boldsymbol{Q}j + \boldsymbol{Q}k$ with the relations; $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. Let ρ be the homomorphism from E to $M_2(\Lambda_{4,-1})$ determined by

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \rho(d) = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},$$

$$\rho(u_\beta) = \begin{pmatrix} x & -x \\ \bar{x} & \bar{x} \end{pmatrix}, \quad \rho(u_\gamma) = \begin{pmatrix} y & -y \\ \bar{y} & \bar{y} \end{pmatrix} \quad \text{and} \quad \rho(u_\delta) = \begin{pmatrix} z & -z \\ \bar{z} & \bar{z} \end{pmatrix},$$

where $x = (1-i)/2$, $y = (1-j)/2$ and $z = (1-k)/2$. It is easy to see that $V_Q(\rho(DQ)) = M_2(\Lambda_{4,-1})$.

We will show that ρ is injective. Suppose that $\operatorname{Ker}\rho \cap DQ \neq 1$. Then $\operatorname{Ker}\rho \cap Z(DQ) \neq 1$. Since $|Z(DQ)| = 2$, $\operatorname{Ker}\rho \supset Z(DQ)$. Therefore $\rho(DQ) \cong DQ/\operatorname{Ker}\rho$ is an abelian group, because $DQ/Z(DQ)$ is an elementary abelian group. However $[\rho(c), \rho(d)] \neq 1$. Thus $\operatorname{Ker}\rho \cap DQ = 1$. We set $\Omega' = \{\rho(C_1), \rho(C_2), \rho(C_3), \rho(C_4), \rho(C_5)\}$. Let $\sigma \in E$. Then $\rho(\sigma)$ induces a permutation $\widetilde{\rho(\sigma)}$ on Ω'. We denote by ϕ the mapping $\rho(\sigma) \to \widetilde{\rho(\sigma)}$. We can easily check that $\phi(\langle\rho(u_\beta), \rho(u_\gamma), \rho(u_\delta)\rangle) \cong PSL(2, 5)$ and $\operatorname{Ker}\phi \supset \rho(DQ)$. Therefore $|\rho(E)| = |\rho(E) : \operatorname{Ker}\phi|\,|\operatorname{Ker}\phi| \geqq |PSL(2, 5)|\,|\rho(DQ)| = |PSL(2, 5)|\,|DQ| = |E|$. Thus

ρ is injective. Hence we can regard E as a subgroup of $M_2(\Lambda_{4,-1})$.

The fact $V_Q(DQ)=M_2(\Lambda_{4,-1})$ and the fact $V_Q(DQ)\subset V_Q(E)\subset M_2(\Lambda_{4,-1})$ imply $V_Q(E)=M_2(\Lambda_{4,-1})$, as desired.

Let G be a perfect subgroup of $M_2(\Delta)$. We assume that G has a normal subgroup N which is isomorphic to DQ.

LEMMA 3.2. *If* $V_Q(G)=M_2(\Delta)$, *then* $|C_G(N)|=2$.

PROOF. In the proof of (3.1) we showed that $DQ\subset M_2(\Lambda_{4,-1})$ and $V_Q(DQ)=M_2(\Lambda_{4,-1})$. Since $DQ/[DQ, DQ]$ is an elementary abelian group of order 16, $\boldsymbol{Q}[DQ/[DQ, DQ]]\cong\boldsymbol{Q}\oplus\boldsymbol{Q}\oplus\cdots\oplus\boldsymbol{Q}$. Because $\dim_Q M_2(\Lambda_{4,-1})=16$, $\boldsymbol{Q}[DQ]\cong\boldsymbol{Q}\oplus\boldsymbol{Q}\oplus\cdots\oplus\boldsymbol{Q}\oplus M_2(\Lambda_{4,-1})$. Therefore $V_Q(N)\cong M_2(\Lambda_{4,-1})$. Let P be a Sylow 2-subgroup of $C_G(N)$. Suppose that P has an element x of order 4. Then $V_Q(N)V_Q(\langle x\rangle)\cong V_Q(N)\otimes_Q V_Q(\langle x\rangle)\supset M_2(\Lambda_{4,-1})\otimes_Q Q(i)\cong M_4(\boldsymbol{Q}(i))$. It contradicts the fact $V_Q(N)V_Q(\langle x\rangle)\subset M_2(\Delta)$. This implies that any element of P is of order $\leqq 2$. Thus by [6] P is an elementary abelian group generated by at most 2 elements. It follows from [7] (3.1) that $C_G(N)$ has a normal 2-complement M. Since $O(G)=1$ by (2.4), $M=1$ and $C_G(N)=P$. If $|C_G(N)|=|P|=4$, then $V_Q(P)\cong\boldsymbol{Q}\oplus\boldsymbol{Q}$, and by [7] (2.3) G has a normal subgroup of index 2. But it is impossible. Therefore $|C_G(N)|=|P|=2$.

The factor group $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$. Since $\mathrm{Aut}(N)\cong\langle\alpha, \beta, \gamma, \delta\rangle DQ/[DQ, DQ]$ and $[G, G]=G$, $G/C_G(N)\cong\langle\beta, \gamma, \delta\rangle DQ/[DQ, DQ]$. We denote this isomorphism by ϕ. Let ρ be the morphism from G to $\mathrm{Aut}(DQ)$ determined by the mapping $x\rightarrow\phi(xC_G(N))$. We put $H=\rho^{-1}(\langle\beta, \gamma, \delta\rangle)$. On the other hand $G/C_G(C_G(N))$ is isomorphic to a subgroup of $\mathrm{Aut}(C_G(N))$. Since $|C_G(N)|=2$ and $[G, G]=G$, we have $G=C_G(C_G(N))$, and so $C_G(N)\subset Z(G)$. Because $H/C_G(N)\cong PSL(2, 5)$, H is a central extension of $PSL(2, 5)$ by $C_G(N)$. It follows that $[H, H]C_G(N)/C_G(N)\cong[PSL(2, 5), PSL(2, 5)]\cong PSL(2, 5)$. If $[H, H]\cap C_G(N)=1$, then $[H, H]\cong PSL(2, 5)$ and $[H, H]\subset M_2(\Delta)$. It is a contradiction (see [8]). Therefore $[H, H]\supset C_G(N)$ and $[H, H]=H$. Thus $H\cong SL(2, 5)$. By the definition of E we have $G=HN\cong E$. Let V be an irreducible $M_2(\Delta)$-module. Put $K=$ the center of $M_2(\Delta)$. Since $[G, G]=G$, by [7] (2.3) the number of all isomorphism classes of irreducible KN-submodules of V is 1. Therefore $V\cong U\oplus U\oplus\cdots\oplus U$ as KN-module, where U is an irreducible KN-module. Let χ be an irreducible complex character corresponding to U. Since $\boldsymbol{Q}[DQ]\cong\boldsymbol{Q}\oplus\boldsymbol{Q}\oplus\cdots\oplus\boldsymbol{Q}\oplus M_2(\Lambda_{4,-1})$, we have $\boldsymbol{C}N\cong\boldsymbol{C}[DQ]\cong\boldsymbol{C}\oplus\boldsymbol{C}\oplus\cdots\oplus\boldsymbol{C}\oplus M_4(\boldsymbol{C})$. This shows $\chi(1)=16$, because χ is faithful character. For any $g\in E$ the irreducible character χ^g has degree 16, and $\chi^g=\chi$, because N has only one irreducible character χ of degree 16. This implies $\chi^G|_N=$

$|G:N|\chi$. Since $(\chi^G, \chi^G)_G=(\chi^G|_N, \chi)_N=|G:N|$, χ^G is decomposed into the irreducible complex characters μ_i of G, $\chi^G=\mu_1+\mu_2+\cdots+\mu_t$, where $t=|G:N|$. Since $1\neq(\mu_i, \chi^G)_G=(\mu_i|_N, \chi)_N$, $\mu_i(1)\geqq\chi(1)$. Thus $|G:N|\chi(1)=\chi^G(1)=\sum_{i=1}^t \mu_i(1)\geqq|G:N|\chi(1)$, which implies $\mu_i(1)=\chi(1)=16$. Let μ be an irreducible complex character corresponding to V. Since $(\mu|_N, \chi)_N\neq 1$, we have $\mu(1)=16$, which shows $\dim_K M_2(\Delta)=16=\dim_Q M_2(\Lambda_{4,-1})=\dim_K M_2(\Lambda_{4,-1}\otimes_Q K)$. Since $M_2(\Delta)\supset M_2(\Lambda_{4,-1}\otimes_Q K)$, we have $M_2(\Delta)=M_2(\Lambda_{4,-1}\otimes_Q K)$.

Hence by (2.2) and (2.7) we have

THEOREM 3.3. *Let Δ be a division algebra. Let G be a perfect subgroup of $M_2(\Delta)$. Then one of the following holds:*

(1) $G\cong SL(2, 5)$ *and* $V_Q(G)\cong\Lambda_{10,-1}$;

(2) $G\cong SL(2, 5)$ *and* $V_Q(G)\cong \boldsymbol{Q}\oplus\Lambda_{10,-1}$;

(3) $G\cong SL(2, 5)\times SL(2, 5)$ *and* $V_Q(G)\cong\Lambda_{10,-1}\oplus\Lambda_{10,-1}$;

(4) $G\cong SL(2, 5)$ *and* $V_Q(G)\cong M_2((\boldsymbol{Q}(\zeta_3), \tau, -1))$

(5) $G\cong SL(2, 9)$ *and* $V_Q(G)\cong M_2((\boldsymbol{Q}(\zeta_3), \tau, -1))$; *or*

(6) $G\cong E$ *and* $V_Q(G)\cong M_2(\Lambda_{4,-1}\otimes_Q K)$ *for some commutative field* K.

§ 4. Non-solvable groups.

In this section we consider non-solvable subgroups of $M_2(\Delta)$.

Let G be a non-solvable subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. Then G has a perfect normal subgroup H such that G/H is solvable. By [7] (2.1) $V_Q(H)\cong\Delta_1$, $\Delta_1\oplus\Delta_2$ or $M_2(\Delta_1)$ for some division algebras Δ_1, Δ_2.

LEMMA 4.1. *Let N be a normal subgroup of G. Assume that $V_Q(N)\cong\Delta_1\oplus\Delta_2$. Then*

(1) *G has a normal subgroup G_0 of index 2.*

(2) *Put $G/G_0=\{G_0, gG_0\}$. Then there exist normal subgroups T_1, T_2 of G_0 and relatively prime integers m, r such that $T_1\cap T_2=1$, $T_1^g=T_2$, $G_0/T_1\cong SL(2, 5)\times G_{m,r}$ and $\Delta\cong\Lambda_{10,-1}\otimes_Q\Lambda_{m,r}$.*

PROOF. By [7] (2.3) G has a normal subgroup G_0 of index 2 such that $V_Q(G_0)\cong\Delta\oplus\Delta$. Moreover G_0 has normal subgroups T_1, T_2 satisfying $T_1\cap T_2=1$, $T_1^g=T_2$ and $G_0/T_1\cong\rho(G_0)$, where $\{1, g\}$ is a set of representatives of G/G_0 in G and ρ is the projection of $V_Q(G_0)$ on the first component of $\Delta\oplus\Delta$. Therefore $G_0/T_1\cong G_0/T_2$. If G_0/T_1 is solvable, then G_0/T_1 and $T_1T_2/T_2\cong T_1$ are solvable. This means that G_0 is solvable. But it is impossible. Therefore G_0/T_1 is non-solvable. Since $V_Q(\rho(G_0))=\Delta$, it follows from (2.1) that $\rho(G_0)\cong SL(2, 5)\times G_{m,r}$ and $\Delta\cong\Lambda_{10,-1}\otimes_Q\Lambda_{m,r}$ for some relatively prime integers m, r.

LEMMA 4.2. *Assume that* $V_Q(H)\cong \Delta_1$ *or* $M_2(\Delta_1)$. *Let* P *be a non-cyclic* 2-*subgroup of* $M_2(\Delta)$ *of order* >4.

(1) *If* $V_Q(P)\cong\Gamma_1$ *or* $\Gamma_1\oplus\Gamma_2$ *for some division algebras* Γ_1, Γ_2, *then* $[H, P]\neq 1$.

(2) *Especially, if* P *is the quaternion group of order* 8 *or an abelian group, then* $[H, P]\neq 1$.

PROOF. (1) By (3.3) $V_Q(H)\supset \Lambda_{4,-1}\ni 1$ or $V_Q(H)\supset M_2(\boldsymbol{Q}(\zeta_8))\supset M_2(\boldsymbol{Q})\ni 1$. First we assume that $V_Q(P)\cong\Gamma_1$. Since P is not cyclic, it follows from (2.1) that P is generalized quaternion and $V_Q(P)\supset\Lambda_{4,-1}\ni 1$. By (2.3) we have that $[H, P]\neq 1$. Next we assume that $V_Q(P)\cong\Gamma_1\oplus\Gamma_2$. In the case where $V_Q(H)\supset M_2(\boldsymbol{Q})\ni 1$, if $[H, P]=1$ then $M_2(\Delta)\supset M_2(\boldsymbol{Q})\otimes_Q(\Gamma_1\oplus\Gamma_2)\cong M_2(\Gamma_1)\oplus M_2(\Gamma_2)$. It is a contradiction. So we may assume that $V_Q(H)\supset \Lambda_{4,-1}\ni 1$. In the case where P is abelian, since P is generated by at most 2 elements, $|P|>4$ implies that P has an element of order 4. Thus $V_Q(P)\supset\boldsymbol{Q}\oplus\boldsymbol{Q}(i)\ni 1$. If $[P, H]=1$, then $M_2(\Delta)\supset\Lambda_{4,-1}\otimes_Q(\boldsymbol{Q}\oplus\boldsymbol{Q}(i))\cong\Lambda_{4,-1}\oplus M_2(\boldsymbol{Q}(i))$, which is a contradiction. Therefore $[P, H]\neq 1$. In the case where P is non-abelian, $\Gamma_1\supset\Lambda_{4,-1}$ or $\Gamma_2\supset\Lambda_{4,-1}$. Thus $V_Q(P)\supset\boldsymbol{Q}\oplus\Lambda_{4,-1}\ni 1$. If $[H, P]=1$, then $M_2(\boldsymbol{Q})\otimes_Q(\boldsymbol{Q}\oplus\Lambda_{4,-1})\cong M_2(\boldsymbol{Q})\oplus M_2(\Lambda_{4,-1})$. Thus $[H, P]\neq 1$.

(2) If P is the quaternion group of order 8 or an abelian group, then $\boldsymbol{Q}P$ does not contain a simple algebla which is isomorphic to $M_2(\Gamma)$ for some division algebra Γ. Thus $V_Q(P)\cong\Gamma_1$ or $\Gamma_1\oplus\Gamma_2$ for some division algebra Γ_1, Γ_2. Therefore by (1) $[H, P]\neq 1$.

We now have

THEOREM 4.3. *Let* Δ *be a division algebra. Let* G *be a non-solvable subgroup of* $M_2(\Delta)$ *such that* $V_Q(G)=M_2(\Delta)$. *Then* G *satisfies one of the following conditions* (1) *and* (2).

(1) G *has a normal subgroup* G_0 *of index* 2. *Put* $G/G_0=\{G_0, gG_0\}$. *Then there exist normal subgroups* T_1, T_2 *of* G_0 *and relatively prime integers* m, r *such that* $T_1\cap T_2=1$, $T_1^g=T_2$, $G_0/T_1\cong SL(2, 5)\times G_{m,r}$ *and* $\Delta\cong\Lambda_{10,-1}\otimes_Q\Lambda_{m,r}$.

(2) *Let* H *be the perfect normal subgroup of* G *such that* G/H *is solvable. Then* H *and* $C_G(H)$ *satisfy the one of the following conditions.*

(i) $H\cong SL(2, 5)$, $SL(2, 9)$ *or* E, *and* $C_G(H)\cong G_{m,r}$ *for some relatively prime integers* m, r.

(ii) $H\cong SL(2, 5)$, $O(C_G(H))\cong G_{m,r}$ *for some relatively prime integers* m, r, *and* $C_G(H)/O(C_G(H))$ *is a cyclic* 2-*group or a dihedral group of order* $2^n\geqq 4$.

PROOF. Let H be the perfect normal subgroup of G such that G/H

is solvable. We assume that G does not satisfy the condition (1). Then (4.1) implies that $V_Q(H)\cong\Delta_1$ or $M_2(\Delta_1)$, $V_Q(C_G(H))\cong\Delta_2$ or $M_2(\Delta_2)$ for some division algebras Δ_1, Δ_2. Since G/H is solvable, $C_G(H)/(H\cap C_G(H))$ is solvable, which implies $C_G(H)$ is solvable.

First we assume that $V_Q(H)\cong M_2(\Delta_1)$. Then it follows from (2.3) $V_Q(C_G(H))\cong\Delta_2$. By (2.1) and (4.2) a Sylow 2-subgroup of $C_G(H)$ is cyclic, and $C_G(H)\cong G_{m,r}$ for some relatively prime integers m, r. By (3.3) $H\cong SL(2,5)$, $SL(2,9)$ or E.

We assume that $V_Q(H)\cong\Delta_1$. In this case $H\cong SL(2,5)$ and $V_Q(H)\cong \Delta_{10,-1}$, by (3.3). If $V_Q(C_G(H))\cong\Delta_2$, then $C_G(H)\cong G_{m,r}$ for some relatively prime integers m, r. Thus we may assume that $V_Q(C_G(H))\cong M_2(\Delta_2)$.

Let P be a Sylow 2-subgroup of $C_G(H)$. Suppose that P is abelian. By [7] (3.1) $C_G(H)/O(C_G(H))\cong P$. If P is a non-cyclic group of order >4, then $[P,H]\neq 1$ by (4.2). It is a contradiction. Thus P is a cyclic group or an elementary abelian group of order 4.

Next we suppose that P is not abelian. We will prove that P is a dihedral group. By (4.2) $V_Q(P)\cong M_2(\Gamma)$ for some division algebra Γ. If Γ is not a commutative field, then $M_2(\Gamma)\supset M_2(\Delta_{4,-1})\ni 1$. Since $V_Q(H)\supset \Delta_{4,-1}\ni 1$, it follows from (2.3) $[P,H]\neq 1$. It is impossible. Thus Γ is a commutative field. If P does not have a cyclic subgroup of index 2, then P has a subgroup P_0 of index 2 such that $V_Q(P_0)\cong\Gamma\oplus\Gamma$. Since Γ is commutative, P_0 is an abelian group. By (4.2) $|P_0|\leqq 4$, and P has a cyclic subgroup of index 2. It is a contradiction. Thus P has a cyclic subgroup of index 2. In the case where $P\cong\langle a,b\,|\,a^{2^n}=1,\ b^2=1,\ bab^{-1}=a^{1+2^{n-1}}\rangle$ $n\geqq 3$, $Z(P)=\langle a^2\rangle$ and $\Gamma\ni i$. Therefore $V_Q(P)\supset M_2(\boldsymbol{Q}(i))\ni 1$. It contradicts the fact $P\subset C_G(H)$ by (2.3). Hence it follows from (4.2) that P is a dihedral group.

We will show that $C_G(H)/O(C_G(H))\cong P$. Suppose that $C_G(H)/O(C_G(H))\not\cong P$. Then $C_G(H)$ has normal subgroups K_0, K_1, K_2 such that $C_G(H)\supset K_2\supset K_1\supset K_0=O(C_G(H))$, K_2/K_1 is an elementary abelian p-group for some odd prime p and K_1/K_0 is a 2-group. If K_1/K_0 is abelian, then by [7] (3.1) K_2 has a normal 2-complement K. Since K is a characteristic subgroup of K_2, $C_G(H)\triangleright K$ and $O(C_G(H))\supset K$, which is a contradiction. Thus K_1/K_0 is not abelian. Since K_1/K_0 is a subgroup of dihedral group, K_1/K_0 is a dihedral group and $\mathrm{Aut}(K_1/K_0)$ is a 2-group. Let L/K_0 be a Sylow p-subgroup of $C_{K_2/K_0}(K_1/K_0)$. Then $[L/K_0, K_1/K_0]=1$ and $L/K_0\cong K_2/K_1$, because $|K_2/K_0\colon C_{K_2/K_0}(K_1/K_0)|\,\big|\,|\mathrm{Aut}(K_1/K_0)|$. Thus we have that $K_2/K_0\cong (L/K_0)\times(K_1/K_0)$. Hence K_2 has a normal 2-complement L, which is a contradiction. Thus we conclude that $C_G(H)/O(C_G(H))\cong P$.

Finally we will prove that $O(C_G(H))\cong G_{m,r}$ for some relatively prime integers m, r. If $V_Q(O(C_G(H)))\cong \Delta_1\oplus\Delta_2$ for some division algebras Δ_1, Δ_2, then by (4.1) G satisfies the condition (1). So $V_Q(O(C_G(H)))$ is a division algebra. It follows from (2.1) that $O(C_G(H))\cong G_{m,r}$ for some relatively prime integers m, r.

THEOREM 4.4. *Let Δ be a division algebra. Let G be a non-solvable subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. Assume that G does not satisfy the condition* (1) *in* (4.3). *Then there exists a chain of normal subgroups of G, $G\supset G_1\supset G_2=O(G)$, which satisfies the following conditions* (1)-(3).

(1) *$G_1/G_2\cong SL(2, 5)P$, $SL(2, 9)$ or E, where P is a cyclic 2-group or a dihedral group of order $2^n\geqq 4$, and $SL(2, 5)P$ is the central product of $SL(2, 5)$ and P.*

(2) *G/G_1 is a 2-group. The order $|G/G_1|\leqq 4$ if $G_1/G_2\cong SL(2, 5)P$, $\leqq 8$ if $G_1/G_2\cong SL(2, 9)$, $\leqq 2$ if $G_1/G_2\cong E$.*

(3) *$O(G)\cong G_{m,r}$ for some relatively prime integers m, r with $(n, t)=1$.*

PROOF. Let H be the perfect normal subgroup of G such that G/H is solvable. Let N be the largest solvable normal subgroup of G. Since G does not satisfy the condition (1) in (4.3), it follows from (4.1) that $V_Q(O(G))$ is a division algebra. By (2.1) $O(G)\cong G_{m,r}$ for some relatively prime integers m, r with $(n, t)=1$, and $N\supset O(G)$.

Suppose that $H\cong SL(2, 5)$ or $SL(2, 9)$. For any $h\in H$, $n\in N$, we have $[h, n]=\pm 1$, because $H\cap N=\{\pm 1\}$. Therefore $n^{-2}hn^2=h$, which implies $|N: C_N(H)|\leqq 2$. Since $C_G(H)$ is a solvable normal subgroup of G by (4.3), we have $N\supseteq C_G(H)$ and $C_N(H)=C_G(H)\supseteq O(G)$. We put $G_1=HC_G(H)$. Since $|\mathrm{Aut}(PSL(2, 5))/PSL(2, 5)|=2$ and $|\mathrm{Aut}(PSL(2, 9))/PSL(2, 9)|=4$, it follows from (1.2) that $|G/HN|\leqq 2$ if $H\cong SL(2, 5)$, $\leqq 4$ if $H\cong SL(2, 9)$. Thus $|G/HC_G(H)|\leqq 4$ if $H\cong SL(2, 5)$, $\leqq 8$ if $H\cong SL(2, 9)$.

Let P be a Sylow 2-subgroup of $C_G(H)$. Then HP is the central product of H and P. By (4.3) if $H\cong SL(2, 5)$, then P is a cyclic group or a dihedral group of order $\geqq 4$. Suppose that $H\cong SL(2, 9)$. By the proof of (4.3) and by (3.3) $V_Q(H)\cong M_2((\boldsymbol{Q}(\zeta_3), \tau, -1))$ and $V_Q(C_G(H))$ is a division algebra. If $C_G(H)$ has an element of order 4, then $V_Q(C_G(H))\supset \boldsymbol{Q}(i)\ni 1$, and $M_2(\Delta)\supset M_2((\boldsymbol{Q}(\zeta_3), \tau, -1))\otimes_Q \boldsymbol{Q}(i)\cong M_4(\boldsymbol{Q}(i))$. But it is impossible. Therefore $|P|=2$ if $H\cong SL(2, 9)$.

We now assume that $H\cong E$. Since $\boldsymbol{Q}[DQ]\cong \boldsymbol{Q}\oplus\boldsymbol{Q}\oplus\cdots\oplus\boldsymbol{Q}\oplus M_2(\Delta_{4,-1})$ we have $V_Q(DQ)\cong M_2(\Delta_{4,-1})$. It follows from (2.3) that $V_Q(C_G(DQ))$ is a division algebra. If $C_G(DQ)$ has an element of order 4, then $V_Q(C_G(DQ))\supset \boldsymbol{Q}(i)\ni 1$, which is a contradiction. Therefore the order of a Sylow 2-subgroup of $C_G(DQ)$ is 2. We set $G_1=EC_G(DQ)$. Since $|\mathrm{Aut}(DQ): (E/[DQ, DQ])|=2$, $|G/EC_G(DQ)|\leqq 2$. Thus we have $O(G)=O(C_G(DQ))$, which means $G_1/G_2\cong E$, because $|E \cap C_G(DQ)|=2$. The proof of the theorem is completed.

§ 5. Additional result.

Let G be a subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. Let P be a Sylow 2-subgroup of G. Then $V_Q(P)\cong \Delta_1$, $\Delta_1 \oplus \Delta_2$ or $M_2(\Delta_1)$, where Δ_1 and Δ_2 are commutative fields or the quaternion algebras $\Lambda_{2^n,-1}$ (see [6]). We put $H_n=\Lambda_{2^n,-1}$. In [7] we considered all finite subgroups of $M_2(\Delta)$ with abelian Sylow 2-groups. So we may assume that P is not abelian. If $V_Q(P)\cong \Delta_1$, then P is a generalized quaternion group.

Here we will prove a proposition which gives an information on G in the case where $V_Q(P)\cong \Delta_1\oplus\Delta_2$ or $M_2(\Delta_1)$.

PROPOSITION 5.1. *Let Δ be a division algebra. Let G be a subgroup of $M_2(\Delta)$ such that $V_Q(G)=M_2(\Delta)$. Let P be a Sylow 2-subgroup of G. Assume that $V_Q(P)$ satisfies one of the following conditions.*

(1) *$V_Q(P)\cong H_n\oplus K$, $n\geqq 3$, where K is a commutative field.*

(2) *$V_Q(P)\cong H_n\oplus H_m$, $n\geqq 3$, $n\geqq m\geqq 2$.*

(3) *$V_Q(P)\cong M_2(H_n)$, $n\geqq 3$.*

Then the Schur index of Δ is 2, and G is a subgroup of $GL(4, \boldsymbol{C})$.

To prove this proposition we will use the following result.

(5.2) (Benard-Schacher [2]). Let χ be an irreducible complex character of finite group. Then $\zeta_m \in \boldsymbol{Q}(\chi)$, if $m_Q(\chi)=m$.

PROOF OF PROPOSITION. Let s be the Schur index of Δ. Then by (5.2) ζ_s is contained in the center of Δ. Thus $V_{Q(\zeta_s)}(P)\subset M_2(\Delta)$. We denote by L_n the center of H_n. Then $L_n=\boldsymbol{Q}(\zeta_a+\zeta_a^{-1})$, where $a=2^n$.

First we show that $\boldsymbol{Q}(\zeta_s)$ is not a splitting field for H_n. Assume that $\boldsymbol{Q}(\zeta_s)$ is a splitting field for H_n. In the case (1), $M_2(\Delta)\supset V_{Q(\zeta_s)}(P)\cong \boldsymbol{Q}(\zeta_s)\otimes_{L_n}H_n\oplus \boldsymbol{Q}(\zeta_s)\otimes_K K\cong M_2(L_n(\zeta_s))\oplus K(\zeta_s)$, which is a contradiction. In the case (2), $M_2(\Delta)\supset V_{Q(\zeta_s)}(P)\cong M_2(L_n(\zeta_s))\oplus \boldsymbol{Q}(\zeta_s)\otimes_{L_m}H_n$, which is a contradiction. If $V_Q(P)\cong M_2(H_n)$, then $V_{Q(\zeta_s)}(P)\cong \boldsymbol{Q}(\zeta_s)\otimes_{L_n}M_2(H_n)\cong M_4(L_n(\zeta_s))$, which implies $V_{Q(\zeta_s)}(P)\not\subset M_2(\Delta)$. Thus $\boldsymbol{Q}(\zeta_s)$ is not a splitting field for H_n.

Next we show that $\boldsymbol{Q}(\zeta_s)$ is a splitting field for H_n if $s>2$. Since $L_n(\zeta_s)\supset \boldsymbol{Q}(\zeta_8+\zeta_8^{-1})=\boldsymbol{Q}(\sqrt{2})$ by the assumption on n, the local degrees of $L_n(\zeta_s)$ at all primes of $L_n(\zeta_s)$ extending the rational prime (2) are even. If $s>2$, then $L_n(\zeta_s)$ is totally imaginary. It follows from [4] that $L_n(\zeta_s)$ is a splitting field for $H_2=\Lambda_{4,-1}$. Thus $H_n\otimes_{L_n}\boldsymbol{Q}(\zeta_s)\cong(\Lambda_{4,-1}\otimes_Q L_n)\otimes_{L_n}\boldsymbol{Q}(\zeta_s)\cong \Lambda_{4,-1}\otimes_Q L_n(\zeta_s)\cong M_2(L_n(\zeta_s))$. Hence we conclude that $s\leqq 2$.

Finally we show that $s=2$. Suppose that $s=1$. Then Δ is a field, and $V_Q(P)\subset M_2(\Delta)\subset M_2(\Delta)\otimes_\Delta C=M_2(C)$. It follows that $V_C(P)\subset M_2(C)$. But it is impossible. In fact, $V_C(P)\cong(H_n\otimes_{L_n}C)\oplus(K\otimes C)\cong M_2(C)\oplus C$ if $V_Q(P)\cong H_n\oplus K$, $V_C(P)\cong(H_n\otimes_{L_n}C)\oplus(H_m\otimes_{L_m}C)\cong M_2(C)\oplus M_2(C)$ if $V_Q(P)\cong H_n\oplus H_m$, and $V_C(P)\cong M_2(H_n)\otimes_{L_n}C\cong M_4(C)$ if $V_Q(P)\cong M_2(H_n)$.

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